The embedded Calabi-Yau problem for minimal surfaces of finite genus

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The Calabi-Yau problem for minimal surfaces (all surfaces are orientable)

- Calabi-Yau (1966-2000): Can a complete embedded minimal surface be contained in a ball (halfspace)?
- Nadirashvili (Inventiones 1996): $\exists M \subset \mathbb{B}(1)$ compl immersed minimal disk.
- Hoffman-Meeks (Inventiones 1990): If $M \subset \mathbb{R}^3$ properly immersed, nonplanar minimal surface $\Rightarrow$ M cannot be contained in a halfspace.

Question (embedded Calabi-Yau problem):

$M \subset \mathbb{R}^3$ complete embedded minimal surface (CEMS). Is M proper?

Theorem 1 (Meeks-P-Ros, JDG 2004)

$M \subset \mathbb{R}^3$ CEMS with finite genus and K_M locally bded $\Rightarrow$ M proper.

Theorem 2 (Colding-Minicozzi, Annals 2008)

$M \subset \mathbb{R}^3$ CEMS with finite topology $\Rightarrow$ M proper.

Theorem 3 (MLCT, Meeks-Rosenberg, Duke 2006)

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The embedded Calabi-Yau problem for finite genus

$M \subset \mathbb{R}^3$ CEMS $\Rightarrow$ M noncompact $\Rightarrow$ $\mathcal{E}(M) = \{\text{ends of } M\} \neq \emptyset$.

Definition 1 (set of ends)

$\mathcal{E}(M) = \mathcal{A}/_\sim$, where $\mathcal{A} = \{\alpha \colon [0,\infty) \to M \text{ proper arc}\}$ and
$\alpha_1 \sim \alpha_2$ if $\forall C \subset M$ cpt set, α_1, α_2 lie eventually in the same compnt of $M - C$.
$E \subset M$ noncpt subdomain, ∂E cpt. E represents $[\alpha] \in \mathcal{E}(M)$ if $\alpha[t_0,\infty) \subset E$ for some t_0.

Theorem 4 (Collin-Kusner-Meeks-Rosenberg, JDG 2004)

If $M \subset \mathbb{R}^3$ proper EMS $\Rightarrow \mathcal{E}(M)$ countable.

Theorem 5 (Meeks-P-Ros, 2018)

$M \subset \mathbb{R}^3$ CEMS with finite genus and countably many ends $\Rightarrow$ M proper.

Definition 2 (limit ends)

$\mathcal{E}(M) \hookrightarrow [0,1]$ embedding. $\mathbf{e} \in \mathcal{E}(M)$ simple end if $\mathbf{e}$ isolated in $\mathcal{E}(M)$.
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The embedded Calabi-Yau problem for finite genus, II

Theorem 6 (Meeks-P-Ros, 2018)

$M \subset \mathbb{R}^3$ CEMS, ∂M cpt, $g(M) < \infty$, $\#(\mathcal{E}(M)) = \infty$.
If M has countably many limit ends $\Rightarrow$ M proper & M has 1 or 2 limit ends.

Sketch of proof of Thm 6:

Take $M \subset \mathbb{R}^3$ as in Thm 6. Baire's Thm $\Rightarrow$ isolated points in $\mathcal{E}_{limit}(M)$ (simple limit ends) are dense. So it suffices to show:

1. If M has 1 or 2 simple limit ends $\Rightarrow$ M proper.
2. M cannot have 3 simple limit ends (thus M has 1 or 2 limit ends, both simple).

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Theorem 7 (Meeks-P-Ros, Inventiones 2004)

If $M \subset \mathbb{R}^3$ PEMS, $\partial M = \varnothing$, $g(M) < \infty$ $\Rightarrow$ M cannot have just 1 limit end.

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Proposition 1 (Christmas tree picture)

E simple limit end of $M \subset \mathbb{R}^3$ CEMS, $g(E)=0$ $\Rightarrow$ E proper and ...

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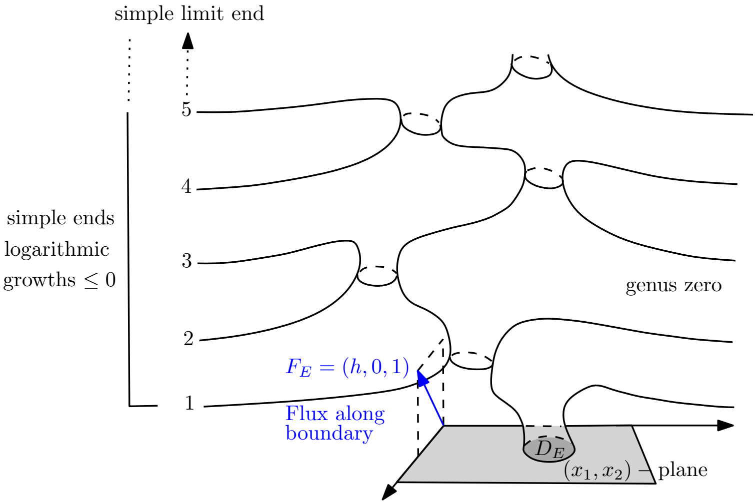
Proposition 1 (Christmas tree picture)

