From Representations of the Symmetric Group To Branched Covers of the Disk.

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- Okounkov and Pandharipande '02
- Buryak, N.Z., Pandharipande, Tessler - in preparation.

Def$^{\underline{n}}$: Given d and conjugacy classes $C_1, C_2, \dots, C_n$

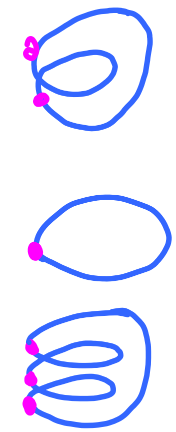
$$\{1\} \neq C_i \subseteq S_d \,, \quad 1 \leq i \leq n$$

the Hurwitz number:

$$H := \frac{1}{d!} \# \{ (\sigma_1, \dots, \sigma_n) \in C_1 \times \dots \times C_n \mid \sigma_1 \sigma_2 \cdots \sigma_n = 1 \} = \frac{1}{d!} \mathbb{E}(X_1 \cdots X_n)$$

$\sigma_i \in \mathrm{Sym}(F)$

$|F| = d$



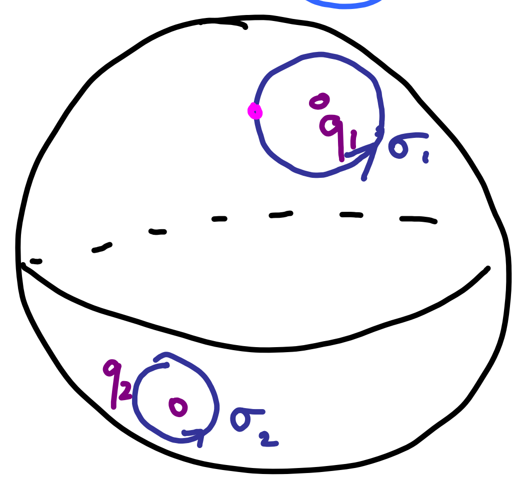
$d=6$

Problem: compute H, describe structure...

$$\sigma \in C_i \iff \sigma = (\cdot\,\cdot)(\cdot\,\cdot\,\cdot)(\cdot\,\cdot\,\cdot)$$

Algebraic Topology: Fix $Q = \{q_1, \dots, q_n\} \subset \mathbb{CP}^1$

$$H = \Big| \Big\{ \underbrace{\check{\Sigma} \xrightarrow{\check{\pi}} \mathbb{CP}^1 \setminus Q}_{\text{deg } d \text{ covers}} \,\Big|\, \sigma_i \in C_i \Big\} \Big/ \mathrm{Iso}^{\underline{m}} \Big| := \sum_{[\check{\pi}]} \frac{1}{|\mathrm{Aut}\, \check{\pi}|}$$

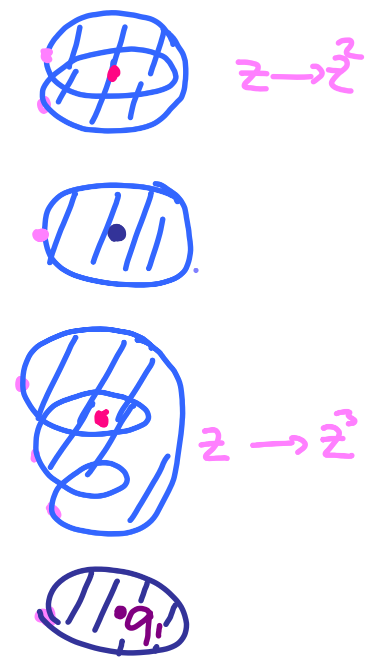


Riemann + Hurwitz: Can fill in the punctures

$\rightsquigarrow \Sigma \xrightarrow{\pi} \mathbb{C}P^1$ branched cover.

ramification point:

$p \in \Sigma$, $\pi_p \simeq z \mapsto z^a$, $a \geq 2$.



Fantasy: $$\mathcal{M} = \left\{ \left(\Sigma \xrightarrow[\substack{\deg d \\ \text{branched}}]{\pi} \mathbb{C}P^1, z_1, \ldots, \underset{\Sigma}{\overset{}{z_i}}, \ldots, z_N \right) \right\} / \text{iso}^{\underline{m}}$$

$$W_i(q,a) = \left\{ (\pi, z_1, \ldots, z_N) \in \mathcal{M} \;\middle|\; \pi(z_i) = q,\ \pi_{z_i} \simeq z \mapsto z^a \right\}$$

$$\hat{H} = \prod_{i=1}^{n} |\mathrm{Sym}(C_i)| \cdot H \overset{?!}{=} \# \bigcap_{i=1}^{N} W_i(q_{j(i)}, a_i)$$

Upshot

$[W_i]$, $\cap$-product.

