# The Brownian motion as the limit of a deterministic system of hard-spheres

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Joint work with I. Gallagher and L. Saint-Raymond

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# Outline

- Linear Boltzmann equation & Brownian motion
- Lanford's strategy & pruning procedure
- Coupling with the Boltzmann hierarchy

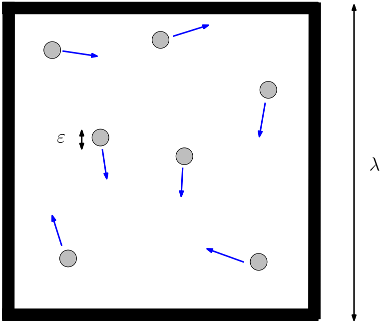
# Hard sphere dynamics

Gas of $N$ hard spheres with deterministic Newtonian dynamics (elastic collisions).

Periodic domain: $\mathbf{T}^d_\lambda = [0,\lambda]^d$
Sphere radius $= \varepsilon$

Dilute gas : $N\varepsilon^{d-1}\lambda^{-d} \equiv 1$

Boltzmann-Grad scaling



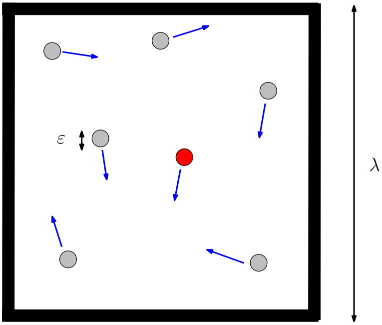
# Hard sphere dynamics

Gas of $N$ hard spheres with deterministic Newtonian dynamics (elastic collisions).

Initial data at equilibrium
and a tagged particle $(x_1, v_1)$

**Questions.**
In the Boltzmann-Grad limit
$N \to \infty, \quad N\varepsilon^{d-1}\lambda^{-d} \equiv 1$

➪ Distribution of the tagged particle $(x_1(t), v_1(t))$ for $t > 0$ ?



# Hard sphere dynamics

Gas of $N$ hard spheres with deterministic Newtonian dynamics (elastic collisions).

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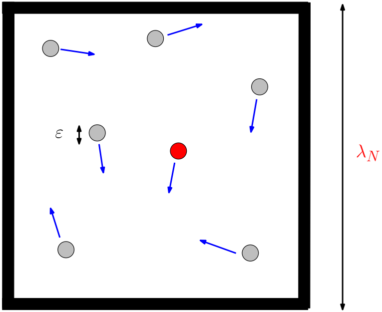



**Questions.**
In the Boltzmann-Grad limit
$N \to \infty, \quad N\varepsilon^{d-1}\lambda^{-d} \equiv 1$

➪ Distribution of the tagged particle $(x_1(t), v_1(t))$ for $t > 0$ ?

➪ Position the tagged particle $\frac{1}{\lambda_N} x_1(\lambda_N^2 \tau)$ for $\tau > 0$ ?

# Hard sphere dynamics

Gas of $N$ hard spheres : $Z_N = \{(x_i(t), v_i(t)\}_{i\leq N}$

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as } \; |x_i(t) - x_j(t)| > \varepsilon\,,$$

and elastic collisions if $|x_i(t) - x_j(t)| = \varepsilon$

$$\begin{cases} v_i(t^+) = v_i(t^-) - \frac{1}{\varepsilon^2}(v_i - v_j)\cdot(x_i - x_j)(x_i - x_j)(t^-) \\ v_j(t^+) = v_j(t^-) + \frac{1}{\varepsilon^2}(v_i - v_j)\cdot(x_i - x_j)(x_i - x_j)(t^-) \end{cases}$$

Liouville equation for the particle density $f_N(t, Z_N)$

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0$$

in the phase space

$$\mathcal{D}_\varepsilon^N := \left\{ Z_N \in \mathbf{T}_\lambda^{dN} \times \mathbb{R}^{dN} \,/\, \forall i \neq j\,, \quad |x_i - x_j| > \varepsilon \right\}$$

with specular reflection on the boundary $\partial\mathcal{D}_\varepsilon^N$.