E simple limit end of $M \subset \mathbb{R}^3$ CEMS, $g(E)=0$ $\Rightarrow$ E proper and after passing to a smaller end representative, translation, rotation & homothety:

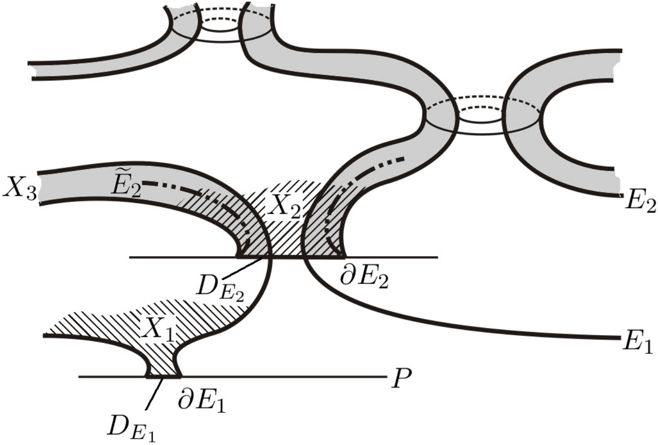
(1) Simple ends of E have FTC & $\log \leq 0$
(2) The limit end of E is the top end
(3) $\partial E = \partial D$, $D \overset{cnvx}{\subset} \{x_3 = 0\}$, $\mathring{D} \cap E = \emptyset$

(4) $\exists f : \mathcal{R}_+ \to E$ orient preserving diffeo ($\mathcal{R}_+$ = top half of a Riemann min example)

The Christmas tree picture for a simple limit end of genus zero



Discarding 3 simple limit ends for a CEMS



Produce $\widetilde{E}_2$ area minimizing in X_3 with cpt bdry, thus with FTC (Fischer-Colbrie) $\Rightarrow$ $\widetilde{E}_2$ has a highest catenoidal end C of positive logarithmic growth.
None of the annular ends of E_1 can lie above C $\Rightarrow$ E_1 lies between two half catenoids !!

Proving the Christmas tree picture: properness

E simple limit end representative of $M \subset \mathbb{R}^3$ CEMS, $g(E) = 0$.

- Topologically, $E \equiv \overline{\mathbb{D}} - \left[\{\frac{1}{2n}\}_n \cup \{0\}\right]$, $\partial E \equiv \partial \mathbb{D} = \mathbb{S}^1$.

We will use:

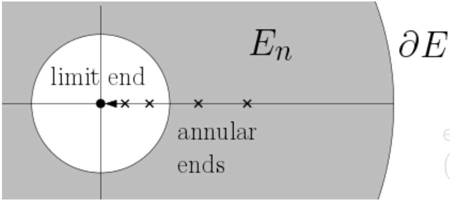
Theorem 7 (MLCT Meeks-Rosenberg, Duke 2006)

$M \subset \mathbb{R}^3$ CEMS with ∂M cpt.

- *If $I_M \geq \varepsilon > 0$ outside some δ-neighb of $\partial M \Rightarrow M$ proper.*
- *If M has finite topology $\Rightarrow I_M \geq \varepsilon > 0$ outside some δ-neighb of ∂M.*

So it suffices to show: $I_E \geq \varepsilon > 0$ outside some δ-neighb of ∂E?

- Description of the annular ends in E:



E_n CEMS, finite topology, ∂E_n cpt $\overset{(\text{Thm } 7)}{\Rightarrow}$ E_n proper $\overset{(Collin)}{\Rightarrow}$ E_n has catenoidal or planar ends, that we will assume horizontal (after rotation indep of n).

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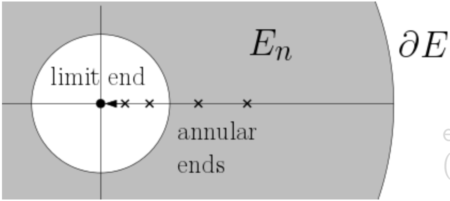

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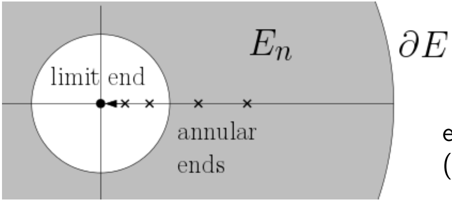

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Proving the Christmas tree picture: properness II

Arguing by contrad, assume $\exists \{q_n\}_n \subset E$ s.t. $d_E(q_n, \partial E) \geq \delta > 0$ and $I_E(q_n) \to 0$.

