Jean Bourgain and quasiperiodic (ergodic) Schrödinger operators

May 23, 2016

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acting on $\ell^2(\mathbb{Z}^d)$,
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Anderson model:
$v(T^n \theta)$ i.i.d.r.v.

Exotic Spectral properties

- Metal-insulator transitions
- Dense point spectrum
- Unusual absolutely continuous spectrum
- Eigenfunctions decaying at the non-Lyapunov rate
- Singular continuous spectrum
- Cantor spectrum
- Unusual eigenvalue statistics

Prizes

- Anderson localization (Nobel, 1977)
- Integer Quantum Hall Effect (Nobel, 1998)
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Quasiperiodic operators: KAM methods

$$(H_{\lambda,\alpha,\theta}\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + \lambda v(T^n\theta)\Psi_n$$

$$v(T^n\theta) = v(\theta + n\alpha)$$

$\theta, \alpha \in \mathbb{T}^d$, Diophantine α

Small coupling

Dinaburg-Sinai (76), Eliasson (91)

- KAM in the momentum space
- Reducibility of transfer-matrix cocycles

Large coupling, $d = 1$

Sinai, Fröhlich-Spencer-Wittwer (late 80s) cos-type

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Direct estimates of the Green's function in the regime of positive Lyapunov exponents (large coupling). No multiple scales!

non-perturbative ac spectrum by dual localization

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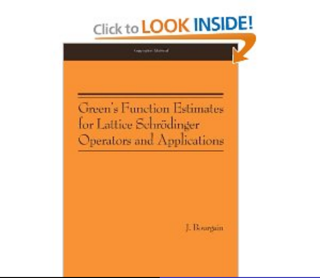
Bourgain (+Goldstein, Schlag, J.) (2000-2005): analytic, multi-frequency, multidimensional
applications e.g. to random NLS (Bourgain, 2005, Bourgain-Wang, 2008)



Other aspects

1. Massiveness of the spectrum (Bourgain, 2005)

Corollary: absolutely continuous spectrum for multi-dimensional quasiperiodic operators at low coupling (Bourgain, J., Parnovsky, 2016)

Other aspects

2. Continuity of the Lyapunov exponent of analytic $SL(2,\mathbb{R})$ cocycles **in frequency** at irrational frequencies (Bourgain-J, 2002)

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- Avila's global theory of quasiperiodic cocycles (Avila, 2009-2015)

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Bourgain, 2005: the same continuity for multifrequency

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Still open in the discrete case!

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Still open for $d > 3$...

Almost Mathieu operators

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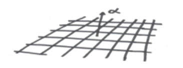
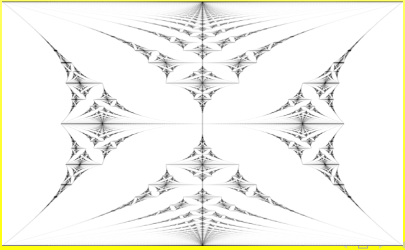
Tight-binding model of 2D Bloch electrons in magnetic fields



Hofstadter butterfly



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Diophantine α : $\beta(\alpha) = 0$,

α-Diophantine θ : $\delta(\alpha,\theta) = 0$

Sharp transitions

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J.- Liu (2015-2016):

- For Diophantine α, there is a sharp transition at $\lambda_0 = e^{\delta(\alpha,\theta)}$
  $|\lambda| < e^{\delta(\alpha,\theta)} \implies H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum
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- For α-Diophantine θ (any α) there is a sharp transition at $\lambda_0 = e^{\beta(\alpha)}$
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  Ten Martini paper: localization for $|\lambda| > e^{16/9\beta(\alpha)}$.

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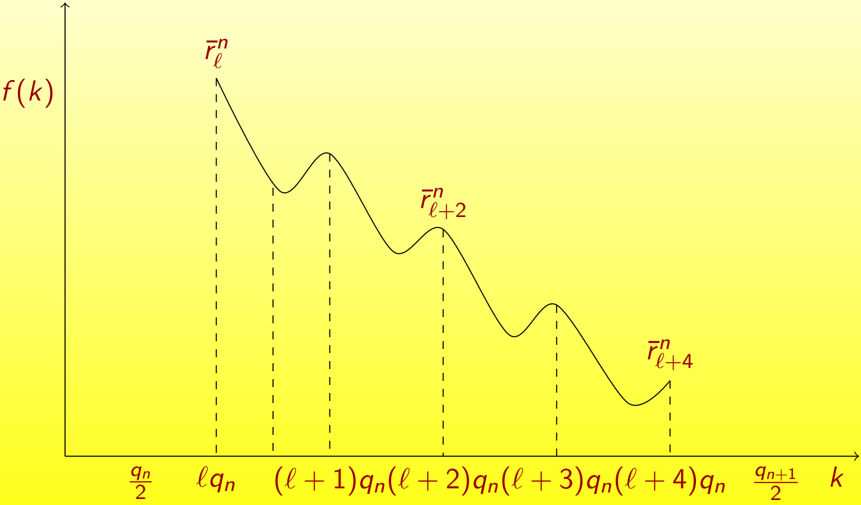
$$f(|k|)e^{-\varepsilon|k|} \leq ||U(k)|| \leq f(|k|)e^{\varepsilon|k|},$$

The behavior of $f(k)$



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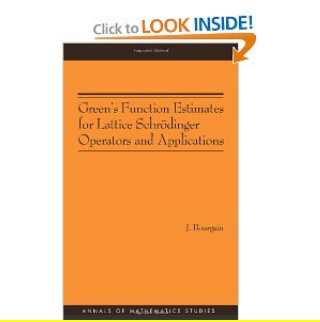
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Green's Function Estimates for Lattice Schrödinger Operators and Applications
J. Bourgain
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Incidentally, I found the reading of Jean's papers as a graduate student to be simultaneously extremely frustrating and extremely rewarding. Decoding an offhand remark or a mysterious step in his paper was often as instructive (and as time-consuming) as reading several pages of arguments by some other authors. (But his papers do become much easier to read once one has internalised enough of his "box of tools"...)

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edited Jul 10 '12 at 20:48

answered Jul 10 '12 at 17:37

Terry Tao
43.4k ● 14 ● 198 ● 282

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