# The tagged particle

Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big( -\frac{\beta}{2} \sum_{i=1}^{N} |v_i|^2 \Big) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Particle $Z_1 = (x_1, v_1)$ is tagged. Initial distribution :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\, \varphi_N^0\big((x_1, v_1)\big)$$

Uniform bound: $\varphi_N^0\big((x_1, v_1)\big) \leq \mu_N$

**Notation: Marginals**

$$t \geq 0, \forall s \geq 1, \qquad f_N^{(s)}(t, Z_s) = \iint f_N(t, Z_N)\, dz_{s+1} \dots dz_N$$

Tagged particle distribution $f_N^{(1)}(t, (x_1, v_1))$

# Linear Boltzmann equation

Large time convergence : [van Beijeren, Lanford, Lebowitz, Spohn] [Lebowitz, Spohn]

$N$-particle system

$$f_N^{(1)}(t, (x_1, v_1))$$

$$\xrightarrow{N\to\infty} \quad \forall t > 0$$

Linear Boltzmann equation

$$g(t, (x_1, v_1))$$

Linear Boltzmann equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g = -Lg, \\ Lg(v) := \iint [g(v)M_\beta(v_1) - g(v')M_\beta(v_1')] \left((v - v_1) \cdot \nu\right)_+ dv_1 d\nu, \\ v' = v + (\nu \cdot (v_1 - v))\,\nu \quad v_1' = v_1 - (\nu \cdot (v_1 - v))\,\nu \end{cases}$$

with $M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} \exp\left(-\frac{\beta}{2}|v|^2\right)$

# Linear Boltzmann equation

Large time convergence : [van Beijeren, Lanford, Lebowitz, Spohn] [Lebowitz, Spohn]

**$N$-particle system**

$$f_N^{(1)}(t,(x_1,v_1))$$

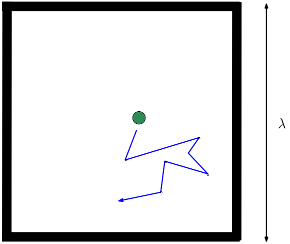
$$\xrightarrow{N\to\infty} \quad \forall t>0$$

**Linear Boltzmann equation**

$$g(t,(x_1,v_1))$$

Markov process on the velocities $\{v(t)\}_{t\geq 0}$ with generator $L$.

**Position** : $x(t) = \int_0^t v(u)du$



# Linear Boltzmann equation

Large time convergence : [van Beijeren, Lanford, Lebowitz, Spohn]
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**$N$-particle system**

$$f_N^{(1)}(t,(x_1,v_1))$$

$$\xrightarrow{N\to\infty} \quad \forall t>0$$

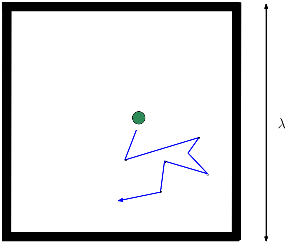
**Linear Boltzmann equation**

$$g(t,(x_1,v_1))$$

- Probabilist approaches :

  Tanaka, Sznitman, Meleard, Graham, Fournier ...

- Quantum brownian motion



# Convergence to a Brownian motion

$N$-particle system $\xrightarrow[\forall t > 0]{N\to\infty}$ Linear Boltzmann equation $\xrightarrow[(\lambda^2\tau, \lambda y)]{\lambda\to\infty}$ Heat equation

Brownian motion

# Convergence to a Brownian motion

$$\boxed{N\text{-particle system}} \xrightarrow[\forall t>0]{N\to\infty} \boxed{\text{Linear Boltzmann equation}} \xrightarrow[(\lambda^2\tau,\lambda y)]{\lambda\to\infty} \boxed{\begin{array}{c}\text{Heat equation}\\ \text{Brownian motion}\end{array}}$$

$$\lambda_N = o\big((\log\log N)^{1/5}\big), \quad N\varepsilon^{d-1}\lambda_N^{-d} \equiv 1, \quad ({\lambda_N}^2\tau, \lambda_N y)$$

**Key : quantitative convergence to the Linear Boltzmann equation**

For $t_N = o\big(\mu_N^{-1/2}\sqrt{\log\log N}\big)$

$$\left\|\lambda^d f_N^{(1)} - g\right\|_{L^\infty([0,t_N]\times\mathbf{T}^d_\lambda\times\mathbb{R}^d)} \leq C\mu_N^2\left(\frac{t_N^2}{\log\log N}\right)^2$$

with $f_N^0(Z_N) = M_{N,\beta}(Z_N)\,\varphi_N^0(z_1)$, $g^0(Z_N) = \lambda^d M_\beta(v_1)\,\varphi^0(z_1)$.