Rescale by topology (Meeks-P-Ros, JDG 2018): $\exists \{p_n\}_n \subset E$, $\varepsilon_n \searrow 0$ s.t. $d_E(q_n, p_n) \to 0$ and

- M_n:=closure of cpnt of $E \cap \mathbb{B}(p_n, \varepsilon_n)$ that contains p_n is cpt, $\partial M_n \subset \partial \mathbb{B}(p_n, \varepsilon_n) - \partial E$,
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- $\{\lambda_n(M_n - p_n)\}_n$ converges to one of the two following cases:

(C1) A PEMS $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$, $I_{M_\infty}(\vec{0}) = 1$, hence a catenoid or Riemann minimal example (Meeks-P-Ros, Annals 2015).

(C2) A minimal parking garage structure with two columns.

How to find a contradiction in cases (C1) and (C2)?

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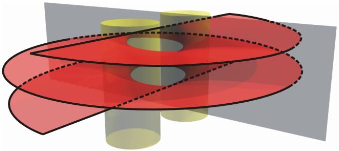
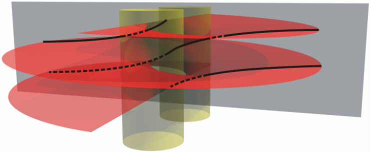
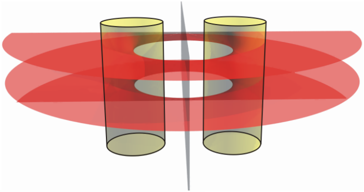
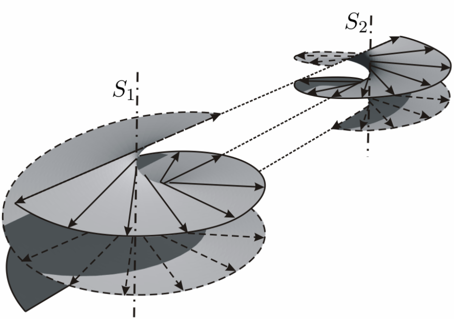
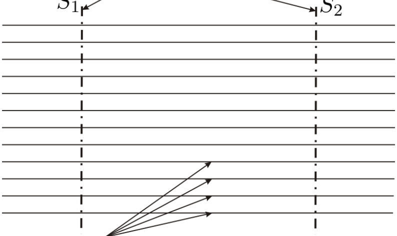




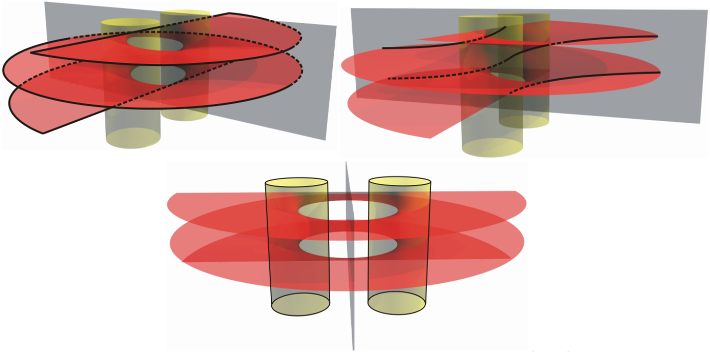


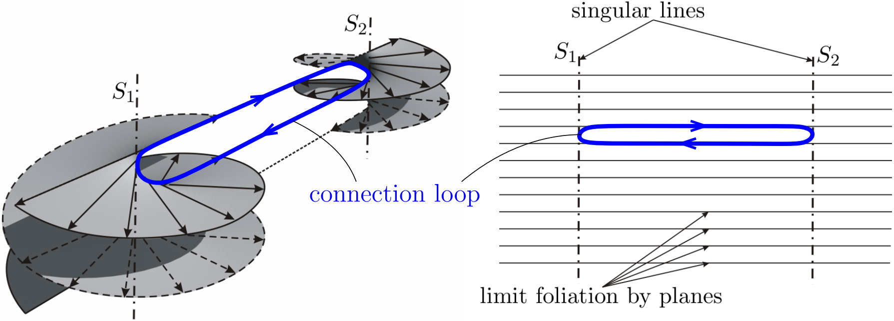


S_2
S_1



singular lines
S_1
S_2
limit foliation by planes




S_1
S_2
connection loop
singular lines
S_1
S_2
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Properness III: Discarding a Riemann minimal example (sketch)

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Lemma 1

For n large, Flux(E, γ_n) is vertical (thus Flux(M_∞, γ) vertical).

