Concatenation of cubic structure

Tamar Ziegler

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Motivation: Polynomial progressions in Primes

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- Also conjecture asymptotics

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Example: number of 4-term arithmetic progressions of primes:

$$\#\{x, m \leq N : x, x+m, x+2m, x+3m \in \mathbb{P}\} \sim \frac{9}{2} \prod_{p \geq 5} (1 - \frac{3p-1}{(p-1)^3}) \frac{N^2}{(\log N)^4}$$

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For example:

$$x,\ x + n^2,\ x + 2n^2 + n + 4;$$

$$x,\ x + nm,\ x + nk,\ x + nm + nk$$

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Working example

$x,\ x+nm,\ x+nk,\ x+nm+nk \qquad n,m,k \leq \sqrt{N}, x \leq N$

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- The former (at least naively) can not !

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Consider a variant : $H = \sqrt{N}$

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Theorem (Matomäki - Radziwill (2015))

$$\mathbb{E}_{x\leq N}(\mathbb{E}_{m\leq H}\mu(x+m))^2=o(1)$$

when $H\to\infty$ as slow as we wish.

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$$x+y,\ x+y+m,\ x+y+k,\ x+y+m+k; \qquad y,m,k \le H, x \le N$$

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then think of $f_{x,H}$ as a function on $\mathbb{Z}/H\mathbb{Z}$

$$\mathbb{E}_{y,u,v \le H} f_x(y) f_x(y+u) f_x(y+v) f_x(y+u+v) \le \sup_{r \le H} |\hat{f}_{x,H}(r)|^2$$

Problem: need to understand shorts scale Fourier coefficients of arithmetic functions !

Example

The Möbius function

$$\mu(n) = \begin{cases} (-1)^k & n = p_1 \dots p_k \quad \text{where } p_i \text{ are distinct primes} \\ 0 & \text{otherwise} \\ 1 & n = 1 \end{cases}$$

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Conjecture: short scale Fourier behavior of Möbius function

$$\frac{1}{NH} \sum_{x \le N} \sup_{\alpha} \Big| \sum_{n \in [x, x+H]} \mu(n) e(n\alpha) \Big| = o(1).$$

where $H \to \infty$ as $N \to \infty$.

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where $H \to \infty$ as $N \to \infty$. Not known even for $H = \sqrt{N}$! would imply

$$\frac{1}{NH^2} \sum_{x \leq N, m, k \leq H} \mu(x)\mu(x+m)\mu(x+k)\mu(x+m+k) = o(1).$$

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- $K = \{n_1 m\}_{m \in \sqrt{N}}$ and
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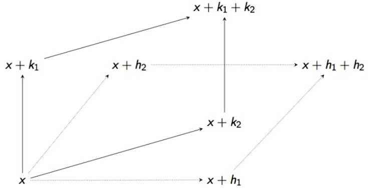
- Look at cubes along K, H:



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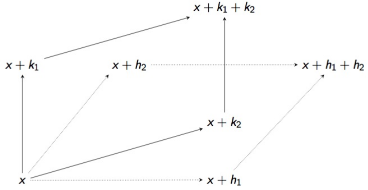

Key question

Can we concatenate "cubic" information along K and H to "cubic" information along $K+H$?

Concatenating cubic structure

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- a-dim cubic averages along H
- b-dim cubic averages along K

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$$\Delta_h \Delta_k f(x) = f(x)\overline{f(x+h)f(x+k)}f(x+h+k)$$

Say that f is a H-polynomial of degree $< m$

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- but not in G; it is of degree < 3 in G.

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Harder: "99% world":

- $\Delta_{h_1} \dots \Delta_{h_a} f = 1$ on $1-\delta$ proportion of H-cubes.
- $\Delta_{k_1} \dots \Delta_{k_b} f = 1$ on $1-\delta$ proportion of K-cubes.

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What if we only have information on a small proportion of cubes ?

Concatenating Gowers norms

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The U^a_H-Gowers norm:

$$\|f\|_{U^a_H}^{2^a} = \mathbb{E}_{x \in G, h_i \in H} \Delta_{h_1} \dots \Delta_{h_a} f(x)$$

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Difficulties:

- No uniqueness: if f is a sum of a few H-polynomials (this is not a polynomial !) then $\|f\|_{U^a_H}^{2^a} \gg 1$.

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Question

Given $K, H < G$ Can we relate $Z_H^a(X)$, $Z_K^b(X)$ and $Z_{H+K}^{a+b-1}(X)$?

Theorem (Tao-Z); ergodic concatentation

$$Z^a_H(X) \wedge Z^b_K(X) \subset Z^{a+b-1}_{H+K}(X)$$

Namely: if f is measurable with respect to $Z^a_H(X)$, and $Z^b_K(X)$ then it is also measurable with respect to $Z^{a+b-1}_{H+K}(X)$.

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Recall: G finite abelian group (say $\mathbb{Z}/N\mathbb{Z}$), H, K are "subgroups".

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If $\|f\|_{U_{H_i}^{a_i}} > \delta$ for many i, then

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Corollary

Let $H_n = \{nm\}_{m \leq \sqrt{N}}$. Suppose $\|f\|_{U^3_{\mathbb{Z}/N\mathbb{Z}}} = o(1)$, then $(2+2-1=3)$

$$\mathbb{E}_{x \leq N, n, m, k \leq \sqrt{N}} \Delta_{nm} \Delta_{nk} f = o(1)$$

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- Can use concatenation technique to get polynomial progressions in primes with gap as small as $(\log N)^{O(1)}$.

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Show that intersection of space "generated" by $D_K f$'s and space "generated" by $D_H f$'s lies in the space "generated" by $D_{H+K} f$'s.