An application of the the Renormalization Group Method to the Navier-Stokes System

Ya.G. Sinai Princeton University

"A deep analysis of nature is the most fruitful source of mathematical discoveries" (Poincaré)

(Collaborators: C. Boldrighini, S. Frigio, D. Li, P. Maponi)

► We shall consider the simplest Navier-Stokes system for incompressible fluids in the *d*-dimensional space: u = (u₁, u₂, ··· , u_d)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \Delta u - \nabla p, \quad (1)$$

$$\nabla \cdot u = 0. \quad (2)$$

The viscosity is 1, p is the pressure, and

$$\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)_i=\frac{\partial u_i}{\partial t}+\sum_k\frac{\partial u_i}{\partial x_k}\cdot u_k,\ 1\leq i\leq d.$$

The equation $\nabla \cdot u = \text{div} u = 0$ is the incompressibility condition.

A big role in the whole theory is played by the energy

$$E(u)=\frac{1}{2}\int |u|^2 dx$$

Real-valued solutions satisfy the energy inequality

$$\int |u|^2(t_1,x)dx \leq \int |u|^2(t_2,x)dx, \quad t_1 > t_2.$$

We shall discuss also the Burgers system

$$\frac{\partial u_i}{\partial t} + \sum \frac{\partial u_i}{\partial x_k} \cdot u_k = \Delta u_i, \quad 1 \le i \le d \quad (3)$$

which does not contain the pressure and the incompressibility condition. For the purpose of this talk it does not contain external forcing. This means that we shall discuss mainly kinematic effects.

- It is well-known that (3) has simple solutions and in some sense is integrable if u = ∇φ. This representation is preserved in time. It will be shown that (3) also has interesting solutions of different type.
- Our main method of analysis is based on the Renormalization Group Method (RGM). Initially it appeared long ago in Quantum Field Theory and Statistical Physics. Then M. Feigenbaum applied RGM to some problems in dynamics and later it led to the appearance of a large new field in dynamics. Some names which can be mentioned on this occasion are P. Coullet, Ch. Tresser, B. Derrida, A. Gervois, Y. Pomeau, O. Lanford, K. Khanin.

Intro to RGM: a motivating example

Recall the Central Limit Theorem: X_n iid, $EX_n = 0$, $EX_n^2 = 1$, then

$$\frac{\sum_{i=1}^{n} X_{i}}{\sqrt{n}} \rightarrow \mathcal{N}(0,1).$$

Q: How to understand it from RGM point of view?

RGM point of view on Central Limit Theorem

- Consider iid X_n with (even) pdf p(x) and $\int x^2 p(x) = 1$.
- Define

$$\xi_m = \frac{X_1 + \dots + X_{2^m}}{2^{m/2}}$$

Then

$$\xi_{m+1} = \frac{\xi_m' + \xi_m''}{\sqrt{2}}.$$

• $p_m(x)$ =density of ξ_m , then

$$p_{m+1}(x) = \sqrt{2} \int_{-\infty}^{\infty} p_m(\sqrt{2}x - y) p_m(y) dy =: F(p_m)$$

• the goal: to study $p_m = F^m(p_1)$ as $m \to \infty$.

The RGM argument: Step 1: fixed pts

fixed points solve

$$q(x) = \sqrt{2} \int_{-\infty}^{\infty} q(\sqrt{2}x - y)q(y)dy$$

This equation is called the Gaussian integral equation
Solution given by Gaussian densities:

$$q_{\sigma}(x) = rac{1}{\sqrt{2\pi}\sigma}e^{-rac{x^2}{2\sigma^2}}.$$

RGM argument. Step 2: linearization and spectrum

The linearized map (Gaussian integral operator!)
L_σh(x) = 2√2 ∫_{-∞}[∞] h(√2x - y)q_σ(y)dy.
σ = 1, eigen-functions of L₁: He_m(x) 1/√2π e^{-x²/2}.
eigen-values λ_m = 1/(2^{m/2-1}), m is even (recall p(x) = p(-x)!)
λ₀ = 2, λ₂ = 1, λ_{2m} < 1.