Proof:

- If γ_n does not wind around 0 $\Rightarrow \gamma_n$ homologous to finitely many loops around annular ends of E, all with vertical flux $\Rightarrow$ Flux(E, γ_n) vertical.
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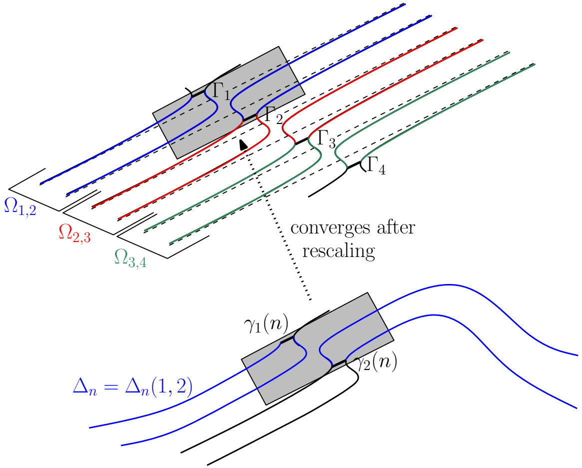

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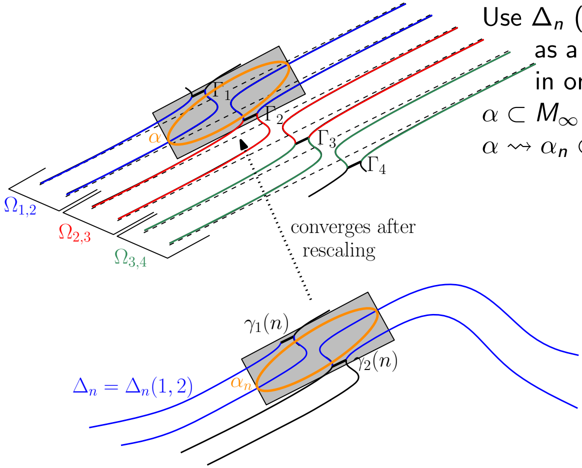

Use Δ_n (suitably modified in a cpt set) as a barrier to solve a Plateau probl in one of its complements: Take $\alpha \subset M_\infty$ enclosing a planar end.

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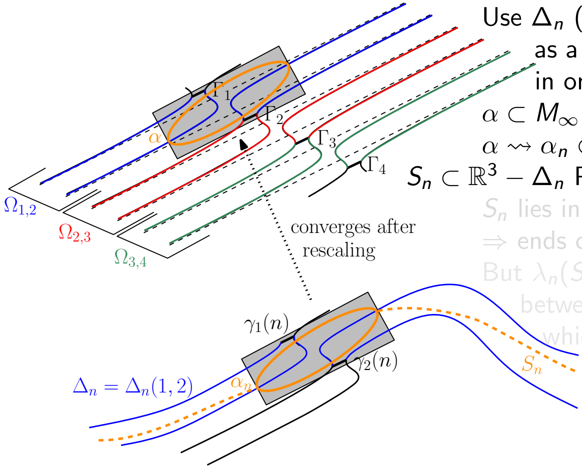

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$S_n \subset \mathbb{R}^3 - \Delta_n$ PEMS stable noncpt $\partial S_n = \alpha_n$

S_n lies in the complement of Δ_n

$\Rightarrow$ ends of S_n are horizontal graphs

But $\lambda_n(S_n - p_n)$ has ends trapped between those of $\lambda_n(\Delta_n - p_n)$, which 'become' tilted planes (limit tgt plane of M_∞) !!

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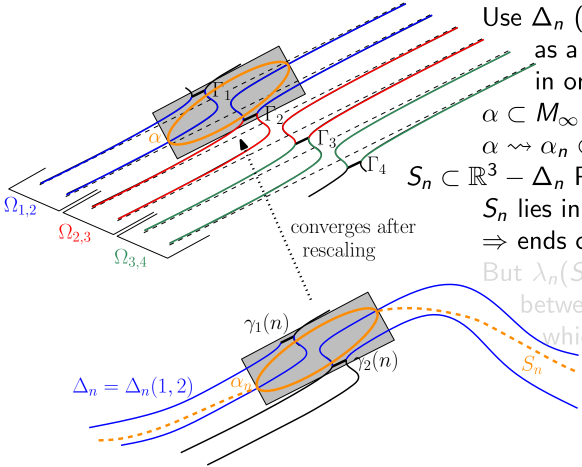

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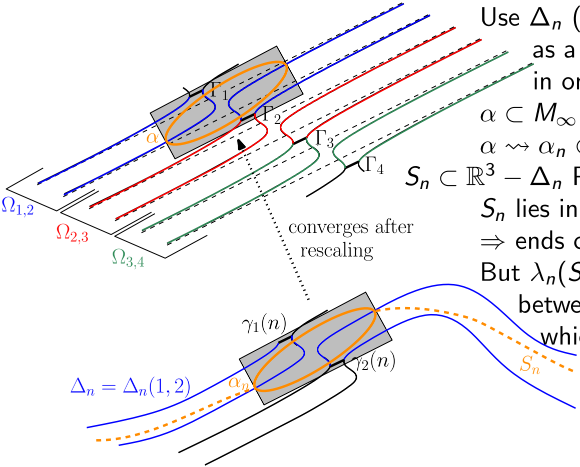

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Properness IV: Discarding a minimal parking garage (sketch)

Suppose (C2) occurs: $\{\lambda_n(M_n - p_n)\}_n$ converges to a minimal parking garage structure with two columns.

