# The susceptibility of the weakly self-avoiding walk in dimension 4

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## The weakly self-avoiding walk (or undirected polymer)

- $X(T)$: continuous-time simple random walk (SRW) on $\mathbb{Z}^d$, $X(0) = 0$
- Intersection local time of $X$ up to time $T$:

$$I(T) = \int_0^T \int_0^T 1_{X(S_1)=X(S_2)}\, dS_1\, dS_2 = \sum_{x \in \mathbb{Z}^d} L^x(T)^2$$

where $L^x(T)$ is the time spent at $x$ by $X[0,T]$ (local time).

- Polymer measure for weakly self-avoiding walk (SAW): Given $g > 0$,

$$P_{g,T}(dX) = \frac{1}{Z_{g,T}} e^{-gI(T)}\, P_{0,T}(dX)$$

where $P_{0,T}$ is the law of SRW of length $T$.

- Universality conjecture: long-time behavior is independent of $g > 0$ (including strictly self-avoiding walk); known for $d = 1$ and $d \geq 5$.

## Distribution of the endpoint

**End-to-end distance**:

$$\left(\mathbb{E}_{g,T}|X(T)|^2\right)^{\frac{1}{2}} \sim [?] \quad \text{as } T \to \infty$$

Fixed $T$ partition functions and their **Laplace transforms**:

- Normalization and **susceptibility**:

$$Z_T = E(e^{-gI(T)}), \quad \chi(\nu) = \int_0^\infty Z_T e^{-\nu T}\, dT$$

- Unnormalized endpoint distribution and **two-point function**:

$$Z_T(x) = E(e^{-gI(T)} 1_{X(T)=x}), \quad G_\nu(x) = \int_0^\infty Z_T(x) e^{-\nu T}\, dT$$

**Critical point.** $Z_{T+S} \leq Z_T Z_S$, as a consequence there is $\nu_c(g)$ such that

$$Z_T = e^{T\nu_c(g)(1+o(1))} \quad \text{as } T \to \infty.$$

## The dependence on dimension

- $d = 1$: **Ballistic behavior.** Trivial for strictly self-avoiding walk, large deviation statements proved for weakly self-avoiding walk (Greven, den Hollander, v.d. Hofstad). In particular,

$$\mathbb{E}_T|X(T)|^2 \sim cT^2.$$

- $d \geq 5$: **Diffusive behavior.** Proved for weakly and strictly self-avoiding walk (Brydges–Spencer, Hara–Slade, Hara):

$$\mathbb{E}_T|X(T)|^2 \sim cT, \quad \chi(\nu_c + \varepsilon) \sim c\varepsilon^{-1}, \quad G_{\nu_c}(x) \sim c|x|^{-(d-2)}.$$

- $2 \leq d \leq 4$: Predicted to be **super-diffusive** and **sub-ballistic**.

  Proved for strictly SAW (Madras$^{(l)}$, Duminil-Copin–Hammond$^{(u)}$):

$$\frac{1}{6}T^{\frac{4}{3d}} \overset{(l)}{\leq} \mathbb{E}_T|X(T)|^2 \overset{(u)}{\leq} o(T^2)$$

  Not proved:

$$T \leq O(\mathbb{E}_T|X(T)|^2), \quad \mathbb{E}_T|X(T)|^2 \leq O(T^{2-\varepsilon})$$

## The dependence on dimension

Non-rigorous predictions in $d = 2$ and $d = 4$:

- $d = 2$: Nienhuis, Lawler–Schramm–Werner:

$$\mathbb{E}_T|X(T)|^2 \sim cT^{\frac{3}{2}}, \quad \chi(\nu_c + \varepsilon) \sim c\varepsilon^{-\frac{43}{32}}, \quad G_{\nu_c}(x) \sim c|x|^{-\frac{5}{24}}.$$

  Evidence that scaling limit is $\mathrm{SLE}_{8/3}$ (Lawler–Schramm–Werner).

