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Jacob Fox

Massachusetts Institute of Technology

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A permutation $\sigma = \sigma_1 \cdots \sigma_n$ *contains* another permutation $\pi = \pi_1 \cdots \pi_k$ if there exists indices $i_1 < \ldots < i_k$ such that $\sigma_{i_j} < \sigma_{i_\ell}$ if and only if $\pi_j < \pi_\ell$. Otherwise, σ is said to *avoid* π.

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Definition

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Theorem: (McMahon 1915, Knuth 1968)

For each 3-permutation π,

$$S_n(\pi) = \frac{1}{n+1}\binom{2n}{n}.$$

Stanley-Wilf conjecture

Conjecture: (Stanley-Wilf 1980)

For each π, there is $L(\pi)$ such that $\lim_{n\to\infty} S_n(\pi)^{1/n} = L(\pi)$.

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For each k-permutation π, $S_n(\pi) \leq C(\pi)^{n\gamma(n)}$, where $\gamma(n)$ is a very slow growing function, related to the Ackermann hierarchy.

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Theorem: (Marcus-Tardos 2004)

For each k-permutation π, $L(\pi)$ exists and satisfies

$$L(\pi) \leq 15^{2k^4\binom{k^2}{k}}.$$

Problem

How large can $L(\pi)$ be for a k-permutation π?

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Theorem: (Claesson-Jelínek-Steingrímsson 2012)

Every layered k-permutation π satisfies $L(\pi) \leq 4k^2$.

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Theorem (F.)

There is a k-permutation π with

$$L(\pi) = 2^{\Omega(k^{1/4})}.$$

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Equivalent to $c(\pi) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,\pi)}{n}$ exists.

Klazar proved $L(\pi) \leq 15^{c(\pi)}$.

Theorem: (Marcus-Tardos 2004)

$$c(\pi) \leq 2k^4 \binom{k^2}{k}.$$

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The *contraction* of two consecutive rows of a matrix replaces the two rows by a single row, with a one in an entry of the new row if at least one of the two entries in the original two rows is a one. Contraction of columns is defined similarly.

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Lemma

$\exists$ ℓ^2-permutation π whose matrix contains J_ℓ as an interval minor.

π is given by $\pi(a\ell + b + 1) = b\ell + a + 1$ for $0 \leq a, b \leq \ell - 1$.

Lower bound construction

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Let $B = (b_{IJ})$ be the $N' \times N'$ matrix with a row for each $I \in V(T_R)$ and a column for each $J \in V(T_C)$ and each entry is one with probability $1 - q$ independently of the other entries.

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For $t = c(\pi)$, this is $S_N(\pi) \leq 2^{O(N)} c(\pi)^{2N}$ and we are done.

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The blocks which are neither wide nor tall each have at most $(k-1)^2$ ones, and the desired inequality follows.