- For n sufficiently large, take $\gamma_n \subset M_n \subset E$ a 'connection loop'.
- Lemma 1 gives that $\text{Flux}(E, \gamma_n)$ is vertical $\Rightarrow$ planes in the limiting parking garage are vertical.
- Adapt the arguments that ruled out a Riemann minimal example.

Properness IV: Discarding a minimal parking garage (sketch)

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Properness V: Discarding a catenoid (sketch)

Suppose (C1) holds with $M_\infty =$ catenoid.

γ: waist circle of $M_\infty \rightsquigarrow \gamma_n \subset M_n$ convex planar curve s.t. $\{\lambda_n(\gamma_n - p_n)\}_n \to \gamma$.

1. M_∞ has vertical flux (by Lemma 1).

2. Three subcases (after subseq):

- D1 γ_n does not wind around 0 & encloses at least two ends of E $\forall n$.
- D2 γ_n winds around 0 ($\Rightarrow \gamma_n, \gamma_{n+k}$ topologically concentric $\forall n, k$).
- D3 γ_n does not wind around 0 & encloses exactly one end of E $\forall n$.

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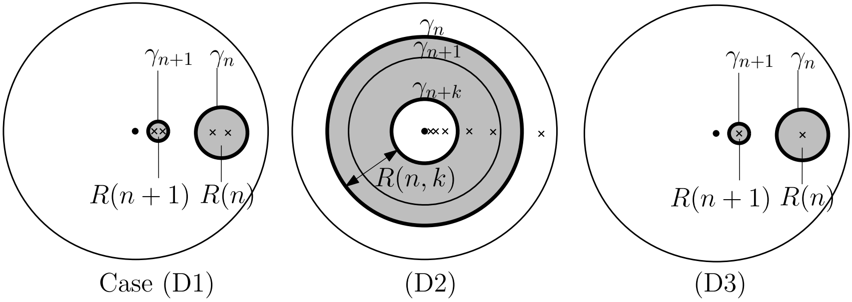

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Strategy:

1. Rule out (D1) (López-Ros deformation argument).
2. Prove Lemma 2 & rule out (D2) (rescaling-by-topology, MLCT, LRST).
3. Discard (D3) (rescaling-by-topology, MLCT, CM 1-sided curv estim).

Lemma 2

If $\{p_n\}_n \to p_\infty \in \mathbb{R}^3$ and (D3) holds $\forall n \Rightarrow E$ lies at one side of $\{x_3 = x_3(p_\infty)\}$.

Properness VI: Discarding a catenoid, case (D1) (sketch)

Suppose (C1) holds with $M_\infty =$ catenoid and (D1) holds $\forall n$.

- $\gamma_n = \partial R(n)$, $R(n) \subset E$ proper, finite topol domain with at least two ends & vertical flux.
- For n large, choose $C_n \subset E$ s.t. $\gamma_n \subset \mathrm{Int}(C_n)$, C_n arbitrarily close to a rescaling of a fixed large cpt unstable piece C of a vertical catenoid,with ∂C_n: two cnvx horiz curves.
- The open planar disks $D_1(n), D_2(n) \subset \mathbb{R}^3$ bded by ∂C_n are disjoint from $R(n)$ (otherwise use a cpt portion of $R(n)$ in the 'interior' of $C_n \cup D_1(n) \cup D_2(n)$ as a barrier to find a stable min annulus with bdry ∂C_n, contradicting Meeks-White).
- $R(n) \cup D(\gamma_n)$ is a properly emb piecewise smooth surface $\Rightarrow$ separates $\mathbb{R}^3$. Apply López-Ros argument to $R(n)$ to find a contradiction.

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$\Rightarrow \{x_3 = x_3(p_\infty)\} - \{p_\infty\}$ intersects E transversely at some interior point
$\Rightarrow$ for n large, $R(n)$ intersects $E - R(n)$!!

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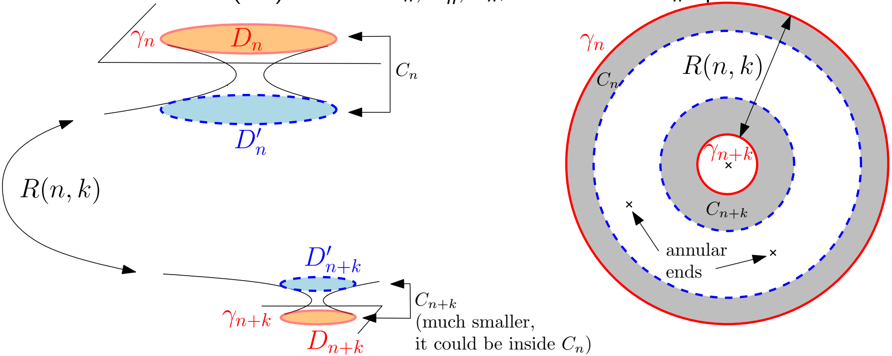
How to rule out case (D2): Define D_n, D'_n, C_n, that bound W_n cpt. Take $k \gg 1$.