- $d = 3$: –
- $d = 4$: Brézin, Le Guillou, Zinn-Justin, ...:

$$\mathbb{E}_T|X(T)|^2 \sim cT(\log T)^{\frac{1}{4}},$$

$$\chi(\nu_c + \varepsilon) \sim c\varepsilon^{-1}(\log \varepsilon^{-1})^{\frac{1}{4}}, \quad G_{\nu_c}(x) \sim c|x|^{-2}.$$

*Heuristically*, two exponents determine the third (“Fisher’s relation”).

## Main result

**Theorem (with Brydges-Slade).** For $d = 4$ and $g > 0$ small,

$$\chi(\nu_c + \varepsilon) \sim A_g \varepsilon^{-1} (\log \varepsilon^{-1})^{\frac{1}{4}} \quad (\varepsilon \downarrow 0). \qquad (*)$$

As $g \downarrow 0$,

$$A_g = (g/(2\pi^2))^{\frac{1}{4}} (1 + O(g)), \qquad \nu_c(g) = -2G_{0,0}(0) g + O(g^2).$$

Immediately related results:

- Brydges–Slade: Proof of $G_{\nu_c}(x) \sim c|x|^{-2}$ in $d = 4$. Proof of theorem for susceptibility extends techniques developed for two-point function.
- Bovier–Felder–Fröhlich (1984): If $G_{\nu_c}(x) \sim c|x|^{-2}$ then there can be at most a logarithmic correction for $\chi$ in $d = 4$.

## Selected related results

Logarithmic corrections:

- Brydges–Evans–Imbrie and Brydges–Imbrie (1992, 2003): Proof of similar results for two-point function and end-to-end distance for weakly self-avoiding walks on 4D hierarchical groups.
- Hara–Tasaki (1986, with method of Gawedzki–Kupiainen): Proof of similar logarithmic corrections for (scalar) $\varphi^4$ spin model on $\mathbb{Z}^4$ (different exponents)
- Lawler (1986, 1995): Loop-erased random walk on $\mathbb{Z}^4$ has logarithmic corrections (but different exponents).

Decay of critical two-point function for related models:

- Gawedzki–Kupiainen ($\varphi^4$), Feldman–Rivasseau–Magnen–Sénéor ($\varphi^4$), Iagolnitzer–Magnen (Edwards model)

## Intersections of simple random walks and criticality of $d = 4$

- Expected intersection time of two independent simple random walks:

$$\int_0^\infty \int_0^\infty \mathbb{E}(1_{X(T)=Y(S)})\, dT\, dS \begin{cases} = \infty & (d \le 4), \\ < \infty & (d > 4). \end{cases}$$

Thus $d = 4$ is critical. As $m^2 \downarrow 0$,

$$\mathsf{B}_{m^2} = \int_0^\infty \int_0^\infty \mathbb{E}(1_{X(T)=Y(S)}) e^{-m^2 T} e^{-m^2 S}\, dT\, dS \sim \begin{cases} cm^{-(4-d)} & (d < 4), \\ c \log m^{-2} & (d = 4), \\ c & (d > 4). \end{cases}$$

- $\mathsf{B}_{m^2}$ is also called the bubble diagram for the simple random walk:

$$\mathsf{B}_{m^2} = \sum_{x \in \mathbb{Z}^d} C_{m^2}(x)^2, \quad C_{m^2}(x) = (-\Delta_{\mathbb{Z}^d} + m^2)^{-1}_{0,x}.$$

## Supersymmetry

Let $A = (A_{xy})_{x,y=1}^{M}$: matrix with positive definite real part. Then:

$$(A^{-1})_{xy} = (\det A)^{-1} \int_{\mathbb{C}^M} \bar{\phi}_x \phi_y \, \exp\left\{ -\sum_{x,y=1}^{M} A_{xy} \bar{\phi}_x \phi_y \right\} \prod_{x=1}^{M} \frac{d\bar{\phi}_x \, d\phi_x}{2\pi i}$$