Local version of CLT by using RGM

Thm: Let
$$p_1(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}(1+h(x))$$
 where

$$\int_{-\infty}^{\infty} h(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = 0, \quad \text{(kill unstable)}$$

$$\int_{-\infty}^{\infty} h(x)x^2\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = 0, \quad \text{(kill neutral)}$$
If $\|h\|_{L^2(e^{-x^2/2}dx)}$ is sufficiently small, then
 $F^m(p_1) \to \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ in the sense of L^2 .

General RG



- The number of parameters in $\mathcal{F}^{(0)}$ equals to $dim(\Gamma^{(u)})$.
- In the theory of dynamical systems, RG is used for the analysis of the number of periodic points of continuous maps (Feigenbaum), smoothness of the invariant measures in one-dimensinoal homeomorphisms and even as some replacement of KAM theory.

- In statistical physics RGM is used for analysis of critical points in systems such as 2D Ising model. Recently deep results were obtained by Fields Medalist S. Smirnov who studied conformal invariance of the probability distribution of the critical Ising model.
- There were many attempts to apply RGM to problems of fluid dynamics and turbulence theory like a series of papers by I. Moise and R. Temam and others. Scaling concepts appeared in ground-breaking works by Kolmogorov and Richardson.
- We shall use the RGM in the style of Feigenbaum. We consider the 3D case. However, other values of dimension are equally possible. The first step is to make Fourier transform. Then NSS takes the form

$$v(k,t) = e^{-t|k|^2} v(k,0) + \int_0^t e^{-(t-s)|k|^2} \int_{\mathbb{R}^d} \langle v(k-k',s),k \rangle P_k v(k',s) dk' ds.$$
(4)

In (4) P_k is the projection to the subspace orthogonal to k:

$$P_k \mathbf{v} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{k} \rangle \mathbf{k}}{\langle \mathbf{k}, \mathbf{k} \rangle}.$$

- We consider (4) in the subspace \langle v(k, t), k \rangle = 0 which is some form of the incompressibility condition.
- Real-valued solutions of (4) are complex-valued solutions of the initial Navier-Stokes system. Such solutions do not satisfy energy inequality and their analysis in many respects is simpler. On the other hand, presumably many properties of complex blow-ups will be valid in real cases (if they exist).
- The equation (4) can be studied in various spaces. The spaces which consist of functions

$$egin{aligned} \mathsf{v}(k,t) &= rac{\mathsf{c}(k,t)}{|k|^lpha}, \, 2 < lpha < 3, \ & \sup |\mathsf{c}(k,t)| < \infty \end{aligned}$$

are of some interest because in these spaces some scaling properties are valid.

There are many deep existence and uniqueness results here which we shall not discuss in detail. See a very good expository paper by M. Cannone ("Harmonic Analysis Tools for Solving the Incompressible Navier-Stokes equation. Handbook of Mathematical Fluid Dynamics, vol. 3, 2002") Instead we shall use an approach which is often used in the theory of dynamical systems. Namely, we consider one-dimensional families of initial conditions

$$v_{\mathcal{A}}(k,0) = \mathcal{A}v(k,0)$$

and analyze the dependence of solutions on A. Little thinking shows that it is convenient to write the unknown function $v_A(k, t)$ in the form

$$v_{A}(k,t) = e^{-t|k|^{2}} Av(k,0) + \int_{0}^{t} e^{-(t-s)|k|^{2}} \sum_{p} A^{p} g_{p}(k,s) ds. \quad (5)$$

For the coefficients g_p(k, s) we have the system of recurrent equations

$$g_{p}(k,s) = \int_{0}^{s} ds_{2} \int_{\mathbb{R}^{3}} \langle v(k-k',0),k \rangle P_{k}g_{p-1}(k',s_{2}) \\ \cdot e^{-s|k-k'|^{2}-(s-s_{2})|k'|^{2}} d^{3}k' \\ + \sum_{\substack{p_{1}+p_{2}=p\\p_{1},p_{2}>1}} \int_{0}^{s} ds_{1} \int_{0}^{s} ds_{2} \int_{\mathbb{R}^{3}} \langle g_{p_{1}}(k-k',s_{1}),k \rangle \\ P_{k}g_{p_{2}}(k',s_{2})e^{-(s-s_{1})|k-k'|^{2}-(s-s_{2})|k'|^{2}} d^{3}k' \\ + \int_{0}^{s} ds_{1} \int_{\mathbb{R}^{3}} \langle g_{p-1}(k-k',s_{1}),k \rangle P_{k}v(k',0) \\ e^{-(s-s_{1})|k-k'|^{2}-s|k'|^{2}} d^{3}k'.$$
(6)

which replaces the initial Navier-Stokes system.

