Integral points on some cubic surfaces

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(joint work with Peter Sarnak)

Definition

By an affine form f in n-variables we mean $f \in \mathbb{Z}[x_1, \ldots, x_n]$ whose leading homogeneous term f_0 is non-degenerate and such that $f - k$ is (absolutely) irreducible for all constants k.

Definition

An affine cubic form F is an affine form in three variables with f_0 a cubic form.

Definition

For $k \neq 0$ and $\mathbb{X} \subset \mathbb{C}^n$, set

$$V_{k,f}(\mathbb{X}) = \{\mathbf{x} \in \mathbb{X} : \ f(\mathbf{x}) = k\},$$

and $\mathfrak{v}_f(k) := |V_{k,f}(\mathbb{Z})|$.

Basic question: For which k is $V_{k,f}(\mathbb{Z}) \neq \emptyset$, or more generally infinite or Zariski dense in $V_{k,f}$?

To measure the richness of representations by f, we say

Definition

1. *f is perfect if $V_{k,f}(\mathbb{Z})$ is Zariski dense in $V_{k,f}$ for all but finitely many admissible k's;*
2. *f is almost perfect if the same holds for almost all admissible k (in the sense of natural density);*
3. *f is full if $\mathfrak{v}_f(k) \to \infty$ as $k \to \infty$ for almost all admissible k's.*

For an affine form the admissible k's are given in terms of a congruence condition.
If all integers are admissible and if f is perfect, then we say it is *universal*.
If f is a homogenous form in n variables and of degree $d \geq n+1$, then Vojta's Conjectures predict that $V_{k,f}(\mathbb{Z})$ lies in a proper Zariski-closed subset of $\mathbb{P}^{n-1}(\mathbb{Q})$.

Examples

- $n = 2$:
  - A generic quadratic is never full (and not absolutely irreducible).
  - For cubic f, Thue (1909), Siegel(1929) show that $V_{k,f}(\mathbb{Z})$ is finite. Moreover only for very few of the admissible k's is $V_{k,f}(\mathbb{Z})$ non-empty (Schmidt 1987).
- (Davenport 1939) The sum of four cubes is full.
- (Hooley 2016) If f is a homogeneous cubic and is nonsingular with $n \geq 5$, then f is full, while conditional on the Riemann Hypothesis for certain Hasse-Weil L-functions, the same is true for $n \geq 4$. Conjectured: any such f with $n \geq 4$ is perfect.
- (Browning/Heath-Brown 2009) If f is a cubic polynomial, $n \geq 10$ and f_0 is nonsingular then f is perfect.

So for cubics, the case $n = 3$ remains to be studied.

Examples: Ternary Cubics

- $F = S$, the sum of three cubes: $S(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$.
  - Congruence obstructions: $V_{k,S}(\mathbb{Z}) = \emptyset$ if $k \equiv 4, 5 \pmod 9$.
  - $V_{1,S}(\mathbb{Z})$ is Zariski dense in $V_{1,S}$ (Lehmer 1956). Infinitely many one-parameter families of solutions constructed, using factorization and the unit group in quadratic fields.
  - Strong approximation in its strongest form fails for $V_{k,S}(\mathbb{Z})$; the global obstruction coming from an application of cubic reciprocity (Cassels 1985, Heath-Brown 1992, Colliot-Thélène/Wittenberg 2012).
- $F = L_1 L_2 L_3$, product of linear forms. Then $V_{k,F}(\mathbb{Z})$ is finite.
- For a $\mathbb{Q}$-anistropic torus given by $N(\mathbf{x}) = Nm_{K/\mathbb{Q}}(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)$, where α_1, α_2, α_3 is a $\mathbb{Z}$-basis of an order in a cubic number field K. Most k are not represented (Odoni 1977):

$$\left|\{|k| \leq X : \mathfrak{v}_N(k) \neq 0\}\right| \sim CX(\log X)^{-\frac{2}{3}}.$$

Examples of F which are not perfect:

- (Mordell 1953) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3 = k$ where $k = 1 - 4w^2$; with $4 \nmid w$ and w without prime factors congruent to 3 mod 4.
- (Cassels/Guy 1966) $5x_1^3 + 9x_2^3 + 10x_3^3 = k$, where $k = 12w^3$.

