Two-bubble dynamics for the equivariant wave maps equation

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Critical wave maps

Wave maps from $\mathbb{R}^{1+2}$ to a Riemannian manifold $\mathcal{N}$:

$$\Box\Psi \perp T_\Psi\mathcal{N}.$$

Special case $\mathcal{N} = \mathbb{S}^2 \subset \mathbb{R}^3$, k-equivariant solutions ($k \in \mathbb{Z}$):

$$\Psi(t, r\cos\theta, r\sin\theta) = \big(\sin(u(t,r))\cos k\theta, \sin(u(t,r))\sin k\theta, \cos(u(t,r))\big).$$

Equation is reduced to a semi-linear one:

$$\begin{cases} \partial_t^2 u(t,r) = \partial_r^2 u(t,r) + \frac{1}{r}\partial_r u(t,r) - \frac{k^2}{2r^2}\sin(2u(t,r)), \\ (u(t_0,r), \partial_t u(t_0,r)) = (u_0(r), \dot u_0(r)). \end{cases} \tag{WMAP}$$

Notation: $\boldsymbol{v} := (v, \dot v)$, $\|\dot v\|_{L^2}^2 := \int_0^{+\infty}(\dot v^2)\,r\,\mathrm{d}r$,
$\|v\|_{\mathcal{H}}^2 := \int_0^{+\infty}\big((\partial_r v)^2 + \frac{1}{r^2}v^2\big)\,r\,\mathrm{d}r$, $\mathcal{E} := \mathcal{H} \times L^2$.

$$E(\boldsymbol{v}) := \pi\int_0^{+\infty}\Big(\dot v^2 + (\partial_r v)^2 + \frac{k^2}{r^2}(\sin(v))^2\Big)\,r\,\mathrm{d}r.$$

Comments

- If $\lim_{r\to 0} u_0(r) = \lim_{r\to\infty} u_0(r) = 0$, then

$$E(\boldsymbol{u}_0) < \infty \quad \Leftrightarrow \quad \boldsymbol{u}_0 \in \mathcal{E}.$$

- Local well-posedness in $\mathcal{E}$ (conditional)
  - Ginibre, Soffer, Velo (1992)
  - Shatah, Struwe (1994)

$$\forall \boldsymbol{u}_0 \in \mathcal{E},\ \exists! \boldsymbol{u} \in C((T_-, T_+); \mathcal{E}), \qquad T_- < t_0 < T_+.$$

- The energy is conserved; the flow is reversible.
- Let $\lambda > 0$. For $\boldsymbol{v} = (v, \dot v) \in \mathcal{E}$ we denote

$$\boldsymbol{v}_\lambda(r) := \Big(v\big(\frac{r}{\lambda}\big), \frac{1}{\lambda}\dot v\big(\frac{r}{\lambda}\big)\Big).$$

We have $\|\boldsymbol{v}_\lambda\|_{\mathcal{E}} = \|\boldsymbol{v}\|_{\mathcal{E}}$ and $E(\boldsymbol{v}_\lambda) = E(\boldsymbol{v})$. Moreover, if $\boldsymbol{u}(t)$ is a solution of (WMAP) on the time interval $[0, T_+)$, then $\boldsymbol{w}(t) := \boldsymbol{u}(\frac{t}{\lambda})_\lambda$ is a solution on $[0, \lambda T_+)$.

Stationary states – k-equivariant harmonic maps

- Explicit radially symmetric solutions of $\partial_r^2 u(r) + \frac{1}{r}\partial_r u(r) - \frac{k^2}{2r^2}\sin(2u(r)) = 0$:

$$Q_\lambda(r) := 2\arctan\left(\frac{r^k}{\lambda^k}\right), \qquad \boldsymbol{Q}_\lambda := (Q_\lambda, 0) \in \mathcal{E}.$$

- $E(\boldsymbol{Q}_\lambda) = 4k\pi$; orbital stability
- $\boldsymbol{Q}_\lambda$ are, up to sign and translation by π, all the equivariant stationary states.
- Threshold elements for nonlinear behavior – Côte, Kenig, Lawrie and Schlag (2015), using ideas of Kenig and Merle (2008).

Theorem – Côte, Kenig, Lawrie, Schlag (2015)

Let $\boldsymbol{u}_0$ be such that $E(\boldsymbol{u}_0) < 4k\pi$. Then the solution $\boldsymbol{u}(t)$ of (WMAP) with initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$ exists globally and scatters in both time directions.

- True also in the non-equivariant setting: Sterbenz and Tataru (2010).

Refined threshold

Theorem – Côte, Kenig, Lawrie, Schlag (2015)

Let $\boldsymbol{u}_0$ be such that $E(\boldsymbol{u}_0) < 8k\pi$ and $\lim_{r\to 0} u_0(r) = \lim_{r\to\infty} u_0(r)$. Then the solution $\boldsymbol{u}(t)$ of (WMAP) with initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$ exists globally and scatters in both time directions.