- Under very simple assumptions the series (5) converges for sufficiently small t and gives a classical solution of (2).
- In the paper "Diagrammatic approach to solutions of the Navier-Stokes system" (Russian Math Surveys, vol 60, No. 5, 2005) a method was proposed which allowed to represent g_p in the form of some diagrams similar in some respect to diagrams in quantum field theory.
- The formula (6) resembles convolutions in probability theory. For example, if C = suppv(k, 0), then

$$suppg_k = \underbrace{C + C + \cdots + C}_{ktimes} = C_k.$$

If g_1 is concentrated on C, then g_k is concentrated on C_k .

• Choose some initial number $k^{(0)}$ and introduce the vector $K^{(r)} = (0, 0, rk^{(0)})$. Then write $k = K^{(r)} + \sqrt{rk^{(0)}}Y$. Thus instead of k we have the new variable Y which in a typical situation takes values O(1).

Propagation of support in k-space



- There is another change of variables connected with variables s₁, s₂. It was a big surprise for us to understand that in the integrals in (6) the main contribution comes from a small neighborhood of the point s. This is true for Navier-Stokes system and presumably is not true for the Euler system (viscosity is zero).
- In the main approximation, time drops out and only the basic nonlinearity remains.

► Put
$$s_j = s(1 - \frac{\theta_j}{p_j^2}), j = 1, 2$$
. Write
 $g_p(K^{(p)} + \sqrt{pk^{(0)}}Y, s) = \tilde{g}_p(Y, s) =$
 $= (pk^{(0)})^{\frac{5}{2}} \sum_{p_1+p_2=p} \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2 \cdot \frac{1}{p_1^2 p_2^2}$
 $\int_{\mathbb{R}^3} \langle g_{p_1}(\frac{Y - Y'}{\sqrt{\gamma}}, (1 - \frac{\theta_1}{p_1^2})s), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}} \rangle$.
 $P_{\kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}} \tilde{g}_{p_2}(\frac{Y'}{\sqrt{1 - \gamma}}, (1 - \frac{\theta_2}{p_2^2})s)$
 $e^{-\theta_1 |\kappa^{(0)} + \sqrt{k^{(0)}} \frac{Y - Y'}{\sqrt{p\gamma}}|^2 - \theta_2 |\kappa^{(0)} + \sqrt{k^{(0)}} \frac{Y'}{\sqrt{p(1 - \gamma)}}|^2}{\sqrt{p(1 - \gamma)}}|^2 d^3Y'.$
(7)

here $\gamma = \frac{p_1}{p}$, $1 - \gamma = \frac{p_2}{p}$, $\kappa^{(0,0)} = (0,0,1)$, $\kappa^{(0)} = (0,0,k^{(0)})$. It is very important that in front of (7) we have the factor $p^{\frac{5}{2}}$ and inside the sum the factor $\frac{1}{p_1^2 p_2^2}$. As p → ∞, the recurrent equation (7) converges to some limiting form which is the equation for the fixed point of RGM:

$$H(Y) = (H_{1}(Y_{1}, Y_{2}), H_{2}(Y_{1}, Y_{2}))$$

$$\frac{\sigma^{(1)}}{2\pi} e^{-\frac{\sigma^{(1)}}{2}|Y|^{2}} H(Y) =$$

$$\int_{0}^{1} d\gamma \int_{\mathbb{R}^{2}} \frac{\sigma^{(1)}}{2\pi\gamma} e^{-\frac{\sigma^{(1)}|Y-Y'|^{2}}{2\gamma}} \cdot \frac{\sigma^{(1)}}{2\pi(1-\gamma)} \cdot e^{-\frac{\sigma^{(1)}|Y'|^{2}}{2(1-\gamma)}}$$