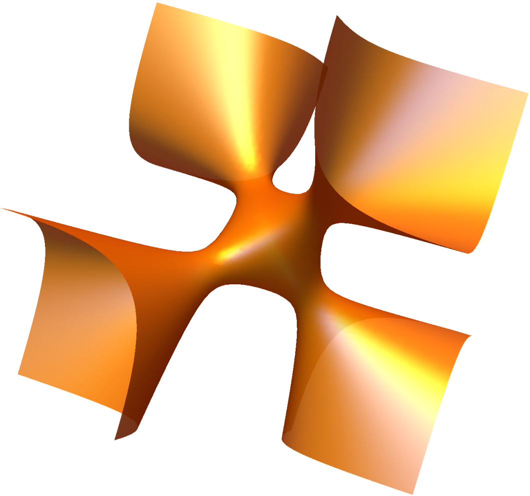
These fail the Hasse Principle for infinitely many k.

Main Topic: Markoff Level Surfaces

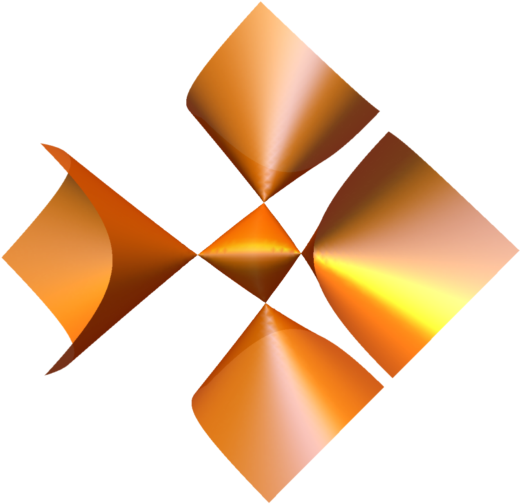
$$M(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3.$$

- (Markoff 1879) $V_{0,M}(\mathbb{Z})$ gives the Markoff triples.
- Local obstructions: $V_{k,M}(\mathbb{Z})$ is empty if $k \equiv 3 \bmod 4$, or $k \equiv \pm 3 \bmod 9$.
- Group Γ of polynomial affine transformations act nonlinearly on $V_{k,M}(\mathbb{Z})$.
- Γ generated by permutations, double sign-changes and three Vieta involutions V_i with $V_1 : (x_1, x_2, x_3) \mapsto (x_2 x_3 - x_1, x_2, x_3)$.
- For $k = 0$, one orbit generated by the point $(3, 3, 3)$ (Markoff). Moreover, $V_{0,M}(\mathbb{Z})$ is Zariski-dense (Corvaja/Zannier 2006).
- (Markoff, Mordell, Hurwitz) For $k \neq 4$, finitely many Γ-orbits.
- $k = 4$ is the Cayley cubic. Infinitely many inequivalent orbits generated by points $(2, a, a)$, $a \geq 0$.

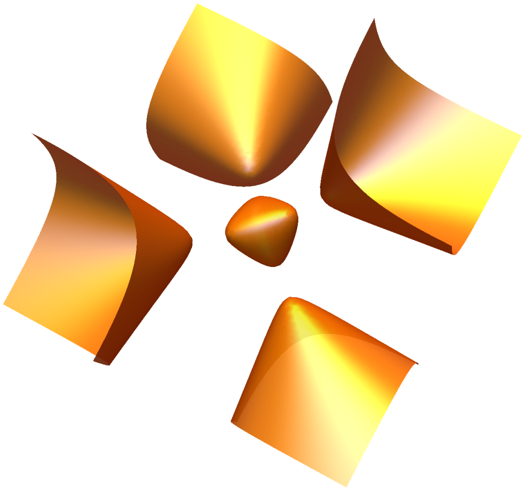
$k = 6$



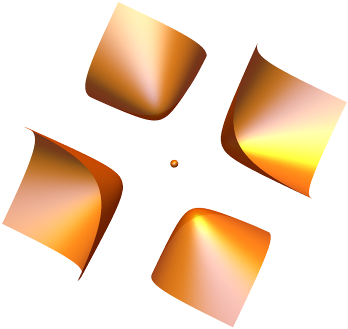
$k = 4$

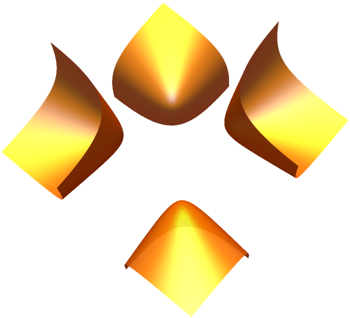


$k = 3$



$k = \frac{1}{10}$ and $k = 0$





Theorem (Parametric solutions)

- *If $k - 4 \neq \square$, there are no one-parameter family of solutions.*
- *If $k = 4 + w^2$, then $(2, t, t + w)$ is a parametric family of solutions.*

Proof.