View $d\phi_x$ and $d\bar{\phi}_x$ as differential forms. Then this can be written as:

$$(A^{-1})_{xy} = \int_{\mathbb{C}^M} \bar{\phi}_x \phi_y \, \exp\left\{ -\sum_{x,y=1}^{M} A_{xy} \left( \bar{\phi}_x \phi_y + \frac{1}{2\pi i} d\bar{\phi}_x \wedge d\phi_y \right) \right\} \qquad \text{(S)}$$

where the integral is over the top degree part of the differential form, and $\wedge$ is the anticommuting wedge product.

The right-hand side of (S) has many properties of the ordinary Gaussian measure, but it is also possible to integrate more general differential forms.

## Supersymmetry

Suppress wedge product $\wedge$ and set

$$\psi_x = (2\pi i)^{-1/2} d\phi_x, \quad \bar{\psi}_x = (2\pi i)^{-1/2} d\bar{\phi}_x.$$

Then (S) reads

$$(A^{-1})_{xy} = \int_{\mathbb{C}^M} \bar{\phi}_x \phi_y \exp\left\{ -\sum_{x,y=1}^{M} A_{xy} \left( \bar{\phi}_x \phi_y + \bar{\psi}_x \psi_y \right) \right\}$$

Interpret $\psi_x$ as anticommuting analogue of $\phi_x$, a Fermionic field. The above is also known as the Berezin integral. The analogues of random variables are differential forms. These are polynomial in $\psi$.

Apply to $A = -\Delta + \nu + iV$, with $-\Delta$ the discrete Laplace operator and $iV$ imaginary potential. On the other hand, $A^{-1}$ can be written as two-point function of simple random walk. Then take $V_x$ i.i.d. Gaussian and expectation.

## Integral representation

Thus (on finite graph $\Lambda$)

$$\int_0^\infty E_x(e^{-g\sum_x L_x(T)^2} 1_{X(T)=y})\, e^{-\nu T}\, dT = \int_{\mathbb{C}^\Lambda} \bar\phi_x \phi_y e^{-\sum_x(\tau_{\Delta,x}+\nu\tau_x+g\tau_x^2)}$$

where

$$\tau_x = \bar\phi_x\phi_x + \bar\psi_x\psi_x, \quad \tau_{\Delta,x} = \bar\phi_x(\Delta\phi)_x + \bar\psi_x(\Delta\psi)_x.$$

RHS is SUSY version of two-point function of $|\phi|^4$ spin model.

SUSY representation can be seen as well-defined implementation of "$n \to 0$" limit of $n$-component $|\varphi|^4$ (or $n$-vector) model of de Gennes (Parisi–Sourlas, McKane).

Resembles $n$-component $|\varphi|^4$ model, but probabilistic tools not available (e.g., reflection positivity and infrared bound of Fröhlich–Simon–Spencer).

Always work on finite torus $\Lambda = \Lambda_N = \mathbb{Z}^d/L^N\mathbb{Z}^d$ (with uniformity in $N$).

## Extension to $|\varphi|^4$ model on $\mathbb{Z}^4$ (with Brydges–Slade)

Let $d = 4$ and $g > 0$ small.

**Susceptibility.** The susceptibility of the $n$-component $|\varphi|^4$ model with $n = 1, 2, 3, \ldots$ satisfies

$$\chi(\nu_c + \varepsilon) \sim A_g \varepsilon^{-1} (\log \varepsilon^{-1})^{\frac{n+2}{n+8}} \quad (\varepsilon \downarrow 0).$$

Formally setting $n = 0$, the exponent of the logarithm becomes $\frac{1}{4}$.

**Other results.** For example, scaling limits to massive free fields with scaling $\nu \downarrow \nu_c$ and $\Lambda \uparrow \mathbb{Z}^d$ simultaneously.