Relative position of W_n, W_{n+k}?

Arguments as those that ruled out (D1) imply now $W_{n+k} \subset \mathrm{Int}(W_n)$ for n, k large.

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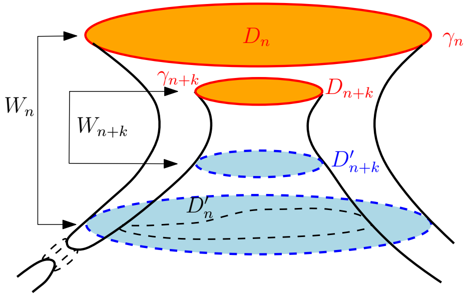
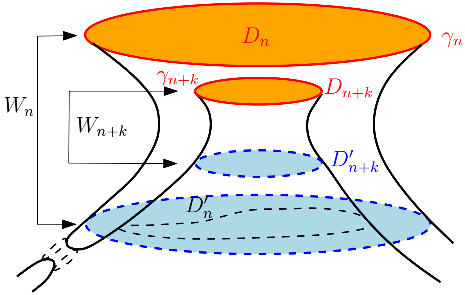


Properness VII: Discarding a catenoid, case (D2) (sketch)

Lemma 2

If $\{p_n\}_n \to p_\infty \in \mathbb{R}^3$ *and (D3) holds* $\forall n \Rightarrow E$ *lies at one side of* $\{x_3 = x_3(p_\infty)\}$.

How to rule out case (D2):
Reindexing and taking subseq,
$W_{n+1} \subset \mathrm{Int}(W_n)\ \forall n$



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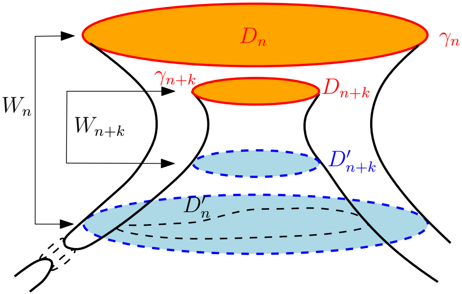

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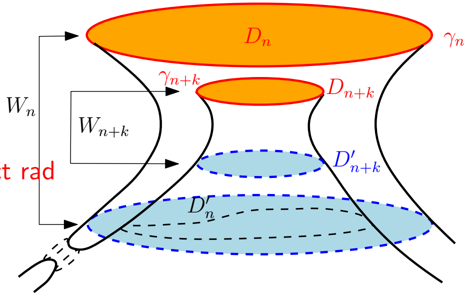
$E \cap [\mathrm{Int}(W_1) - \{c_\infty\}]$ locally posit inject rad

(otherwise rescale by topology, find a vertical catenoid as limit and



- Discard (D1) by López-Ros,
- Discard (D2) because 'catenoids' are concentric,
- Discard (D3) by Lemma 2 !!)

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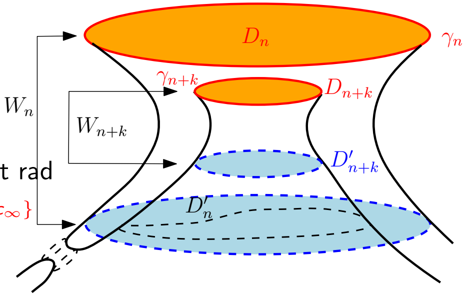

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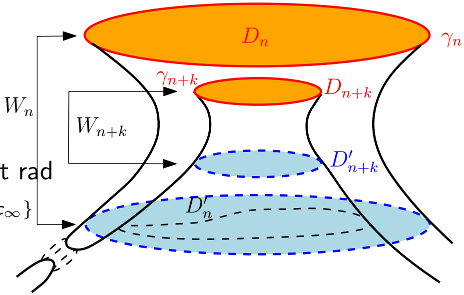
without limit leaves in a neighb of c_∞



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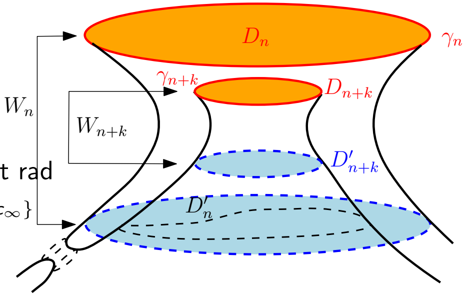
without limit leaves in a neighb of c_∞

$\Rightarrow$ in some cpt neighb U of c_∞,

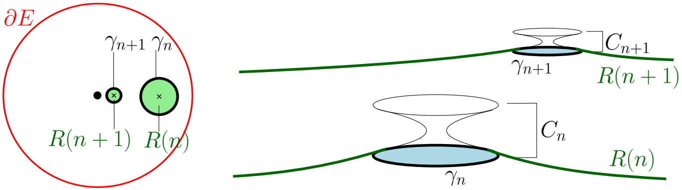


$\mathcal{L} \cap U$ is a PEMS of genus zero $\overset{\text{(LRST)}}{\Rightarrow}$ $\mathcal{L} \cap U$ extends smoothly across c_∞ !!