Apply the group Γ to the parametric family and observe a descent in the degrees of the polynomial coordinates. □

Theorem (Zariski Dense)

If k is not a square, and if $V_{k,M}(\mathbb{Z}) \neq \emptyset$, then $V_{k,M}(\mathbb{Z})$ is Zariski-dense.

Proof.

Use two Vieta transformations to get a $SL(2, \mathbb{Z})$ element of infinite order. Slicing the surface along a hyperplane gives a conic section with infinitely many points. Extension of Corjava/Zannier. □

Results extend to F's of the form $F = cx_1x_2x_3 + G$, where $G = \sum_{i,j} a_{ij}x_ix_j + \sum_i a_ix_i + a$, with $a_{jj} = \pm 1$ for $j = 1, 2, 3$ and $c, a, a_{ij}, a_i \in \mathbb{Z}$. There is a corresponding group action.

Theorem (Universal Ternary Cubic Affine Forms)

U_1 and U_2 are universal. Here

$$U_1(x_1, x_2, x_3) = x_1 + M(x_1, x_2, x_3),$$

and

$$U_2(x_1, x_2, x_3) = x_2(x_3 - x_1) + M(x_1, x_2, x_3).$$

Proof.

$V_{k,U_j}(\mathbb{Z})$ is Zariski-dense for all but finitely many k; moreover for every k, there is a one-parameter family of solutions. □

Hasse Failures of M

Theorem (M is not perfect)

For the following choices of k, $V_k(\mathbb{Z})$ is empty but $V_k(\mathbb{Z}_p)$ is non-empty for all primes p :

1. *Let ν have all of its prime factors lie in the congruence classes $\{\pm 1\}$ modulo 8 with the additional requirement that $\nu \in \{0, \pm 3, \pm 4\}$ modulo 9. Then choose $k = 4 \pm 2\nu^2$.*
2. *Suppose $\ell \geq 13$ is a prime number with $\ell \equiv \pm 4 \pmod 9$. Then choose $k = 4 + 2\ell^2$.*

The smallest positive k here is 342.

Proof.

Use quadratic reciprocity. Similar to Mordell's proof. □

The related $F = x_1^2 + x_2^2 - x_3^2 - x_1x_2x_3$ is also not perfect. It has an infinity of Hasse failures.

Descent and Fundamental Sets of M

Recall: if $k \neq 4$, there are a finite number $\mathfrak{h}_M(k)$ of inequivalent Γ-orbits, with $\mathfrak{h}_M(k) = 0$ if $V_{k,M}(\mathbb{Z})$ is empty.

Definition

k is generic if k is admissible but not of the form (i) $k = u^2 + v^2$ or (ii) $4(k-1) = u^2 + 3v^2$. These latter are exceptional.

- In $|k| \leq K$, there are $O(K(\log K)^{-\frac{1}{2}})$ exceptional k's.
- There are $\sim \frac{7}{12}K$ generic k's.

Theorem (Fundamental sets)

1. *Let* $k \geq 5$ *be generic and consider the compact set*

$$\mathfrak{F}_k^+(\mathbb{R}^3) = \{\mathbf{u} : 3 \leq u_1 \leq u_2 \leq u_3\,,\ u_1^2 + u_2^2 + u_3^2 + u_1 u_2 u_3 = k\}\,.$$

   The points in $\mathfrak{F}_k^+(\mathbb{Z})$ *are* Γ*-inequivalent, and any* $\mathbf{x} \in V_{k,M}(\mathbb{Z})$ *is* Γ*-equivalent to a unique point* $\mathbf{u}' = (-u_1, u_2, u_3)$ *with* $\mathbf{u} = (u_1, u_2, u_3) \in \mathfrak{F}_k^+(\mathbb{Z})$.

2. *Let* $k < 0$ *be admissible and consider the compact set*

$$\mathfrak{F}_k^-(\mathbb{R}^3) = \{\mathbf{u} : 3 \leq u_1 \leq u_2 \leq u_3 \leq \frac{u_1 u_2}{2}, u_1^2 + u_2^2 + u_3^2 - u_1 u_2 u_3 = k\}.$$

   The points in $\mathfrak{F}_k^-(\mathbb{Z})$ *are* Γ*-inequivalent, and any* $\mathbf{x} \in V_{k,M}(\mathbb{Z})$ *is* Γ*-equivalent to a unique point* $\mathbf{u} = (u_1, u_2, u_3) \in \mathfrak{F}_k^-(\mathbb{Z})$.