## Approximation by Gaussian measure

Ignoring Fermions (differential forms) for the moment,

$$e^{-\sum_x (|\nabla \phi_x|^2 + \nu |\phi_x|^2 + g|\phi_x|^4)} = e^{-\sum_x (|\nabla \phi_x|^2 + \nu |\phi_x|^2)} e^{-\sum_x g|\phi_x|^4}$$

But careful: $\nu_c < 0$.

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More generally, for any $m^2 > 0$,

$$e^{-\sum_x (|\nabla \phi_x|^2 + \nu |\phi_x|^2 + g|\phi_x|^4)} = e^{-\sum_x (|\nabla \phi_x|^2 + m^2 |\phi_x|^2)} e^{-\sum_x ((\nu - m^2)|\phi_x|^2 + g|\phi_x|^4)}$$

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But careful: $\nu_c < 0$.

More generally, for any $m^2 > 0$, $z_0 > -1$,

$$\begin{aligned} e^{-\sum_x(|\nabla\phi_x|^2+\nu|\phi_x|^2+g|\phi_x|^4)} &= e^{-\sum_x(|\nabla\varphi_x|^2+m^2|\varphi_x|^2)}e^{-\sum_x(z_0|\nabla\varphi_x|^2+\nu_0|\varphi_x|^2+g_0|\varphi_x|^4)} \\ &= e^{-(\bar{\varphi},A\varphi)}Z_0(\varphi) \end{aligned}$$

with $A = -\Delta + m^2$ and

$$\varphi = (1+z_0)^{-1/2}\phi, \qquad g_0 = g(1+z_0)^2, \qquad \nu_0 = (1+z_0)\nu - m^2.$$

The Gaussian measures with covariances $(-\Delta+m^2)^{-1}$ will indeed play a fundamental role, but $m^2$ and $z_0$ are not known yet.

## Generating functional. Translation of Gaussian measure.

Recall:

$$e^{-\sum_x (|\nabla\phi_x|^2+\nu|\phi_x|^2+(g|\phi_x|^4)} = e^{-(\bar\varphi, A\varphi)} Z_0(\varphi), \quad \varphi = (1+z_0)^{-1/2}\phi.$$

Study the generating functional (where $C = A^{-1} = (-\Delta + m^2)^{-1}$):

$$\begin{aligned}\Sigma(f) &= \int e^{(\bar f,\phi)+(f,\bar\phi)} e^{-(\bar\phi, A\phi)} Z_0(\phi) \\ &= e^{(f,Cf)} \int e^{-(\bar\phi,A\phi)} Z_0(\phi + Cf) = e^{(f,Cf)} \mathbb{E}_C Z_0(\phi + Cf)\end{aligned}$$

The susceptibility can be written in terms of $\Sigma$ as

$$\chi(g,\nu) = \sum_x \int \bar\phi_0 \phi_x e^{-\sum_x (|\nabla\phi_x|^2+\nu|\phi_x|^2+(g|\phi_x|^4)} = \frac{1+z_0}{|\Lambda|} D^2\Sigma(0;1,1),$$

where 1 is the constant test function $1_x = 1$ for all $x \in \Lambda$.

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Convolution:

$$Z_N(\phi) = \mathbb{E}_C\theta Z_0(\phi) := \mathbb{E}_C Z_0(\phi + \zeta)$$

Then:

$$\chi(g,\nu) = \frac{1+z_0}{|\Lambda|} D^2\Sigma(0;1,1) = \frac{1+z_0}{m^2} + \frac{1+z_0}{|\Lambda| m^4} D^2 Z_N(0;1,1)$$

where 1 is the constant test function $1_x = 1$ for all $x \in \Lambda$.

## Semigroup structure of Gaussian measures

Covariance $C = A^{-1} \longrightarrow P_C$: Gaussian measure with covariance $C$.

Semigroup property: $C + C' \longrightarrow P_C * P_{C'}$. (Remains true with Fermions.)