# Convergence to a Brownian motion

Rescaled position of the tagged particle

$$\mathcal{X}(\tau) = \frac{1}{\lambda_N} x_1(\lambda_N^2 \tau) \quad \text{with} \quad \lambda_N = o\big((\log\log N)^{1/5}\big)$$

Initial distribution

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\, \mu_N\, \varphi_N^0\left(\frac{x_1}{\lambda_N^{1-\zeta}}\right), \quad \text{with} \quad \zeta \in \left]0, \frac{1}{4d}\right[$$

**Theorem. [B, Gallagher, Saint-Raymond]**

$\mathcal{X}$ converges weakly to a brownian motion with variance $\kappa$

The distribution $\lambda_N^d f_N^{(1)}(\lambda_N^2 \tau, \lambda_N y, v)$ of the tagged particle converges as $N \to \infty$ to $M_\beta\, \rho$

$$\partial_\tau \rho = \kappa \Delta_y \rho \quad \text{on } \mathbb{R}^+ \times [0,1]^d, \qquad \rho_{|\,\tau=0} = \delta_0$$

# Boltzmann equation

**Theorem. [Lanford; Uchiyama; Cercignani, Illner, Pulvirenti; Gallagher, Saint-Raymond, Texier]**

For chaotic initial data $f_N^0(Z_N) \simeq \prod_{i=1}^N f^0(z_i)$ the density of the particle system converges up to a time $t > 0$ to the solution of the Boltzmann equation when $N \to \infty$, $N\varepsilon^{d-1}\lambda^{-d} \equiv 1$

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f \\ &\quad = \iint_{\mathbf{S}^{d-1} \times \mathbb{R}^d} [f(v')f(v_1') - f(v)f(v_1)] \left((v - v_1) \cdot \nu\right)_+ dv_1 d\nu \end{aligned}$$

with $v' = v + \nu \cdot (v_1 - v)\,\nu\,, \quad v_1' = v_1 - \nu \cdot (v_1 - v)\,\nu$

➪ Lanford's strategy leads to a short time convergence which depends on $f^0$. The convergence time remains short even if initially the system starts from equilibrium $f_N^0 = M_{N,\beta}$ !!

# BBGKY hierarchy for the marginals

For $s < N$ and on $\mathcal{D}^s_\varepsilon = \left\{ Z_s = (x_i, v_i)_{i \le s} \,\middle|\, \quad i \neq j, \quad |x_i - x_j| > \varepsilon \right\}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) f_N^{(s)}(t, Z_s) = \left( C_{s,s+1} f_N^{(s+1)} \right)(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned}
&\left( C_{s,s+1} f_N^{(s+1)} \right)(Z_s) \\
&:= (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v_i^*, \dots, x_i + \varepsilon\nu, v_{s+1}^*) \Big( (v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu dv_{s+1} \\
&- (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon\nu, v_{s+1}) \Big( (v_{s+1} - v_i) \cdot \nu \Big)_- d\nu dv_{s+1}
\end{aligned}$$

where $\mathbf{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

# Duhamel Formula

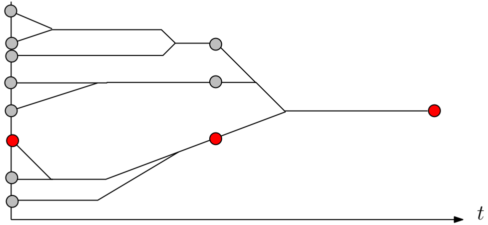
Denote by $\mathbf{S}_s$ the semi-group associated to free transport in $\mathcal{D}^s_\varepsilon$