$S^1 \circlearrowright$

Fixed-Point Formula!

Problem

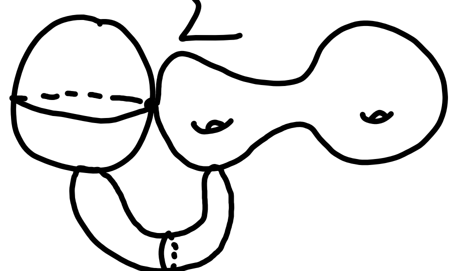
Can't move cycles:

$$d!H_d(C_1, C_2) = \begin{cases} |C_1| & \text{if } C_1 = C_2 \\ 0 & \text{if } C_1 \neq C_2 \end{cases}$$

Unique S^1-inv't branched cover $\mathbb{P}^1 \xrightarrow{d:1} \mathbb{P}^1$

Space of branched covers is not compact. $\rightsquigarrow$ stable maps:

Σ

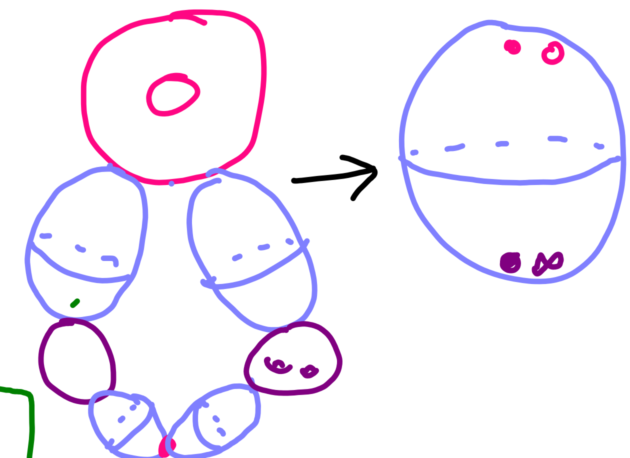


$\xrightarrow{f} \mathbb{CP}^1$

Resolution

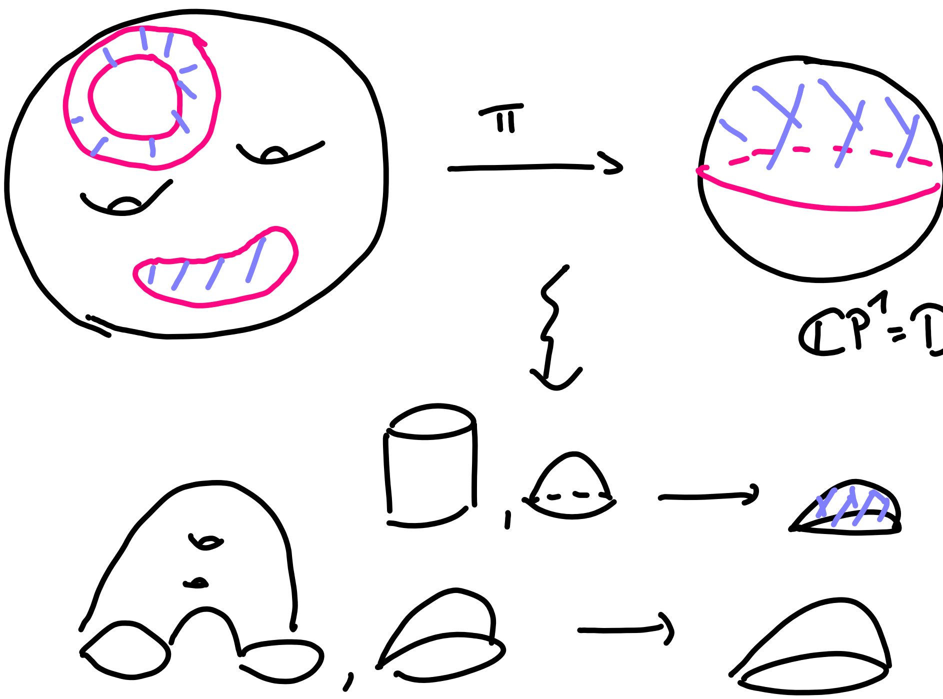
$\cap W_i \supsetneq$ branched covers

More stable S^1-inv't maps:



Open GW - Hurwitz.

$$(S, \partial S) \longrightarrow (D^2, \partial D^2)$$



$$\begin{tikzcd}
\boxed{\text{disc covers}} \arrow[r, "\text{glue}"] \arrow[d, phantom, "\cap" description] & \boxed{\text{closed covers}} \arrow[d, phantom, "\cap" description] \\
\boxed{\text{open GW}} \arrow[r, "\text{glue}"'] \arrow[ur, phantom, "\text{compactify to}"] & \boxed{\text{closed GW}}
\end{tikzcd}$$

+ homological decomp.

+ fixed-point formula