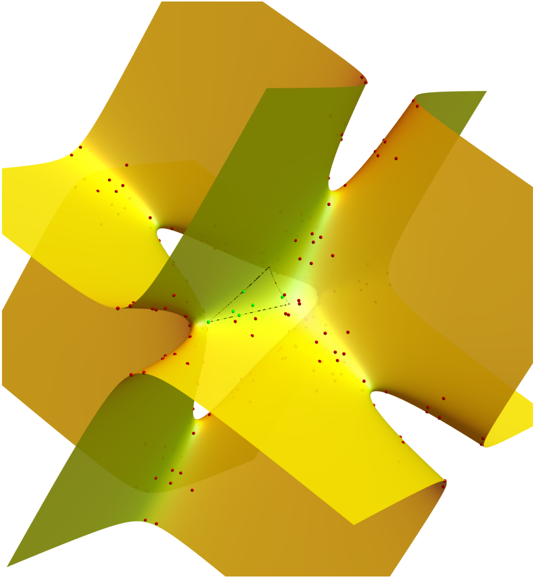


Figure: Lattice points and fundamental set (triangular) for $k = 3685$.

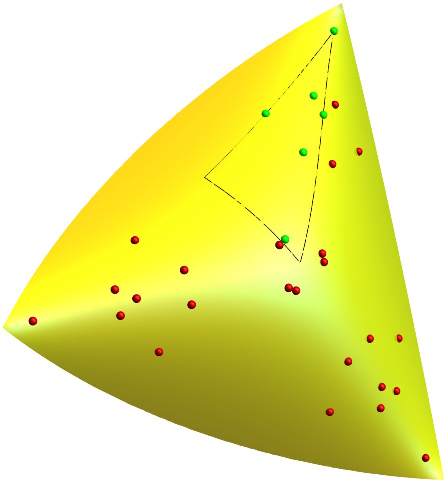


Figure: Closeup of fundamental set (triangular) for $k = 3685$.

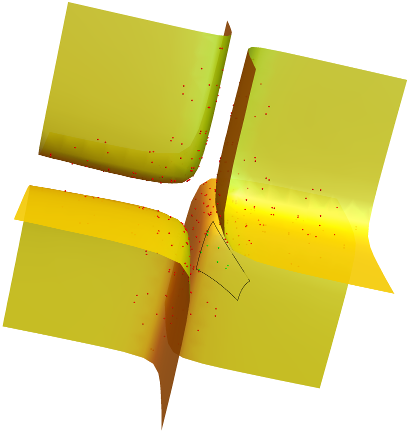


Figure: Lattice points and fundamental set for $k = -3691$.

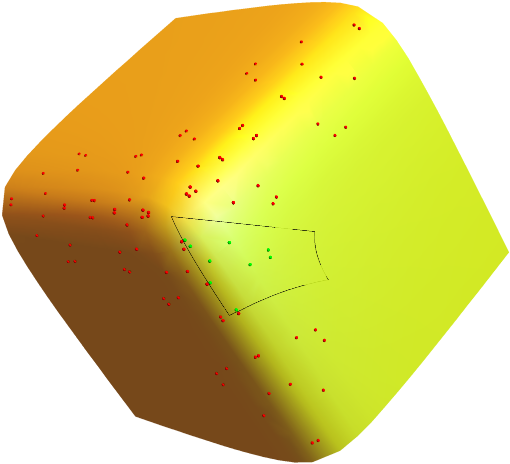


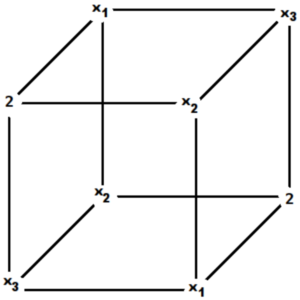
Figure: Closeup of fundamental set for $k = -3691$.