This discards case (D2).

Properness VIII: Discarding a catenoid, case (D3) (sketch)

Suppose (C1) holds with M_∞ vertical catenoid and (D3) holds $\forall n$.



$\gamma_n = \partial R(n)$, $R(n) \subset E$ proper graphical ann, $\log(R(n)) \nearrow 0$ (Length$(\gamma_n) \to 0$)

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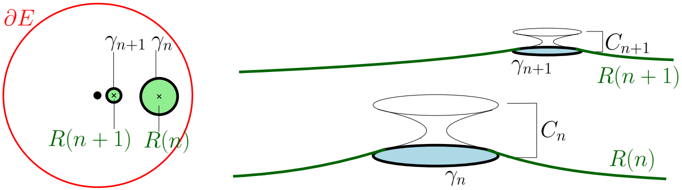

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- E is locally simply cnn in $\mathbb{R}^3$: Otherwise $\exists p_\infty \in \mathbb{R}^3$ s.t. E not loc simply cnn in any neighb of $p_\infty \overset{\text{(Lemma 2)}}{\Rightarrow} E$ lies at one side S of $\Pi(p_\infty) = \{x_3 = x_3(p_\infty)\}$ & $\Pi(p_\infty) = \lim_n R(p_\infty, n)$, $R(p_\infty, n) \subset E$ proper graphical annuli.

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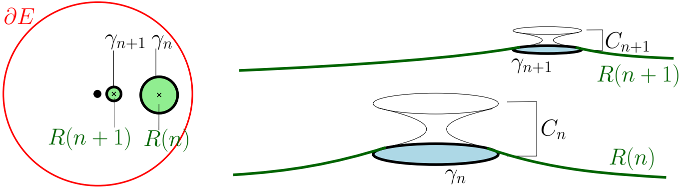
Suppose (C1) holds with M_∞ vertical catenoid and (D3) holds $\forall n$.



$\gamma_n = \partial R(n)$, $R(n) \subset E$ proper graphical ann, $\log(R(n)) \nearrow 0$ ($\text{Length}(\gamma_n) \to 0$)

- *E* is locally simply cnn in $\mathbb{R}^3$: Otherwise $\exists p_\infty \in \mathbb{R}^3$ s.t. E not loc simply cnn in any neighb of $p_\infty \overset{\text{(Lemma 2)}}{\Rightarrow} E$ lies at one side S of $\Pi(p_\infty) = \{x_3 = x_3(p_\infty)\}$ & $\Pi(p_\infty) = \lim_n R(p_\infty, n)$, $R(p_\infty, n) \subset E$ proper graphical annuli.
  E loc simply cnn around every in $\mathbb{R}^3 - \{p_\infty\}$: otherwise we contradict Lemma 2 or that $R(p_\infty, n) \cap R(q, n) \neq \emptyset$ for n large (provided that E not loc simply cnn around $q \in \Pi(p_\infty) - \{p_\infty\}$)

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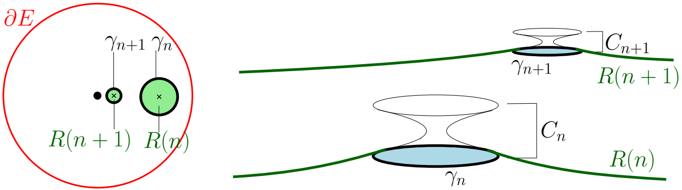
Suppose (C1) holds with M_∞ vertical catenoid and (D3) holds $\forall n$.



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  E loc simply cnn around every in $\mathbb{R}^3 - \{p_\infty\}$
  $\Rightarrow \mathcal{L}_1 := \overline{E - \partial E}^{\mathbb{R}^3 - \{p_\infty\}}$ min lamination of $\mathbb{R}^3 - (\partial E \cup \{p_\infty\})$ (MLCT)

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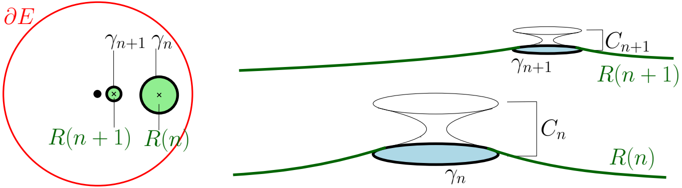

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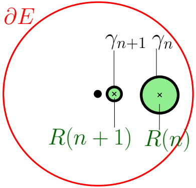
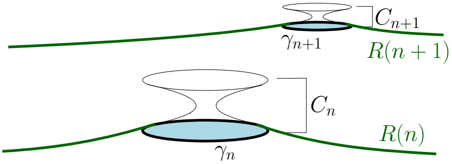
  E properly emb in S: otherwise $\exists L_1 \in \mathcal{L}_1$ limit leaf hence horiz plane & $E \subset \text{slab}[\Pi(p_\infty), L_1]$!!

