The Green-Tao theorem and a relative Szemerédi theorem

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Joint work with David Conlon (Oxford) and Jacob Fox (MIT)

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Green–Tao Theorem (arXiv 2004; Annals 2008)

The primes contain arbitrarily long arithmetic progressions.

Examples:

- 3, 5, 7
- 5, 11, 17, 23, 29
- 7, 37, 67, 97, 127, 157
- Longest known: 26 terms

Green–Tao Theorem (2008)

The primes contain arbitrarily long arithmetic progressions (AP).

Szemerédi's Theorem (1975)

Every subset of $\mathbb{N}$ with positive density contains arbitrarily long APs.

(upper) density of $A \subset \mathbb{N}$ is $\limsup_{N\to\infty} \frac{|A \cap [N]|}{N}$

$[N] := \{1, 2, \ldots, N\}$

$P =$ prime numbers

Prime number theorem: $\frac{|P \cap [N]|}{N} \sim \frac{1}{\log N}$

Proof strategy of Green–Tao theorem

$P =$ prime numbers, $Q =$ "almost primes"

$P \subseteq Q$ with relative positive density, i.e., $\dfrac{|P \cap [N]|}{|Q \cap [N]|} > \delta$

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Relative Szemerédi theorem (informally)

If $S \subset \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of S of positive density contains long APs.

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Relative Szemerédi theorem (informally)

If $S \subset \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of S of positive density contains long APs.

Step 2: Construct a superset of primes that satisfies the conditions.

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What pseudorandomness conditions?

Green–Tao:

1. Linear forms condition
2. Correlation condition

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A natural question (e.g., asked by Green, Gowers, . . .)

Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

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Green–Tao:

1. Linear forms condition
2. Correlation condition $\leftarrow$ no longer needed

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Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

Our main result

Yes! A weak linear forms condition suffices.

Szemerédi's theorem

Host set: $\mathbb{N}$

Relative Szemerédi theorem

Host set: some sparse subset of integers

Conclusion: relatively dense subsets contain long APs

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Random host set

- Kohayakawa–Łuczak–Rödl '96 — *3-AP,* $p \gtrsim N^{-1/2}$
- Conlon–Gowers '10+
Schacht '10+ — *k-AP,* $p \gtrsim N^{-1/(k-1)}$

Pseudorandom host set

- Green–Tao '08 — *linear forms + correlation*
- Conlon–Fox–Z. '13+ — *linear forms*

Conclusion: relatively dense subsets contain long APs

Roth's theorem

Roth's theorem (1952)

If $A \subseteq [N]$ is 3-AP-free, then $|A| = o(N)$.

$[N] := \{1, 2, \dots, N\}$

3-AP $=$ 3-term arithmetic progression

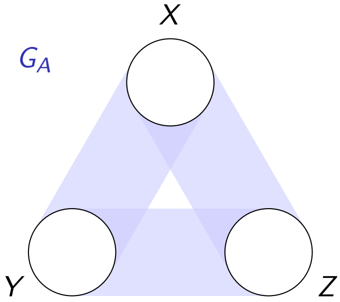
It'll be easier (and equivalent) to work in $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$.

Proof of Roth's theorem

Roth's theorem (1952)

If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Given A, construct tripartite graph G_A with vertex sets $X = Y = Z = \mathbb{Z}_N$.



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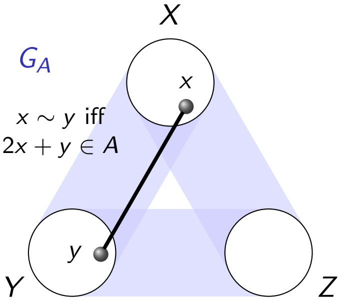

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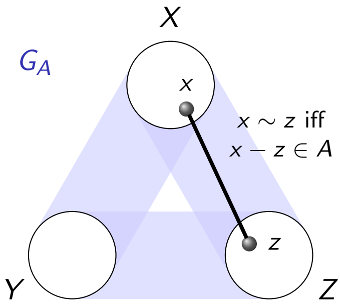