**Duhamel Formula**

$$f_N^{(1)}(t) = \mathbf{S}_1(t) f_N^{(1)}(0) + \int_0^t \mathbf{S}_1(t-t_1) C_{1,2} f_N^{(2)}(t_1)\, dt_1 \,,$$

Collision tree :

- Transport operator $\mathbf{S}_s$ (possible recollisions)
- Addition of a particle to the tree after each collision



# Duhamel Formula

Denote by $\mathbf{S}_s$ the semi-group associated to free transport in $\mathcal{D}_\varepsilon^s$

**Duhamel Formula**

$$f_N^{(1)}(t) = \mathbf{S}_1(t) f_N^{(1)}(0) + \int_0^t \mathbf{S}_1(t-t_1) C_{1,2} f_N^{(2)}(t_1)\, dt_1 \,,$$

**Iterated Duhamel formula**

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

with

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dt_n \dots dt_1\, \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1 - t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

Key issue. Convergence of the series when $N \to \infty$

# Continuity estimates for the collision operators

Weighted norms :

$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}_\varepsilon^k} \left| f_k(Z_k) \ \exp\left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2\right) \right| < \infty$

The initial data satisfy : $\lambda^{ds} \| f_N^{0,(s)} \|_{\varepsilon,s,\beta} \leq \mu_N^2 \ C^s$

Collision operators estimates

$$\left\| Q_{s,s+n}(t) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} \left( C_d(\beta) \lambda^d t \right)^n \| f_{s+n} \|_{\varepsilon,s+n,\beta}$$

Thus

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

can be controlled only up to short times

# $\mathbb{L}^\infty$ bound

Initial distribution :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\, \varphi_N^0\big((x_1, v_1)\big)$$

Uniform bound: $\varphi_N^0\big((x_1, v_1)\big) \leq \mu_N$

The measure $M_{N,\beta}(Z_N)$ is stationary thus the maximum principle implies bounds **uniform in time**

For any $s \geq 1$

$$\sup_{t\geq 0} f_N^{(s)}(t, Z_s) \leq \mu_N M_{N,\beta}^{(s)}(Z_s) \leq \lambda^{-ds} \mu_N \left(1 - \varepsilon \kappa_d\right)^{-s} \; M_\beta^{\otimes s}(V_s)$$

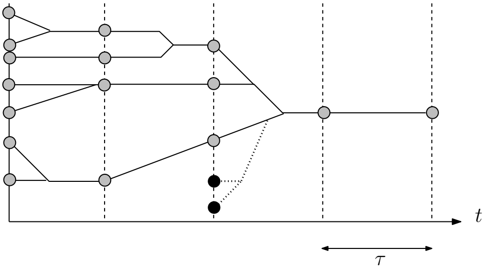
In this way the cancellations in the collision operator are recovered.

# Pruning procedure

Decompose : $[0,t] = \bigcup_{k=1}^{K} [(k-1)\tau, k\tau]$ for some $\tau > 0$

**Good collision trees.**

Less than $n_k = 2^k$ collisions during $[(K-k)\tau, (K-k+1)\tau]$




In each time interval $[(K-k)\tau, (K-k+1)\tau]$

$$\left\| Q_{s,s+n}(\tau) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} \left( C_d(\beta) \lambda^d \tau \right)^n \| f_{s+n} \|_{\varepsilon,s+n,\beta}$$

# Pruning procedure

**Truncated iterated Duhamel formula:**

$$f_N^{(1)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)} + R_N^K(t)$$

with $\quad J_\ell = 1 + j_1 + \dots + j_\ell$

- The main contribution is given by the good collision trees with $j_k \leq 2^k$ during the time interval $[(K-k)\tau, (K-k+1)\tau]$
- The contribution of the large trees $R_N^K(t)$ is controlled

$$\|R_N^K(t)\|_{\mathbb{L}^\infty} \leq \mu_N \frac{t^2}{K}$$

➪ If $t$ is large, then $K$ has to be very large and $\tau$ very small.

