Reeb dynamics in dimension 3

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Thank you!

Thank you for inviting me to give this talk!

Reeb vector fields

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- A *contact form* on Y is a differential one-form satisfying

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If Y is a closed manifold with a contact form, then the Reeb vector field always has at least one closed orbit.

A closed orbit of the Reeb vector field is called a *Reeb orbit*.

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The rest of this talk will be about the three-dimensional Weinstein conjecture and its generalizations.

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Theorem 1 (Taubes)

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This also implies nontriviality (step 3). Nontriviality of Seiberg-Witten Floer cohomology is known by work of Kronheimer and Mrowka. Actually they show that $\widehat{HM}^{-*}(Y)$ has infinite rank as a $\mathbb{Z}$-module.

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- (CG., Hutchings, '12) Any contact form on any three-manifold has at least 2 embedded Reeb orbits (*cf.* Ginzburg-Hein-Hryniewicz-Macarini).

Future directions

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Speculation

If Y is not a lens space, then the Reeb flow has infinitely many embedded Reeb orbits.

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To make sense of this, define the *action* of a Reeb orbit γ

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The action of a Reeb orbit is equal to its period. Also define the *volume* of (Y, λ) by

$$\mathrm{vol}(Y, \lambda) = \int_Y \lambda \wedge d\lambda.$$

Based on limited experimental evidence, we have the following optimistic conjecture:

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Progress:

- If Y has exactly two embedded Reeb orbits then this is true (CG., Hutchings).
- Another promising case is the case where (Y, λ) is *dynamically convex*. This holds, for example, on the boundary of a convex domain in $\mathbb{R}^4$. Here there is a supporting "open book".

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ECH capacities

One such family are the *embedded contact homology capacities*, which are a sequence of nonnegative real numbers

$$0 = c_0(M,\omega) \leq c_1(M,\omega) \leq \ldots \leq c_k(M,\omega) \leq \ldots \leq \infty,$$

defined for any symplectic 4-manifold. They are defined using ECH, and they are sharp in interesting cases, for example embeddings of one ellipsoid into another (McDuff) and many symplectic ball packing problems.

Volume axiom

The ECH capacities also satisfy a *volume axiom*, which is connected to the existence of 2 Reeb orbits.

Theorem 2

(CG., Gripp, Hutchings) If (X, ω) *is a 4-dimensional Liouville domain with all ECH capacities finite, then*

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$$\lim_{k \longrightarrow \infty} \frac{c_k(X, \omega)^2}{k} = 4 \operatorname{vol}(X, \omega).$$

The definition of Liouville domain forces X to have boundary, and any ECH capacity bounds the length of the shortest Reeb orbit from above, hence the motivation for thinking about the conjecture.

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Good news/bad news: A contact form and almost complex structure determine a metric, and other invariants of this metric might also be relevant in bounding the shortest orbit by this approach.