Sketch of Proof for Fundamental Sets

Standard descent argument gets down to the compact sets. To show the points are inequivalent, we use the function

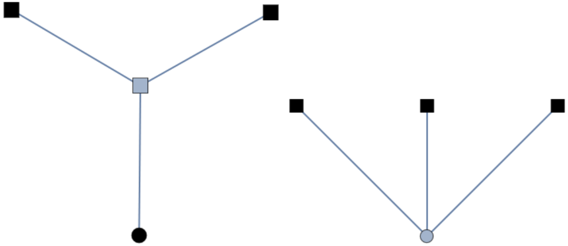
$$\Delta(\mathbf{x}) = (2 + x_1 + x_2 + x_3)(2 + x_2 - x_1 - x_3)\times (2 + x_3 - x_1 - x_2)(2 + x_1 - x_2 - x_3)\,.$$

Originally derived by considering the Bhargava cube



- Δ is invariant under permutations and double sign-changes.
- $$\Delta \circ V_1(\mathbf{x}) - \Delta(\mathbf{x}) = x_2x_3(x_2x_3 - 2x_1)\left[2(k-4) + (x_2^2 - 4)(x_3^2 - 4)\right].$$

Analyse the tree generated by nodes



Properties of $\mathfrak{h}_M(k)$

- $\mathfrak{h}_M(46) = 0$, the first positive Hasse failure .
- $k = -4$ is the first negative Hasse failure .
- $\mathfrak{h}_M(k) \ll_\varepsilon |k|^{\frac{1}{3}+\varepsilon}$ as $k \to \pm\infty$.
- Let $\mathfrak{h}_M^{\pm}(k) = |\mathfrak{F}_k^{\pm}(\mathbb{Z})|$ where $\pm = \mathrm{sgn}(k)$, this being defined for any k.
  For generic k, $\mathfrak{h}_M^{\pm}(k) = \mathfrak{h}_M(k)$ while otherwise $\mathfrak{h}_M(k) \leq \mathfrak{h}_M^{\pm}(k)$. Then

$$\frac{1}{K} \sum_{\substack{k \neq 4 \\ |k| \leq K}} \mathfrak{h}_M^{\pm}(k) \sim C^{\pm}(\log K)^2,$$

  where $C^{\pm} > 0$ and $K \to \infty$.

Main Result

Theorem

M is almost perfect. That is

- $V_{k,M}(\mathbb{Z})$ *is Zariski dense for any admissible* k *if* $V_{k,M}(\mathbb{Z})$ *is non-empty;*
- $$\#\{|k| \leq K : k \text{ admissible}, \ \mathfrak{h}_M(k) = 0\} = o(K),$$

  as $K \to \infty$ *i.e.* M *is full;*
- *Consequently, if* $t \geq 0$ *is fixed, then*

  $$\#\{0 \leq |k| \leq K : \mathfrak{h}_M(k) = t, \ k \text{ generic}\} = o(K),$$

  as $K \to \infty$.

Sketch of Proof

The proof is a variation of the work of Sarnak and of Bourgain-Fuchs on Apollonian packings.
We choose any a in a set $\mathcal{A}$, each of size a suitable power of $A \approx \log K$ and consider the quadratics

$$g_a(x_1, x_2) = x_1^2 + x_2^2 + ax_1x_2 \quad \text{and} \quad f_a(x_1, x_2) = g_a(x_1, x_2) + a^2.$$

Restricting the variables suitably, for each a, we seek the value distribution of f_a. Setting $d = a^2 - 4$, define the sector $\mathcal{S}_d$ in the plane as

$$\mathcal{S}_d = \left\{ (x_1, x_2) : \begin{array}{c} x_1,\ x_2 \geq 0,\ 0 \leq g_a(x_1,\ x_2) \leq \frac{1}{4}, \\ \frac{1}{2}\left(2\sqrt{d} - a\right) x_2 \leq x_1 \leq \frac{1}{2}\left(3\sqrt{d} - a\right) x_2 \end{array} \right\}.$$

For $a \in \mathcal{A}$ and $k \leq K$ let

$$r_a(k) = \#\left\{(x_1, x_2) \in \sqrt{4K}\mathcal{S}_d \cap \mathbb{Z}^2 : f_a(x_1, x_2) = k\right\} .$$

We now set

$$b_{\mathcal{A}}(k) = \sum_{a \in \mathcal{A}} r_a(k),$$

and we are interested in this as a function of k for $1 \leq k \leq K$.
We define our variance

$$V(K) = \sum_{k \leq K} \left(b_{\mathcal{A}}(k) - C(\log A)\delta^{(m)}(k)\right)^2 .$$

Here m is a parameter, $\delta^{(m)}(k)$ a density function.
Expanding and evaluating all the terms asymptotically, using results of Blomer-Granville on the value distribution of positive definite quadratic forms, versions of the circle method by Heath-Brown and Niedermowwe, and some detailed analysis of local densities, shows that $V(K) = o(K)$, from which the result follows.