- If $E(\boldsymbol{u}_0) \leq 8k\pi$, then the assumption $\lim_{r\to 0} u_0(r) = \lim_{r\to\infty} u_0(r)$ is equivalent to the topological degree of u_0 being equal to 0.
- For any $\eta > 0$ there exists $\boldsymbol{u}_0$ such that $E(\boldsymbol{u}_0) < 8k\pi + \eta$ and the solution with initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$ blows up in finite time.
- We are interested in a classification of solutions having the threshold energy $E(\boldsymbol{u}) = 8k\pi = 2E(\boldsymbol{Q})$.
- The *threshold theorem* is a weakened version of what would be a *soliton resolution theorem*.

It turns out that there exist non-scattering solutions of threshold energy.

Theorem 1 – J. (2016)

Let $k \geq 3$. There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{E}$ of (WMAP),

$$\lim_{t \to -\infty} \left\| \boldsymbol{u}(t) - \left(-\boldsymbol{Q} + \boldsymbol{Q}_{\kappa |t|^{-\frac{2}{k-2}}} \right) \right\|_{\mathcal{E}} = 0, \qquad \kappa \text{ constant } > 0.$$

- An analogous result holds for the critical radial Yang-Mills equation (exponential concentration rate); the same would be the case for (WMAP) with $k = 2$.
- Related works – concentration of one bubble:
  - Krieger, Schlag and Tataru (2008) (and extensions)
  - Raphaël and Rodnianski (2012).
- Strong interaction of bubbles: the second bubble could not concentrate without being "pushed" by the first one,
  - Martel and Raphaël (2015)
  - Nguyen Tien Vinh (2017).

There is only one possible dynamical behavior of a non-scattering solution.

Theorem 2 – J. and Lawrie (2017)

Fix any equivariance class $k \geq 2$. Let $\boldsymbol{u}(t) : (T_-, T_+) \to \mathcal{E}$ be a solution of (WMAP) such that

$$E(\boldsymbol{u}) = 2E(\boldsymbol{Q}) = 8\pi k.$$

Then $T_- = -\infty$, $T_+ = +\infty$ and one of the following alternatives holds:

- $\boldsymbol{u}(t)$ scatters in both time directions,
- $\boldsymbol{u}(t)$ scatters in one time direction; in the other time direction, there exist $\iota \in \{-1, 1\}$ and continuous functions $\mu(t), \lambda(t) > 0$ such that

$$\|\boldsymbol{u}(t) - \iota(-\boldsymbol{Q}_{\mu(t)} + \boldsymbol{Q}_{\lambda(t)})\|_{\mathcal{E}} \to 0,$$
$$\mu(t) \to \mu_0 \in (0, +\infty), \qquad \lambda(t) \to 0 \text{ (at a specific rate).}$$

Comments

- We obtain $\lambda(t) \sim |t|^{-\frac{2}{k-2}}$ for $k \geq 3$ and $\exp(-Ct) \leq \lambda(t) \leq \exp(-t/C)$ for $k = 2$
- In particular, the two-bubble solutions from Theorem 1 scatter in forward time, which provides an example of an orbit connecting different types of dynamical behavior for positive and negative times
- Non-existence of solutions which form a pure two-bubble in both time directions is reminiscent of the work of Martel and Merle for gKdV and seems to be a typical feature of models which are not completely integrable
- We conjecture that there exists a unique (up to rescaling and sign change) non-scattering solution of threshold energy
- Probably the only (almost) complete dynamical classification in a setting allowing more than one bubble, except for completely integrable models.

Modulation method – Part 1

- We want to understand the evolution of solutions *close to a two-bubble*, that is $\inf_{\mu,\lambda>0}\left(\|\boldsymbol{u}(t)-(-\boldsymbol{Q}_\mu+\boldsymbol{Q}_\lambda)\|_{\mathcal{E}}+\lambda/\mu\right)\leq\eta\ll 1$.
- We decompose $\boldsymbol{u}(t)=-\boldsymbol{Q}_{\mu(t)}+\boldsymbol{Q}_{\lambda(t)}+\boldsymbol{g}(t)$; for the moment, we do not specify how $\mu(t)$ and $\lambda(t)$ are chosen. We write $\boldsymbol{g}(t)=(g(t),\dot g(t))$.
- Some notation:

$$\Lambda v := -\frac{\partial}{\partial\lambda}\Big(v\Big(\frac{\cdot}{\lambda}\Big)\Big) = r\partial_r v,$$

$$\Lambda_0\dot v := -\frac{\partial}{\partial\lambda}\Big(\frac{1}{\lambda}\dot v\Big(\frac{\cdot}{\lambda}\Big)\Big) = \dot v + r\partial_r\dot v,$$

$$L_\lambda := -\partial_r^2 - \frac{1}{r}\partial_r + k^2\frac{\cos 2Q_\lambda}{r^2},$$

$$\langle v, w\rangle := \int_0^\infty v(r)w(r)\,r\,\mathrm{d}r.$$

- $-L_\lambda$ is the linearization of $\partial_r^2 u+\frac{1}{r}\partial_r u-\frac{k^2}{2r^2}\sin(2u)$ around $u=Q_\lambda$ and it follows that $L_\lambda(\Lambda Q_\lambda)=0$.