Equivalently: if $\phi \sim P_C$ and $\phi' \sim P_{C'}$ (independent) then $\phi + \phi' \sim P_{C+C'}$.

Finite range decomposition for $(-\Delta_{\Lambda_N} + m^2)^{-1}$ with $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$:

$$(-\Delta_{\Lambda_N} + m^2)^{-1} = C_1 + C_2 + \cdots + C_N$$

exists, with smooth dependence in $m^2 > 0$, and the following properties (B., 2013; also BGM 2003):

- $C_j$ is positive definite,
- $C_j$ is independent of $N$ if $j < N$,
- $[C_j]_{xy} = 0$ if $|x - y| > \frac{1}{2} L^j$,
- $|[\nabla^s C_j]_{xy}| \le O(L^{-(d-2)(j+s-1)})$.

(B., 2013) Construction using finite propagation speed of wave equation for general elliptic operators (continuum) and discrete elliptic operators.

# Wilson's renormalization group

As formal power series,

$$\mathbb{E}_{C_1}\theta e^{-\sum_x (z_0|\nabla\phi_x|^2+\nu_0|\phi_x|^2+(g_0|\phi_x|^4)} \approx \mathbb{E}_{C_1}\theta\left(1-\sum_x g_0|\phi_x|^4+\cdots\right) \qquad \text{(P)}$$

$$\approx e^{-\sum_x (z_1|\nabla\phi_x|^2+\nu_1|\phi_x|^2+(g_1|\phi_x|^4)+\text{remainder}}$$

where remainder consists of non-local and higher-order terms.

**Wilson's insight:** Terms that are sufficiently non-local or high-degree are contractive (irrelevant).

Example: In $d=4$, since $[C_j]_{xy}\approx L^{-(d-2)j}=L^{-2j}$, $\phi_x\approx \mathrm{Var}(\phi_x)^{\frac{1}{2}}\approx L^{-j}$,

$$\sum_{|x|\le L^j}|\phi_x|^p \approx L^{dj}L^{-\frac{d-2}{2}jp}=L^{-(p-4)j}\begin{cases}\text{irrelevant} & (p>4),\\ \text{marginal} & (p=4),\\ \text{relevant} & (p<4).\end{cases}$$

RHS of (P) is general local field polynomial respecting symmetries, consisting only of relevant and marginal monomials.

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Normalization: $\mu_j = L^{2j}\nu_j$.

**Dynamical systems picture:**

$$(g_j,\mu_j,z_j,K_j)\mapsto(g_{j+1},\mu_{j+1},z_{j+1},K_{j+1})$$

where $V_j=(g_j,\mu_j,z_j)$ is expanding or marginal and $K_j$ is contracting.

How can $K_j$ be controlled? How does this dynamical system behave?

# Remainder estimate

Kind of Taylor remainder formula with **local** structure:

$$Z_j(\phi) = \sum_{X \subset \Lambda} I_j(\Lambda \setminus X, \phi) K_j(X, \phi), \quad Z_N(\phi) = I_N(\Lambda, \phi) + K_N(\Lambda, \phi).$$

$I_j$ is second order in $V_j$ ($\sim$ perturbative analysis), $K_j$ is estimated by a complicated norm in which it is contractive and third-order.

**Locality properties:** $X$ are unions of blocks of side length $L^j$ and

$I_j(X)$ factors over blocks, $K_j(X)$ factors over connected components.

Such estimates were developed to study the two-point function by Brydges–Slade. We use this.