Computations: Counting Hasse Failures

K	# Hasse Fail	Predictor	% error
6,552,000	388,485	388,474	0.00279494
13,104,000	738,402	738,476	-0.0100959
19,656,000	1,074,038	1,074,075	-0.00351784
26,208,000	1,400,385	1,400,458	-0.00526837
————	————	————	—————-
78,624,000	3,845,160	3,845,601	-0.0114887
85,176,000	4,138,458	4,138,557	-0.00241157
91,728,000	4,429,888	4,429,563	0.00732315
98,280,000	4,718,612	4,718,766	-0.00326508
————	————	————	—————-
157,248,000	7,256,456	7,257,091	-0.00876333
163,800,000	7,532,631	7,533,279	-0.00860614
170,352,000	7,807,978	7,808,490	-0.00656096
176,904,000	8,082,302	8,082,764	-0.00572446
————	————	————	—————-
255,528,000	11,313,674	11,312,152	0.0134518
262,080,000	11,577,887	11,576,836	0.00907272
268,632,000	11,841,388	11,840,928	0.00388283
275,184,000	12,104,565	12,104,442	0.00101294

The data in the table suggests that (at least for K in the range $\{10^7, 50 * 10^7\}$) in the interval $[0, K]$,

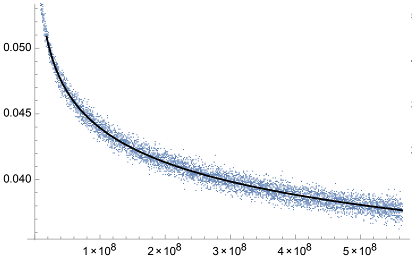
$$\# \text{ of Hasse Failures} \sim C\ K^{f(K)},$$

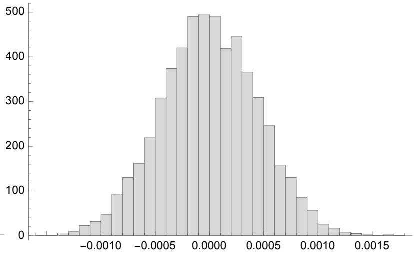
with

$$f(K) \approx 0.887516 - 8.06653\ L^{-2} - 21.8923\ L^{-3} + 2.38097\ L^{-4} + \dots,$$

for some constant $C > 0$; here $L = \log K$.
The error is smaller than 0.1% for $K \geq 10^7$ and gets better for larger values of K.

Average of HF in subintervals of length $h \approx 100,000$





Dark curve is given by
$g(x) = 0.2353x^{-0.0908047} - 46.7396x^{-0.661084}$.

Sample class numbers $\mathfrak{h}(k)$, $k > 0$

k	$\mathfrak{h}(k)$	Fundamental points
54	1	(3, 3, 3)
70	1	(3, 3, 4)
88	1	(3, 3, 5)
108	1	(3, 3, 6)
133	1	(3, 4, 6)
154	1	(3, 3, 8)
166	1	(4, 5, 5)
9230	3	(3, 28, 59), (7, 17, 52), (11, 25, 28)
9234	2	(3, 15, 75), (9, 9, 63)
9253	3	(3, 42, 44), (8, 9, 66), (12, 18, 35)
9260	9	(3, 7, 86), (3, 19, 70), (3, 29, 58), (5, 19, 58), (5, 31, 42)
		(6, 23, 47), (7, 31, 33),(9, 13, 53), (9, 22, 37)
9261	1	(6, 15, 60)
9268	1	(6, 32, 36)
9288	2	(3, 30, 57), (6, 12, 66)
9289	1	(3, 24, 64)
9296	1	(10, 11, 55)
9302	3	(4, 21, 61), (5, 9, 76), (11, 19, 36)
9304	5	(3, 13, 78), (9, 14, 51), (9, 27, 31), (13, 18, 33), (14, 21, 27)
9308	3	(5, 27, 47), (9, 11, 58), (10, 23, 33)

Distribution of $\mathfrak{h}(k)$, generic $0 < k \leq 10^7$

$\mathfrak{h}(k)$	$\mathfrak{n}(\mathfrak{h}(k))$ occurrences
0	574,778
1	423,094
2	346,019
3	259,787
4	202,111
5	157,726
6	124,744
7	100,431
8	81,243
9	66,794
10	54,942
11	45,898
12	38,719
13	32,886
14	28,001
15	23,954
16	20,930
17	17,932
18	15,970
19	13,748
20	12,105
21	10,434

s	$\mathfrak{n}(s+1)/\mathfrak{n}(s)$
0	0.7361
1	0.81783
2	0.750788
3	0.777987
4	0.780393
5	0.790891
6	0.805097
7	0.808943
8	0.822151
9	0.822559
10	0.83539
11	0.843588
12	0.84935
13	0.851457
14	0.855469
15	0.873758
16	0.856761
17	0.890587
18	0.860864
19	0.880492
20	0.861958
21	0.888921