Modulation method – Part 2

- Evolution of the error term; let $f(u) := \frac{k^2}{2}\sin(2u)$. Using $\partial_r^2 Q_\mu + \frac{1}{r}\partial_r Q_\mu = \frac{1}{r^2}f(Q_\mu)$ and $\partial_r^2 Q_\lambda + \frac{1}{r}\partial_r Q_\lambda = \frac{1}{r^2}f(Q_\lambda)$ we have

$$\begin{aligned}\partial_t \dot g = \partial_t^2 u &= \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{1}{r^2}f(u)\\ &= \partial_r^2 g + \frac{1}{r}\partial_r g - \frac{1}{r^2}\left(f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)\right)\\ &\simeq -L_\lambda g - \frac{1}{r^2}\left(f(-Q_\mu + Q_\lambda) - f(Q_\lambda) + f(Q_\mu)\right) + \ldots\end{aligned}$$

- Since $L_\lambda(\Lambda Q_\lambda) = 0$, it is natural to compute $\frac{\mathrm{d}}{\mathrm{d}t}\langle \frac{1}{\lambda}\Lambda Q_\lambda, \dot g\rangle$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{1}{\lambda(t)}\Lambda Q_{\lambda(t)}, \dot g(t)\right\rangle = -\left(8k^2 + o(1)\right)\frac{\lambda(t)^{k-1}}{\mu(t)^k} + O\left(\frac{\|\boldsymbol{g}(t)\|_{\mathcal{E}}^2}{\lambda(t)}\right).$$

- By the conservation of energy, we only get $\|\boldsymbol{g}(t)\|_{\mathcal{E}}^2 \lesssim \frac{\lambda(t)^k}{\mu(t)^k}$, so the equation above is useless.

Part 3 – Raphaël-Szeftel virial correction

- We define an auxiliary function

$$b(t) := -\Big\langle \frac{1}{\lambda(t)}\Lambda Q_{\lambda(t)}, \dot g(t)\Big\rangle - \Big\langle \dot g(t), \text{``}\frac{1}{\lambda(t)}\Lambda_0\text{''} g(t)\Big\rangle.$$

- Using specific structure of the quadratic terms in the equation for $\partial_t \dot g(t)$, we obtain a cancellation of the main terms and obtain

$$b'(t) \geq (8k^2 - c)\frac{\lambda(t)^{k-1}}{\mu(t)^k}, \quad c \text{ small.} \tag{1}$$

- If we choose the orthogonality condition $\langle \Lambda Q_{\lambda(t)}, g(t)\rangle = 0$, then standard computations yield $\lambda'(t) \sim b(t)$.
- Together with (1), this allows to obtain a lower bound on $\lambda(t)$, starting from an initial time t_0 such that $\frac{\mathrm{d}}{\mathrm{d}t}(\lambda(t_0)/\mu(t_0)) \geq 0$ (in fact we need $b(t_0) \geq -c\|\boldsymbol{g}(t_0)\|_{\mathcal{E}}$, c small).
- Bounds on $\mu(t)$ and upper bounds on $\lambda(t)$ are much easier to obtain.

Proof of Theorem 1 – "backward time" construction

Theorem 1 (weakened version)

Let $k \geq 3$. There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{E}$ of (WMAP) and continuous functions $\mu(t), \lambda(t) > 0$ such that

$$\lim_{t \to -\infty} \left\| \boldsymbol{u}(t) - \left(-\boldsymbol{Q}_{\mu(t)} + \boldsymbol{Q}_{\lambda(t)}\right) \right\|_{\mathcal{E}} = 0, \qquad \kappa \text{ constant } > 0,$$

with $\mu(t) \sim 1$ and $\lambda(t) \sim |t|^{-\frac{2}{k-2}}$.