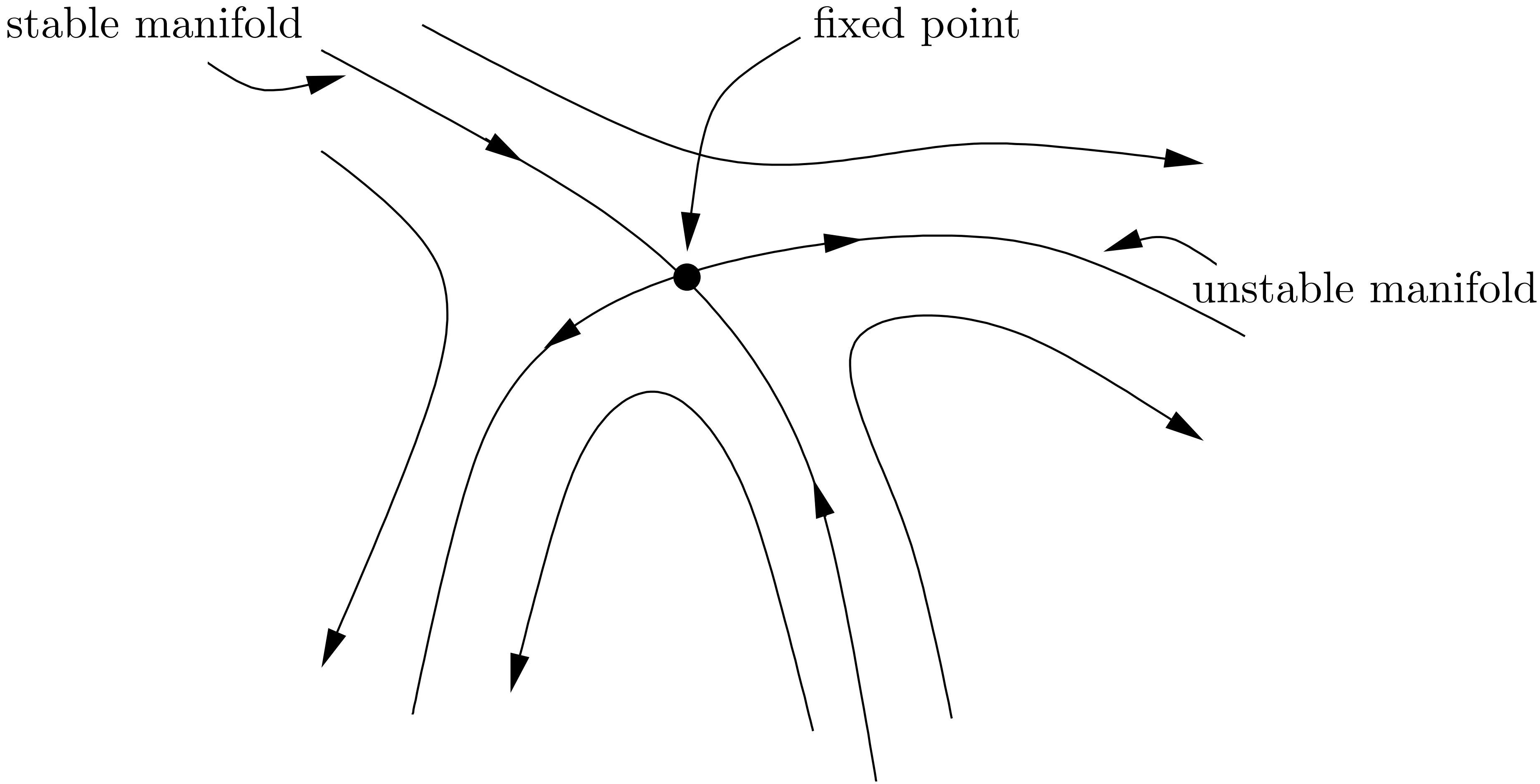
Such approximations understood in many problems but still challenging. Most relevant references: Dipole gas (Brydges–Yau), Hierarchical model (Brydges–Evans-Imbrie), Dipole Gas Park City Lectures (Brydges), Coulomb gas, quantum field theory, $\nabla\phi$ interface models, ...

## Phase portrait

Study the dynamical system: $(V_{j+1}, K_{j+1}) = \Phi_j^{m^2}(V_j, K_j)$, for each $m^2$.

Fixed point: $\Phi_j(0,0) = (0,0)$, free field or simple random walk.