Properness VIII: Discarding a catenoid, case (D3) (sketch)

Suppose (C1) holds with M_∞ vertical catenoid and (D3) holds $\forall n$.





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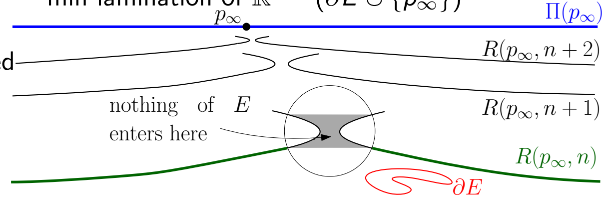
  E properly emb in S

  $\Rightarrow \partial E$ cannot be joined to the portion of $E - R(p_\infty, n)$ above $R(p_\infty, n)$!!



Properness VIII: Discarding a catenoid, case (D3) (sketch)

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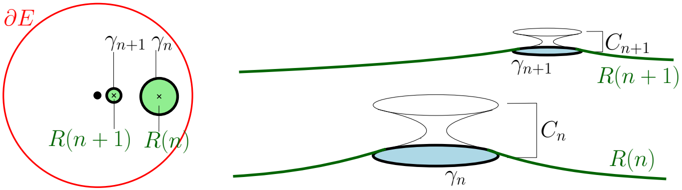

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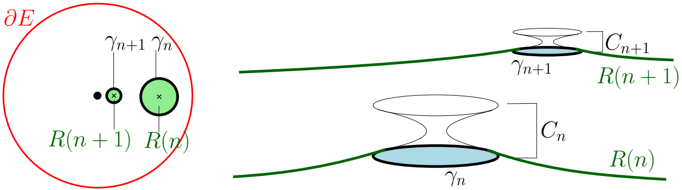

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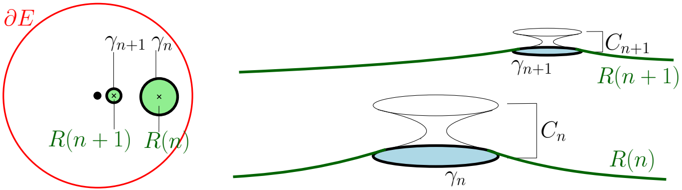

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- $\mathcal{L}'$ consists of a single plane Π: If $\exists \Pi \neq \Pi' \in \mathcal{L}' \Rightarrow E \subset \text{slab}[\Pi, \Pi']$!!

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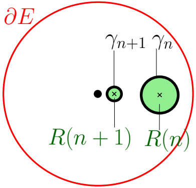

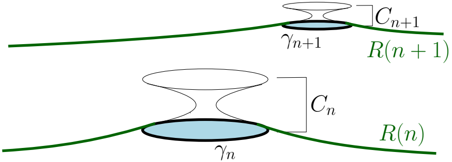

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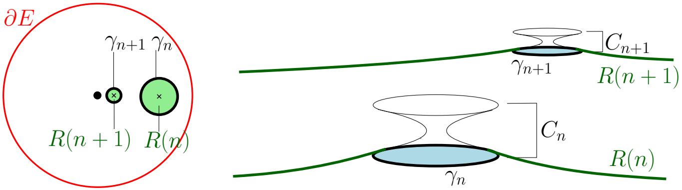

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- Meeks-Rosenberg (CM 1-sided curv estim) $\Rightarrow I_E$ not bded away from zero in any $\{x_3(\Pi) - \delta < x_3 < x_3(\Pi)\}$, $\delta > 0$.

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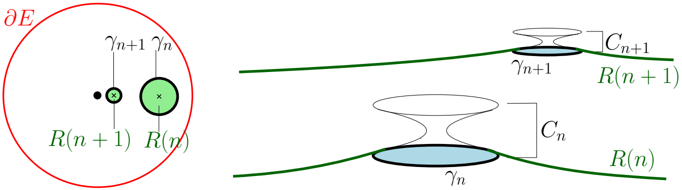

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- Meeks-Rosenberg (CM 1-sided curv estim) $\Rightarrow I_E$ not bded away from zero in any $\{x_3(\Pi) - \delta < x_3 < x_3(\Pi)\}$, $\delta > 0$.
- Adapt the separation arguments above to show that ∂E cannot be joined to the portion of $E - R(n)$ above $R(n)$ for n large !!   (final contradiction)