Proof of Theorem 1 – "backward time" construction

- Key idea: construct a sequence of solutions $\boldsymbol{u}_n(t)$ converging to a non-scattering solution.
- Let $\lambda_{\mathrm{app}}(t) := |t|^{-\frac{2}{k-2}}$. Take $T_n \to -\infty$, and let $\boldsymbol{u}_n(t)$ be the solution of (WMAP) for the initial data $\boldsymbol{u}_n(T_n) = -\boldsymbol{Q} + \boldsymbol{Q}_{\lambda_{\mathrm{app}}(T_n)}$.
- Then $\boldsymbol{u}_n(t) \simeq -\boldsymbol{Q}_{\mu_n(t)} + \boldsymbol{Q}_{\lambda_n(t)}$ for $t \in [T_n, T_0]$, and we have lower and upper bounds on $\mu_n(t)$ and $\lambda_n(t)$, with T_0 independent of n.
- After extraction of a subsequence, $\mu_n(t) \to \mu(t)$, $\lambda_n(t) \to \lambda(t)$ for all $t \leq T_0$ and $\boldsymbol{u}_n(T_0) \rightharpoonup \boldsymbol{u}_0$ weakly in $\mathcal{E}$.
- Let $\boldsymbol{u}(t)$ be the solution of (WMAP) for the initial data $\boldsymbol{u}(T_0) = \boldsymbol{u}_0$. Using weak continuity properties of the flow, we obtain that $\boldsymbol{u}(t)$ exists for $t \in (-\infty, T_0]$ and

$$\lim_{t \to -\infty} \|\boldsymbol{u}(t) - (-\boldsymbol{Q}_{\mu(t)} + \boldsymbol{Q}_{\lambda(t)})\|_{\mathcal{E}} = 0.$$

- Time reversibility of the flow is crucial. This scheme of proof goes back to the works of Merle (1990) and Martel (2005).

Proof of Theorem 2

Theorem 2

Fix any equivariance class $k \geq 2$. Let $\boldsymbol{u}(t) : (T_-, T_+) \to \mathcal{E}$ be a solution of (WMAP) such that

$$E(\boldsymbol{u}) = 2E(\boldsymbol{Q}) = 8\pi k.$$

Then $T_- = -\infty$, $T_+ = +\infty$ and one the following alternatives holds:

- $\boldsymbol{u}(t)$ scatters in both time directions,
- $\boldsymbol{u}(t)$ scatters in one time direction; in the other time direction, there exist $\iota \in \{-1, 1\}$ and continuous functions $\mu(t), \lambda(t) > 0$ such that

$$\|\boldsymbol{u}(t) - \iota(-\boldsymbol{Q}_{\mu(t)} + \boldsymbol{Q}_{\lambda(t)})\|_{\mathcal{E}} \to 0,$$
$$\mu(t) \to \mu_0 \in (0, +\infty), \qquad \lambda(t) \to 0 \text{ (at a specific rate)}.$$

Proof of Theorem 2 – Part 1

- Let $\boldsymbol{u} : [T_0, T_+) \to \mathcal{E}$ be a non-scattering solution such that $E(\boldsymbol{u}) = 2E(\boldsymbol{Q})$. By works of Struwe, Côte, and Jia and Kenig, we know that for some sequence $T_n \to T_+$ we have

$$\lim_{n\to\infty} \inf_{\mu,\lambda} \|\boldsymbol{u}(T_n) - \iota(-\boldsymbol{Q}_\mu + \boldsymbol{Q}_\lambda)\|_{\mathcal{E}} = 0.$$

- The main difficulty now is to exclude the possibility that a solution approaches a two-bubbles configuration an infinite number of times. We need a "one-pass lemma" (terminology of Nakanishi and Schlag).
- Convexity argument based on the localized virial identity:

$$\int_{\tau_1}^{\tau_2} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \leq |\langle \partial_t u, \chi_R r\partial_r u\rangle(\tau_1)| + |\langle \partial_t u, \chi_R r\partial_r u\rangle(\tau_2)| + \int_{\tau_1}^{\tau_2} \Omega_R(\boldsymbol{u}(t))\,\mathrm{d}t$$

- The last term comes from localizing the virial identity and has to be absorbed by the left hand side.

Proof of Theorem 2 – Part 2

$$\int_{\tau_1}^{\tau_2} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \leq |\langle \partial_t u, \chi_R r \partial_r u\rangle(\tau_1)| + |\langle \partial_t u, \chi_R r \partial_r u\rangle(\tau_2)| + \int_{\tau_1}^{\tau_2} \Omega_R(\boldsymbol{u}(t))\,\mathrm{d}t$$

- Suppose that the one pass lemma fails. We take τ_1 and τ_2 such that $\boldsymbol{u}(\tau_1)$ and $\boldsymbol{u}(\tau_2)$ are close to two-bubble configurations. The time interval in between is divided into regions where $\boldsymbol{u}(t)$ is close to a two-bubble ("bad" intervals) and regions where it is not ("good" intervals).
- On the union of the good intervals, the solution has a compactness property, which allows us to deal with the error term $\Omega_R(\boldsymbol{u}(t))$.
- On each bad interval, we use the modulation method and estimates on the growth of the modulation parameters.

Thank you!