# Boltzmann Hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$\Big(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}\Big) g^{(s)}(t, Z_s) = \big(C^0_{s,s+1}\big) g^{(s+1)}(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned}
&\big(C^0_{s,s+1} g^{(s+1)}\big)(Z_s) \\
&:= \cancel{(N-s)\varepsilon^{d-1}} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} g^{(s+1)}(\dots, x_i, v_i^*, \dots, x_i \cancel{+\varepsilon\nu}, v^*_{s+1}) \Big( (v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu \, dv_{s+1} \\
&\quad - \cancel{(N-s)\varepsilon^{d-1}} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} g^{(s+1)}(\dots, x_i, v_i, \dots, x_i \cancel{+\varepsilon\nu}, v_{s+1}) \Big( (v_{s+1} - v_i) \cdot \nu \Big)_- d\nu \, dv_{s+1}
\end{aligned}$$

This is the **limit** hierarchy when $\varepsilon \to 0$ and $N \to \infty$.

# Boltzmann Hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) g^{(s)}(t, Z_s) = \left(C^0_{s,s+1}\right) g^{(s+1)}(t, Z_s)$$

**Iterated Duhamel formula**

$$g^{(1)}(t) = \sum_{n=0}^{\infty} Q^0_{1,1+n}(t) g^{(1+n)}(0)$$

Explicit solution : $g^{(s)}(t) = g(t, z_1) \prod_{i=2}^{s} M_\beta(v_i)$

with $g(t, z_1)$ solution of the **Linear Boltzman equation**

$$\partial_t g + v \cdot \nabla_x g = -Lg$$

and $M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} \exp\left(-\frac{\beta}{2}|v|^2\right)$

# Comparing the BBGKY and Boltzmann hierarchies

As $N \to \infty$ in the scaling $N\varepsilon^{d-1}\lambda^{-d} \equiv 1$,

$$\Big| \Big( \lambda^{ds} f_N^{0(s)} - g_N^{0(s)} \Big) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon} \Big| \leq C\varepsilon \, \mu_N^2 \, s M_\beta^{\otimes s}$$

for the **initial distributions**

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\, \varphi_N^0(z_1), \quad g_N^{0(s)}(Z_s) = \Big( \prod_{i=1}^{s} M_\beta(v_i) \Big) \varphi_N^0(z_1)$$

and

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big( -\frac{\beta}{2} \sum_{i=1}^{N} |v_i|^2 \Big) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

## Main Goal

$$\|\lambda^d f_N^{(1)} - g_N^{(1)}\|_{L^\infty([0,t]\times \mathbf{T}_\lambda^d \times \mathbb{R}^d)} \to 0\,, \quad \text{as } N \to \infty$$

# Comparing the truncated hierarchies

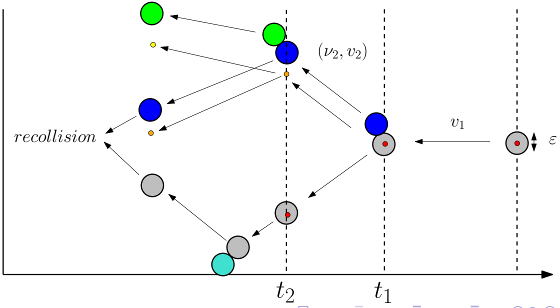
$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)}$$

$$g_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} Q^0_{1,J_1}(\tau) Q^0_{J_1,J_2}(\tau) \dots Q^0_{J_{K-1},J_K}(\tau)\, g_N^{0(J_K)}$$

Geometric interpretation of the collisions operators:

Backward dynamics

**Coupling**
trajectories

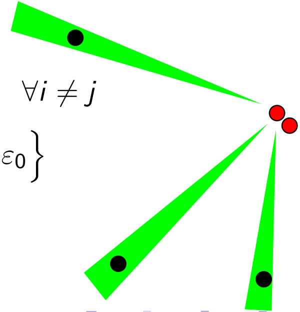


# Removing recollisions

BBGKY and Boltzmann trajectories can be coupled if there are no recollisions

- Truncating high velocities
- Truncating collisions in short time intervals
- Recursive construction of the good trajectories

$$\mathcal{G}_k(\varepsilon_0) = \Big\{ Z_k \in \mathbf{T}_\lambda^{dk} \times \mathbb{R}^{dk} \,/\, \forall s \in [0,t], \quad \forall i \neq j \quad d(x_i - sv_i, x_j - sv_j) \geq \varepsilon_0 \Big\}$$