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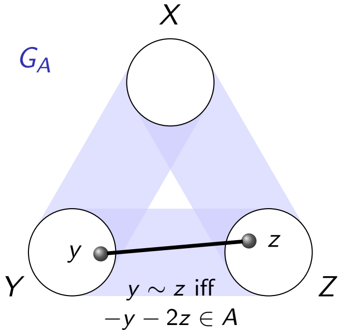

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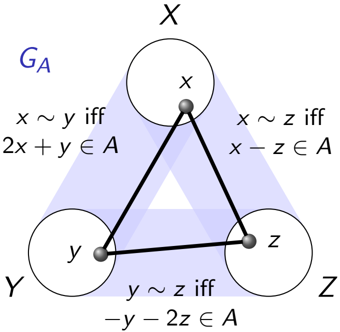

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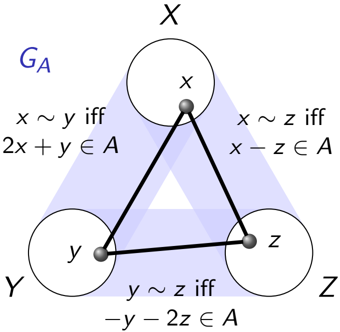
Triangle xyz in $G_A \iff$

$$2x + y,\ x - z,\ -y - 2z\ \in A$$



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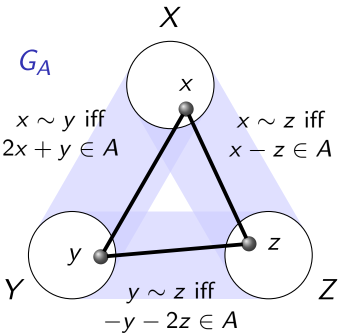

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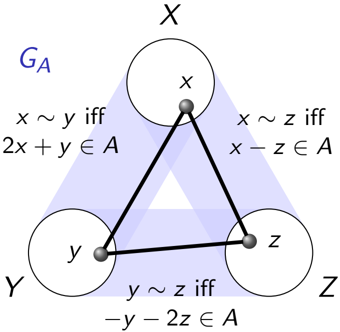

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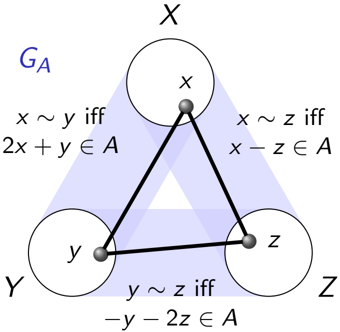

~~No triangles?~~ Only triangles $\longleftrightarrow$ trivial 3-APs with diff 0.

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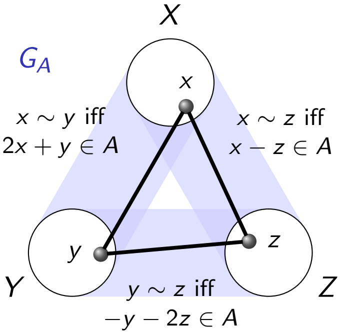

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Every edge of the graph is contained in exactly one triangle (the one with $x + y + z = 0$).

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If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Constructed a graph with

- $3N$ vertices
- $3N|A|$ edges
- every edge in exactly one triangle

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Theorem (Ruzsa & Szemerédi '76)

If every edge in a graph $G = (V, E)$ is contained in exactly one triangle, then $|E| = o(|V|^2)$.

(a consequence of the *triangle removal lemma*)

So $3N|A| = o(N^2)$. Thus $|A| = o(N)$.

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If $S \subseteq \mathbb{Z}_N$ satisfies the 3-linear forms condition,
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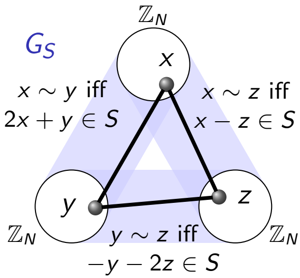

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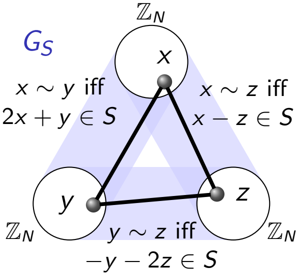
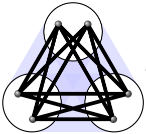
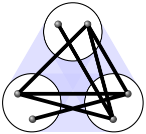