Distribution of $\mathfrak{h}(k)$, generic $0 < k \leq 10^7$

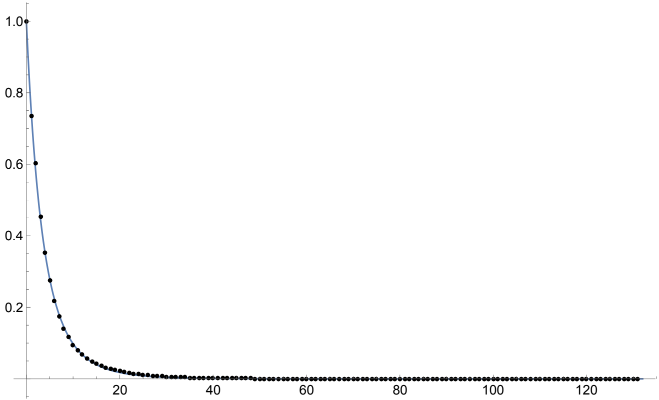


Figure: Occurences of relative number of orbits: $\mathfrak{n}(\mathfrak{h}(k))/\mathfrak{n}(0)$, generic $k \leq 10^7$. Approximation curve $\mathfrak{n}(h) = \mathfrak{n}(0)(6.86293 + 4.62621h + 0.0576149h^2)e^{-1.92905\sqrt{h+1}}$.

Conjectures

Conjecture

For any $\varepsilon > 0$

$$\mathfrak{h}(k) \ll_{\varepsilon} |k|^{\varepsilon}.$$

Conjecture

The number of Hasse failures for $0 \leq k \leq K$ *satisfies*

$$\left| \{ 0 \leq k \leq K : \mathfrak{h}(k) = 0 \text{ and } k \text{ admissible} \} \right| \sim C_0 K^{\theta},$$

for some $C_0 > 0$ *and some* $\frac{1}{2} < \theta < 1$.
More generally, for $t \geq 1$

$$\left| \{ 0 \leq k \leq K : \mathfrak{h}(k) = t \} \right| \sim C_t K^{\theta},$$

with $C_t > 0$.

The values of C_t above are illustrated in the previous slide, suggesting an exponential decay in t.

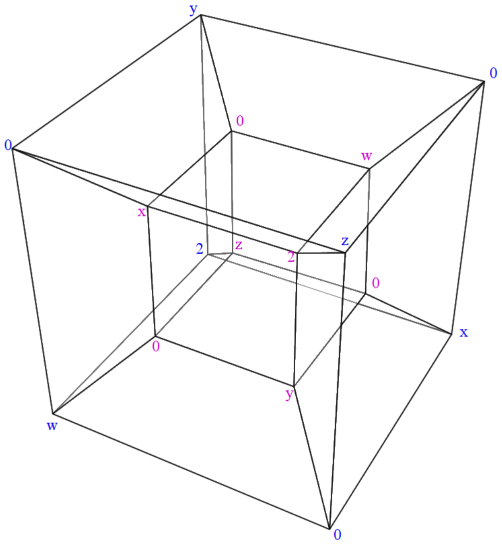
(Almost) THE END

Hurwitz: $x_1^2 + \ldots + x_4^2 - x_1x_2x_3x_4 = k$

.

- No congruence obstructions. $k = 7$ is a Hasse failure.
- If $k-1$ is admissible for Markoff, then k allows integral solutions of Hurwitz. Hence almost all of the $\frac{7}{12}$ values of k allow solutions.
- Suppose $k \geq 7$ and k is *generic* (here k is generic means k is not a sum of three squares and $k-1$ is not Markoff admissible). Hence those k's of the form $k = 4^{a+1}(8b+7)$, or $k = 24b + 7$ with $3 \nmid (b+1)$.
  Then, if there are solutions, then there is descent and using Δ one can determine a fundamental set.
- One can use $\Delta = \prod (2 \pm x_1 \pm x_2 \pm x_3 \pm x_4)$, even number of minus signs. Alternatively, one can use an invariant derived from a $2 \times 2 \times 2 \times 2$ hypercube.