Phase portrait of dynamical system near a hyperbolic fixed point:




Difficulty: Fixed point is not hyperbolic, but picture remains true.

## Massless and massive stable trajectories

Usual heuristics of renormalization group:

- critical point (massless) $\leftrightarrow$ stable manifold.
- subcritical points (massive) $\leftrightarrow$ unstable trajectories.

Our implementation is slightly different (a hybrid solution):

- To remain close to the fixed point (perturbative regime), we consider only stable trajectories in the infinite volume limit.

  To study the the subcritical case, rather than moving off the critical trajectory, we perturb the dynamical system to find new critical trajectories corresponding to massive free fields.

- On the other hand, changes of the stable manifold with the mass parameter are difficult to study directly (e.g., loss of regularity).

  Solution: in finite volume (thus finite trajectories), study infinitesimal change of trajectories (derivatives) in initial condition for fixed mass.

## The mass parameter

For any $m^2 > 0$, $z_0 > -1$, setting $g_0 = (1+z_0)^2 g$, $\nu_0 = (1+z_0)\nu - m^2$,

$$\chi(g,\nu) = (1+z_0) D^2 \Sigma(0;1,1) = \frac{1+z_0}{m^2} + \frac{1+z_0}{|\Lambda_N| m^4} D^2 Z_N(0;1,1).$$

where $Z_N(\phi) = \mathbb{E}_C \theta e^{-\sum_x (z_0 |\nabla \phi_x|^2 + \nu_0 |\phi_x|^2 + g_0 |\phi_x|^4)}$ and $C = (-\Delta_\Lambda + m^2)^{-1}$.

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**Stable manifold.** For $g_0 > 0$, $m^2 > 0$, there exist $(z_0, \nu_0) = (z_0^c, \nu_0^c)$ with

$$\lim_{N\to\infty} |\Lambda_N|^{-1} D^2 Z_N(0;1,1) = 0 \quad \text{(equality!)} \qquad (*)$$

**Derivative orthogonal to stable manifold.**

$$\lim_{N\to\infty} \frac{\partial}{\partial \nu_0} |\Lambda_N|^{-1} D^2 Z_N(0;1,1) \sim c(\log m^{-2})^{-\frac{1}{4}} \quad (m^2 \downarrow 0). \qquad (**)$$

Leads to:

$$\chi'(\nu) \sim c\chi(\nu)^2(-\log\chi(\nu))^{-\frac{1}{4}} \quad \text{as } \nu \downarrow \nu_c.$$

## Source of logarithmic correction

Dynamical system:

$$g_{j+1} = g_j - \beta_j g_j^2 + \cdots \qquad \text{[marginal]}$$

$$\mu_{j+1} = L^2 \left(1 - \frac{1}{4}\beta_j g_j\right) \mu_j + \cdots \qquad \text{[expanding]}$$

The coefficients are $m^2$-dependent and

$$\sum_{j=0}^{\infty} \beta_j = 8B_{m^2} \sim \frac{\log m^{-2}}{2\pi^2}, \quad g_\infty \sim \frac{1}{8B_{m^2}} \sim \frac{2\pi^2}{\log m^{-2}} \qquad (m^2 \downarrow 0)$$

$$1 - \frac{1}{4}\beta_j g_j = (1 - \beta_j g_j)^{\frac{1}{4}} (1 + O(g_j)) = \left(\frac{g_{j+1}}{g_j}\right)^{\frac{1}{4}} (1 + O(g_j)).$$

Thus:

$$\frac{\partial}{\partial \mu_0} \frac{D^2 Z_N(0;1,1)}{|\Lambda_N|} = \lim_{j\to\infty} L^{-2j} \frac{\partial \mu_j}{\partial \mu_0} \sim c \left(\frac{g_\infty}{g_0}\right)^{\frac{1}{4}} \sim c(\log m^{-2})^{-\frac{1}{4}} \qquad (m^2 \downarrow 0).$$