Up to a small set of velocities, the system is stable by addition of the $k+1$ particle

# Removing recollisions

Choosing the velocities such that the particles in both hierarchies remain at distance less than $2^K\varepsilon$ and that there are no recollisions This boils down to remove a set of velocities with small probability.
**Quantitative controls : [Gallagher, Saint-Raymond, Texier]**

$$\mathit{Error} \leq \Big(Ct\Big)^{\mathcal{N}_t}\varepsilon \quad \text{with} \quad \mathcal{N}_t \quad \text{particles in the tree at time } t$$

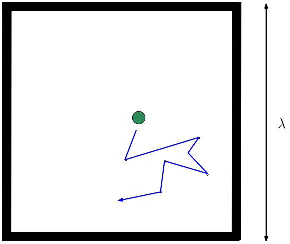
Estimates are valid up to times : $t_N = o\left(\mu_N^{-1/2}\sqrt{\log\log(N)}\right)$

$$\left\|\lambda^d f_N^{(1)} - g_N^{(1)}\right\|_{L^\infty([0,t_N]\times\mathbf{T}_\lambda^d\times\mathbb{R}^d)} \leq C\mu_N^2\left(\frac{t_N^2}{\log\log N}\right)^2$$

# Coupling both hierarchies

**Position** : $x_1^0(t) = \int_0^t v(u)du$

Markov process on the velocities $\{v(t)\}_{t\geq 0}$ with generator $\mathcal{L}$




$$\begin{cases} \mathcal{L}g(v) := \iint M_\beta(v_1)[g(v') - g(v)] \left((v - v_1)\cdot\nu\right)_+ dv_1 d\nu, \\ v' = v + (\nu\cdot(v_1 - v))\,\nu \quad v_1' = v_1 - (\nu\cdot(v_1 - v))\,\nu \end{cases}$$

**Central limit Theorem** for additive functionals of Markov chains ($\mathcal{L}$ has a spectral gap)

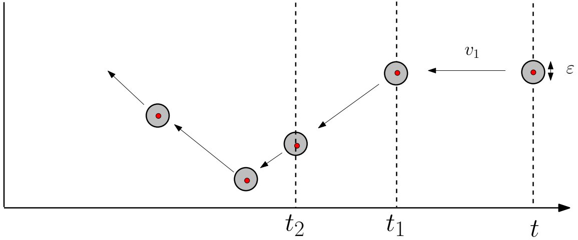
$$\lim_{N\to\infty} \mathbb{E}\Big(h\big(\frac{1}{\lambda_N} x_1^0(\lambda_N^2\,\tau)\big)\Big) = \mathbb{E}\Big(h\big(B(\tau)\big)\Big)$$

$$\lim_{N\to\infty} \mathbb{E}\Big( h_1\big(\frac{1}{\lambda_N} x_1^0(\lambda_N^2 \tau_1)\big) \dots h_\ell\big(\frac{1}{\lambda_N} x_1^0(\lambda_N^2 \tau_\ell)\big)\Big)$$
$$= \mathbb{E}\Big( h_1\big(B(\tau_1)\big) \dots h_\ell\big(B(\tau_\ell)\big)\Big)$$

The diffusion coefficient $\kappa$ is given by

$$\kappa = \lim_{t\to\infty} \frac{1}{d} \mathbb{E}_{M_\beta} \left[ \left( \frac{1}{\sqrt{t}} \int_0^t v_1^0(s)\, ds \right)^2 \right]$$

Coupling the trajectories $x_1$ and $x_1^0$ to get estimates at different times



# Conclusion

- Brownian motion for deterministic hard sphere dynamics

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- Brownian motion for deterministic hard sphere dynamics
- Linearized Boltzmann equation for large times
  [work in progress]