$2 \times 2 \times 2 \times 2$ hypercube



$$
\begin{aligned}
\Delta(\mathbf{x}) = {} & 256 - 256w^2 + 96w^4 - 16w^6 + w^8 - 256x^2 + 64w^2x^2 \\
& + 16w^4x^2 - 4w^6x^2 + 96x^4 + 16w^2x^4 + 6w^4x^4 - 16x^6 \\
& - 4w^2x^6 + x^8 - 256y^2 + 64w^2y^2 + 16w^4y^2 - 4w^6y^2 \\
& + 64x^2y^2 - 96w^2x^2y^2 + 4w^4x^2y^2 + 16x^4y^2 + 4w^2x^4y^2 \\
& - 4x^6y^2 + 96y^4 + 16w^2y^4 + 6w^4y^4 + 16x^2y^4 + 4w^2x^2y^4 \\
& + 6x^4y^4 - 16y^6 - 4w^2y^6 - 4x^2y^6 + y^8 - 768wxyz + 192w^3xyz \\
& + 192wx^3yz + 192wxy^3z - 256z^2 + 64w^2z^2 + 16w^4z^2 - 4w^6z^2 \\
& + 64x^2z^2 - 96w^2x^2z^2 + 4w^4x^2z^2 + 16x^4z^2 + 4w^2x^4z^2 \\
& - 4x^6z^2 + 64y^2z^2 - 96w^2y^2z^2 + 4w^4y^2z^2 - 96x^2y^2z^2 \\
& + 216w^2x^2y^2z^2 + 4x^4y^2z^2 + 16y^4z^2 + 4w^2y^4z^2 + 4x^2y^4z^2 \\
& - 4y^6z^2 + 192wxyz^3 + 96z^4 + 16w^2z^4 + 6w^4z^4 + 16x^2z^4 \\
& + 4w^2x^2z^4 + 6x^4z^4 + 16y^2z^4 + 4w^2y^2z^4 + 4x^2y^2z^4 + 6y^4z^4 \\
& - 16z^6 - 4w^2z^6 - 4x^2z^6 - 4y^2z^6 + z^8.
\end{aligned}
$$

$$
\begin{aligned}
\Delta\circ V_1(\mathbf{x}) - \Delta(\mathbf{x}) &= xyz(2w+xyz)\\
&\times (512 - 48w^4 + 4w^6 - 128x^2 + 32w^2x^2 - 12w^4x^2 + 16x^4\\
&+ 12w^2x^4 - 4x^6 - 128y^2 + 32w^2y^2 - 12w^4y^2 - 96x^2y^2 + 8w^2x^2y^2\\
&+ 4x^4y^2 + 16y^4 + 12w^2y^4 + 4x^2y^4 - 4y^6 - 96w^3xyz + 12w^5xyz\\
&+ 32wx^3yz - 24w^3x^3yz + 12wx^5yz + 32wxy^3z - 24w^3xy^3z\\
&+ 8wx^3y^3z + 12wxy^5z - 128z^2 + 32w^2z^2 - 12w^4z^2 - 96x^2z^2\\
&+ 8w^2x^2z^2 + 4x^4z^2 - 96y^2z^2 + 8w^2y^2z^2 + 120x^2y^2z^2\\
&- 112w^2x^2y^2z^2 + 22w^4x^2y^2z^2 + 16x^4y^2z^2 - 28w^2x^4y^2z^2\\
&+ 6x^6y^2z^2 + 4y^4z^2 + 16x^2y^4z^2 - 28w^2x^2y^4z^2 + 4x^4y^4z^2\\
&+ 6x^2y^6z^2 + 32wxyz^3 - 24w^3xyz^3 + 8wx^3yz^3 + 8wxy^3z^3\\
&- 64wx^3y^3z^3 + 24w^3x^3y^3z^3 - 16wx^5y^3z^3 - 16wx^3y^5z^3 + 16z^4\\
&+ 12w^2z^4 + 4x^2z^4 + 4y^2z^4 + 16x^2y^2z^4 - 28w^2x^2y^2z^4 + 4x^4y^2z^4\\
&+ 4x^2y^4z^4 - 16x^4y^4z^4 + 16w^2x^4y^4z^4 - 4x^6y^4z^4 - 4x^4y^6z^4 + 12wxyz^5\\
&- 16wx^3y^3z^5 + 6wx^5y^5z^5 - 4z^6 + 6x^2y^2z^6 - 4x^4y^4z^6 + x^6y^6z^6)
\end{aligned}
$$