3-linear forms condition:

G_S has asymp. the expected number of embeddings of $K_{2,2,2}$ & its subgraphs (compared to random graph of same density)

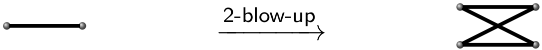


$K_{2,2,2}$ & subgraphs, e.g.,



Analogy with quasirandom graphs

Chung-Graham-Wilson '89 showed that in constant edge-density graphs, many quasirandomness conditions are equivalent, one of which is having the correct C_4 count



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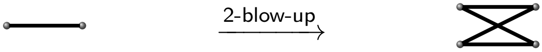

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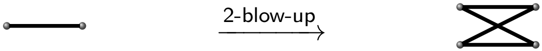

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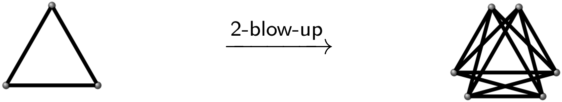
Our results can be viewed as saying that:

Many extremal and Ramsey results about H (e.g., $H = K_3$) in sparse graphs hold if there is a host graph that behaves pseudorandomly with respect to counts of the 2-blow-up of H.



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Fix $k \geq 3$. If $S \subseteq \mathbb{Z}_N$ satisfies the k-linear forms condition, and $A \subseteq S$ is k-AP-free, then $|A| = o(|S|)$.

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$k = 4$: build a 4-partite 3-uniform hypergraph

Vertex sets $W = X = Y = Z = \mathbb{Z}_N$

- $xyz \in E \iff 3w + 2x + y \in S$
- $wxz \in E \iff 2w + x - z \in S$
- $wyz \in E \iff w - y - 2z \in S$
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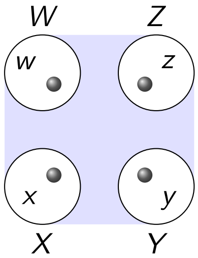

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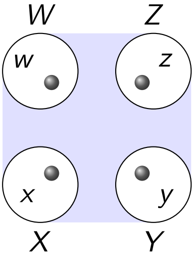

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4-linear forms condition: correct count of the 2-blow-up of the simplex $K_4^{(3)}$ (as well as its subgraphs)

Two approaches

Conlon, Fox, Z.
A relative Szemerédi theorem
20pp

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Conlon, Fox, Z. **More general**

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Z. **More direct**

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Roth's theorem: from one 3-AP to many 3-APs

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By an averaging argument (Varnavides), we get many 3-APs:

Roth's theorem (counting version)

$\forall \delta > 0 \; \exists c > 0$ so that for sufficiently large N,
every $A \subset \mathbb{Z}_N$ with $|A| \geq \delta N$ contains at least cN^2 many 3-APs.

Transference

Start with

$$\text{(sparse)} \qquad A \subset S \subset \mathbb{Z}_N, \qquad |A| \geq \delta\,|S|$$

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$$\geq cN^2 \qquad \text{[By Roth's Theorem]}$$

$\Longrightarrow$ relative Roth theorem

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Roth's theorem (weighted version)

$\forall\delta > 0\ \exists c > 0$ so that for sufficiently large N,
every $f : \mathbb{Z}_N \to [0, 1]$ with $\mathbb{E}f \geq \delta$ satisfies

$$AP_3(f) := \mathbb{E}_{x,d\in\mathbb{Z}_N}[f(x)f(x+d)f(x+2d)] \geq c.$$

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Sparse setting: some sparse host set $S \subset \mathbb{Z}_N$.
More generally, use a normalized measure:

$$\nu \colon \mathbb{Z}_N \to [0, \infty) \qquad \text{with} \qquad \mathbb{E}\nu = 1.$$

E.g., $\nu = \frac{N}{|S|} 1_S$ normalized indicator function.

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The subset $A \subset S$ with $|A| \ge \delta|S|$ corresponds to

$$f\colon \mathbb{Z}_N \to [0,\infty), \qquad \mathbb{E}f \ge \delta$$

and f majorized by ν, meaning that $f(x) \le \nu(x)\ \forall x \in \mathbb{Z}_N$.

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Relative Roth theorem (Conlon, Fox, Z.)