$$\left[-(1-\gamma)^{\frac{3}{2}} (\frac{Y_{1}-Y_{1}'}{\sqrt{\gamma}} H_{1}(\frac{Y-Y'}{\sqrt{\gamma}}) + \frac{Y_{2}-Y_{2}'}{\sqrt{\gamma}} H_{2}(\frac{Y-Y'}{\sqrt{\gamma}})) + \frac{\gamma^{\frac{1}{2}}(1-\gamma)(\frac{Y_{1}'}{\sqrt{1-\gamma}} H_{1}(\frac{Y-Y'}{\sqrt{\gamma}})) + \frac{Y_{2}'}{\sqrt{1-\gamma}} H_{2}(\frac{Y-Y'}{\sqrt{\gamma}})) \right] H(\frac{Y'}{\sqrt{1-\gamma}}) d^{2}Y'. \quad (8)$$

The equation looks complicated but it can be studied in detail due to the system of recurrent equation which it implies

Thm: Let $\sigma^{(1)}, \sigma^{(2)} > 0$ and $h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}$ be sufficiently small. Then the equation for the fixed point has a solution

$$G(Y_{1}, Y_{2}, Y_{3}) = \frac{1}{2\pi\sigma^{(1)}} e^{-\frac{|Y_{1}|^{2} + |Y_{2}|^{2}}{2\sigma^{(1)}}} \cdot \frac{1}{\sqrt{2\pi\sigma^{(2)}}} e^{-\frac{|Y_{3}|^{2}}{2\sigma^{(2)}}} \cdot \frac{1}{\sqrt{\sigma^{(1)}}} H^{(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})} (\frac{1}{\sqrt{\sigma^{(1)}}} Y_{1}, \frac{1}{\sqrt{\sigma^{(2)}}} Y_{2})$$

where H is written as a series wrt Hermite polynomials

 It is possible to show that the coefficients of the series decay fast enough so it is converging (absolutely)

• In our situation, we take
$$\sigma^{(1)} = \sigma^{(2)} = 1$$
,
 $h_{12}^{(1)} = h_{21}^{(1)} = h_{30}^{(1)} = 0$, and
 $H(Y) = H(Y_1, Y_2) = (-2Y_1, -2Y_2)$.

The next step in RGM is to study the spectrum of the linearized transformation near the fixed point. This can be done in our case and here is the result:

- 1. there is a four-dimensional subspace generated by four unstable eigen-vectors;
- 2. there is a six-dimensional subspace generated by six neutral eigen-values (i.e. the corresponding eigen-value is 1)
- 3. there is a linear subspace of co-dimension 10 where the spectrum is stable.



Main Theorem. Let the moment of blowup t_{cr} be chosen. Then there exists a 10-parameter local submanifold of initial conditions such that for some point from this submanifold the solution develops blowup at t_{cr} .

Geometric meaning: Initial conditions are concentrated in a small ball away from the origin. Under the action of nonlinear terms the modes get concentrated along the line generated by the initial conditions and decay there in a power-like manner.

This construction in particular gives some hints which can be interpreted as giving the frequency of blowups.

Corollary: Let E(t) be the energy of solution at the moment t (before t_{cr}), then

$$E(t) \sim \frac{C}{(t_{cr}-t)^5}.$$
 (9)

- This asymptotics is the same for all cases treated by our method and it does not depend on the dimension of the system. This is connected with the fact that in the critical regime all modes become concentrated along a single direction.
- The singularity in the x-space is concentrated near the origin in the x-space and dependence in time becomes very complicated. In this sense the solution resembles a tornado and can be called kinematic tornado because the temperature is not involved.