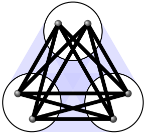
$\forall \delta > 0\ \exists c > 0$ so that for sufficiently large N, if

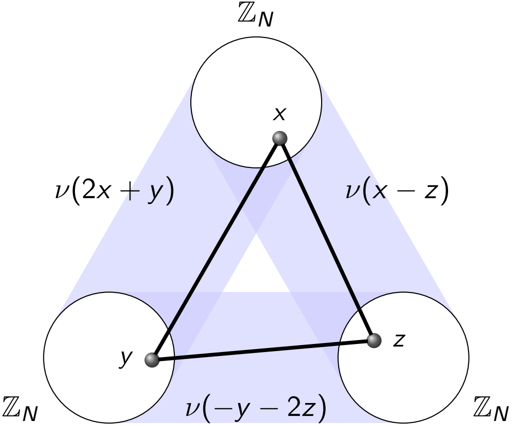
- $\nu \colon \mathbb{Z}_N \to [0,\infty)$ satisfies the 3-linear forms condition, and
- $f \colon \mathbb{Z}_N \to [0,\infty)$ majorized by ν and $\mathbb{E}f \geq \delta$, then

$$AP_3(f) \geq c.$$

Recall $AP_3(f) = \mathbb{E}_{x,d \in \mathbb{Z}_N}[f(x)f(x+d)f(x+2d)]$

3-linear forms condition

The density of $K_{2,2,2}$  in



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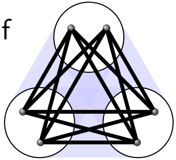

$$\begin{aligned}
&\mathbb{E}[\nu(2x+y)\nu(2x'+y)\nu(2x+y')\nu(2x'+y')\cdot \\
&\quad \nu(x-z)\nu(x'-z)\nu(x-z')\nu(x'-z')\cdot \\
&\quad \nu(-y-2z)\nu(-y'-2z)\nu(-y-2z')\nu(-y'-2z')] = 1 + o(1)
\end{aligned}$$

as well as if any subset of the 12 factors were deleted.

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Start with $f \leq \nu$

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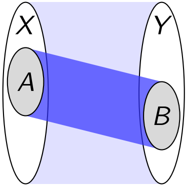
Using cut norm:

- Cheaper dense model theorem
- Trickier counting lemma

Cut norm for weighted bipartite graph (Frieze-Kannan):
$g\colon X \times Y \to \mathbb{R}$

$$\|g\|_{\square} := \frac{1}{|X|\,|Y|} \sup_{\substack{A \subset X \\ B \subset Y}} \left| \sum_{\substack{x \in A \\ y \in B}} g(x, y) \right|$$



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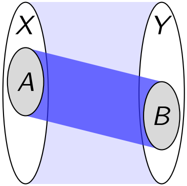

Cut norm for $\mathbb{Z}_N$: $f\colon \mathbb{Z}_N \to \mathbb{R}$

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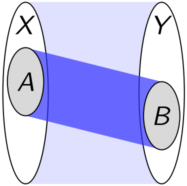

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Dense model theorem

Assume $\nu \colon \mathbb{Z}_N \to [0,\infty)$ satisfies $\|\nu - 1\|_\square = o(1)$.
Then $\forall\ 0 \le f \le \nu$, $\exists\ \widetilde{f} \colon \mathbb{Z}_N \to [0,1]$ s.t. $\|f - \widetilde{f}\|_\square = o(1)$.

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Proof of the general dense model theorem

1. Regularity-type energy-increment argument
   (Green–Tao, Tao–Ziegler)
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Specialized/simplified for the cut norm on $\mathbb{Z}_N$ (Z.)

Higher cut norms

For 4-AP

3-uniform weighted hypergraph $g \colon X \times Y \times Z \to \mathbb{R}$, define

$$\|g\|_{\square} = \frac{1}{|X|\,|Y|\,|Z|} \sup_{\substack{A \subset Y \times Z \\ B \subset X \times Z \\ C \subset X \times Y}} \left| \sum_{\substack{x \in X, y \in Y, z \in Z \\ (y,z) \in A \\ (x,z) \in B \\ (x,y) \in C}} g(x, y, z) \right|.$$

i.e., supremum taken over all 2-graphs between X, Y, Z

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Weighted graphs $g, \widetilde{g} \colon (X \times Y) \cup (X \times Z) \cup (Y \times Z) \to \mathbb{R}$
Triangle density $t(g) := \mathbb{E}_{x,y,z}[g(x,y)g(x,z)g(y,z)]$

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Assume $0 \le g, \widetilde{g} \le 1$. If $\|g - \widetilde{g}\|_\square \le \epsilon$, then

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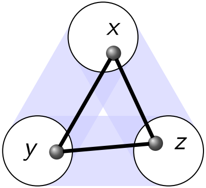

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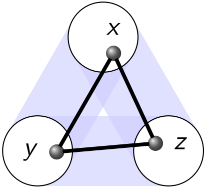

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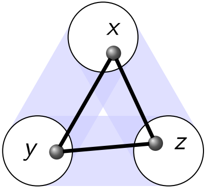

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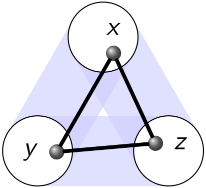

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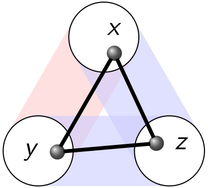

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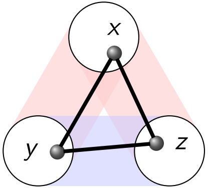

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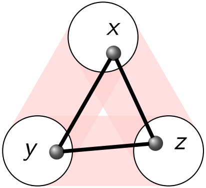

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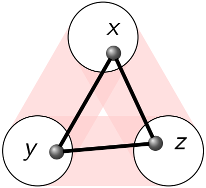

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This argument doesn't work in the sparse setting (g unbounded)

Sparse counting lemma

Sparse triangle counting lemma (Conlon, Fox, Z.)

Assume that ν satisfies the 3-linear forms condition.
If $0 \le g \le \nu$, $0 \le \widetilde{g} \le 1$ and $\|g - \widetilde{g}\|_\square = o(1)$, then

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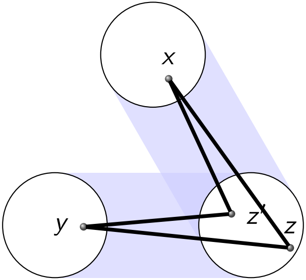
Proof ingredients

1. Cauchy-Schwarz
2. **Densification**
3. Apply cut norm/discrepancy (as in dense case)

Densification



$$\mathbb{E}[g(x,z)g(y,z)g(x,z')g(y,z')]$$

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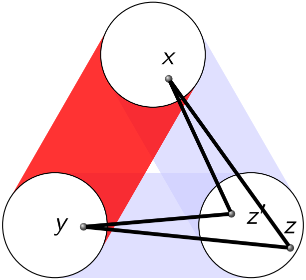
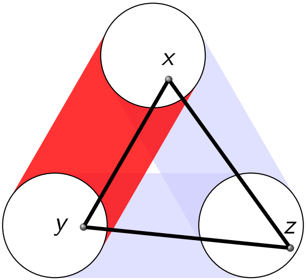


$$\mathbb{E}[g(x,z)g(y,z)g(x,z')g(y,z')]$$

Set $g'(x,y) := \mathbb{E}_{z'}[g(x,z')g(y,z')]$, i.e., codegrees

$g'(x,y) \lesssim 1$ for almost all (x,y)

Densification



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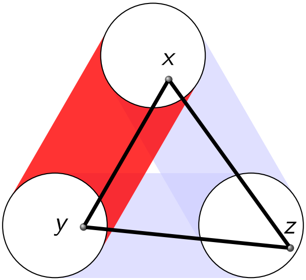

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Made $X \times Y$ dense. Now repeat for $X \times Z$ & $Y \times Z$.
Reduce to dense setting.

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$\Longrightarrow$ relative Roth theorem

COMING SOON

The Green-Tao theorem: an exposition

Coming Soon

The Green-Tao theorem: an exposition

- A gentle exposition giving a complete & self-contained proof of the Green-Tao theorem (assuming Szemerédi's theorem)
- $\sim$ 25 pages

THANK YOU!