Numerical results

C. Boldrighini (Universita di Roma), S. Frigio and P. Maponi (Universita di Camerino) studied numerically the 2-dimensional Burgers system: u = (u₁, u₂)

$$\frac{\partial u}{\partial t} + \sum u_k \frac{\partial u}{\partial x_k} = \Delta u.$$

- This system is integrable on the space of gradient-like solutions (due to Hopf-Cole substitution). For real-valued solutions O.
 Ladyzenskaya proved global existence and uniqueness result. It turns out that complex-valued solutions differ drastically from real-valued solutions.
- In the paper by BFM the authors found complex-valued initial conditions for which the solutions develop singularity in finite time in complete agreement with the theory presented above. In particular in their solutions the energy grows according to (9). More recent results were obtained for complex-valued 3-dim Navier-Stokes system

083101-9 Boldrighini, Frigio, and Maponi

J. Math. Phys. 53, 083101 (2012)



FIG. 2. Case 1, $\delta_t = 2^{-8} \times 10^{-4}$, $\delta_k = 1$, implicit integration method. Plot of the energy density $e(k, t) = \frac{|v(k,t)|^2}{2}$ at times 12.00 × 10⁻⁴, 12.02 × 10⁻⁴, 12.03 × 10⁻⁴.



Figure 1: Initial energy E(0) = 200. T_0 is the estimated critical time.



Figure 6: Plot of the log of the marginal energy density in **k**-space vs k_z . Initial energy E(0) = 1. Initial bumps at distance L = 20, radius $\sigma = 17$. Time $T = 40 \times \delta_t, \delta_t = 10^{-5}$.

Why the theory of real-valued solutions is more difficult

- In all cases of complex-valued solutions initial conditions were concentrated in small domains away from the origin.
- For real-valued solutions we need conditions which are odd or even functions in the Fourier space. There can be two types of behavior of solutions:
 - 1. the modes get concentrated at infinity and this leads to finite-time singularities;
 - 2. the modes remain concentrated in a compact part of the space and in this case there are no singularities.
- We have no possibility to decide whether the first case is possible.

Bifurcation of viscous vortices

- Dong Li and I studied bifurcations in equations of fluid dynamics. We proposed a new approach to the whole set of problems here. So far it works mainly in the 2-dim case and its extension to the multi-dimensional situation is a very interesting problem.
- Let us write NSS for the stream function $\psi(t, x, y)$ which is connected with the velocity by the relation $(u_1, u_2) = (-\partial_y \psi, \partial_x \psi) = \nabla^{\perp} \psi,$ $\frac{\partial \psi}{\partial t} + \Delta^{-1} \left(\frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Delta \psi}{\partial x} \right) = \Delta \psi$

For simplicity assume periodic boundary conditions.

Let the initial condition \u03c6(0, x, y) be a smooth function on \u03c7² and is a Morse function on \u03c7² having two critical points: the point of maximum and the point of minimum. In the non-degenerate case level sets near the critical points are closed curves. Since the velocity vector u is tangent to the level set, it is natural to call critical points as (local) viscous vortices. This notion can be used even in the linear cases. The fluid trajectories resemble in some sense hurricanes.



- In our joint paper with Dong Li we proved the following theorems.
- Thm 1(Splitting of vortices): There exists an open set A_1 in the space of stream functions such that for any $\psi_0 \in A_1$ there is an open neighborhood U of the origin, two moments of time $0 < t_1 < t_2$ so that
 - 1. for any $0 \le t < t_1$, ψ has only one critical point in U;
 - 2. for $t = t_1$, the stream function ψ has two critical points in U;
 - 3. for $t_1 < t < t_2$, the stream function ψ has three critical points in U
- Simpler situation:
 - 1 nondegenerate critical point
 - \rightarrow 1 degenerate critical point
 - \rightarrow 3 critical points

Hump of a young camel



- Thm2(Merging of vortices): There exists an open set A_2 in the space of stream functions such that for any $\psi_0 \in A_1$ there is an open neighborhood U of the origin, two moments of time $0 < t_1 < t_2$ so that
 - 1. for any $0 \le t < t_1$, ψ has three critical points in U;
 - 2. for $t = t_1$, the stream function ψ has two critical points in U;
 - 3. for $t_1 < t < t_2$, the stream function ψ has only one critical point in U

The reversed picture (merging)



Contrary to the usual bifurcation theory, deformations of solutions (versal deformations) are produced by solution of the NSS system. This process is also interesting for linear systems.