Universality of the local regime for some types of random band matrices

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Two models:

We consider the following two models:

- Random band matrices

$\{H_n\}$ are Gaussian random matrices whose entries are numerated by indices $i, j \in \Lambda = [1, n]^d \cap \mathbb{Z}^d$ and:

$$\langle H_{jk} \rangle = 0, \quad \langle H_{ij} H_{lk} \rangle = \delta_{ik} \delta_{jl} J_{ij}, \quad J_{ij} = \left(-W^2 \Delta + 1\right)_{ij}^{-1},$$

where Δ is the discrete Laplacian with Neumann boundary conditions on $[1, n]^d$.

For 1D RBM the variance of the matrix elements is exponentially small when $|i - j| \gg W$. Hence W can be considered as the width of the band.

- Block band matrices

  Assign to every site $j \in \Lambda$ one copy $K_j \simeq \mathbb{C}^W$ of an W-dimensional complex vector space, and set $K = \oplus K_j \simeq \mathbb{C}^{|\Lambda|W}$. From the physical point of view, we are assigning W valence electron orbitals to every atom of a solid with hypercubic lattice structure.

  We start from the matrices $M : K \to K$ belonging to GUE, and then multiply the variances of all matrix elements of H connecting K_j and K_k by the positive number J_{jk}, $j, k \in \Lambda$ (which means that H becomes the matrix constructed of $W \times W$ blocks, and the variance in each block is constant).

  Such models were first introduced and studied by Wegner.

  Note that $P_N(dH_N)$ is invariant under conjugation $H_N \to U^* H_N U$ by $U \in \mathcal{U}$, where $\mathcal{U}$ is the direct product of all the groups of unitary transformations in the subspaces K_j. This means that the probability distribution $P_N(dH_N)$ has a local gauge invariance.

We consider

$$J = 1/W + \alpha\Delta/W, \quad \alpha < 1/4d.$$

This model is one of the possible realizations of the Gaussian random band matrices, for example for $d = 1$ they correspond to the band matrices with the width of the band $2W + 1$.

Density of states for both models:

Denote by $\lambda_1, \dots, \lambda_N$ the eigenvalues of the random matrix H_N.

NCM and the density of states:

$$\mathcal{N}_N[\Delta] = N^{-1}\sharp\{\lambda_i \in \Delta\} \to \mathcal{N}(\Delta), \quad \rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \quad \lambda \in [-2,2].$$

The key physical parameter of RBM models is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length ℓ is comparable with the matrix size, and it is called localized otherwise.

Conjecture in 1d case (Casati, Molinari, Israilev, Mirlin, Fyodorov, 1990-1991):

$\ell \sim W^2$, which means that varying W we can see the crossover: for $W \gg \sqrt{N}$ the eigenvectors are expected to be delocalized, and for $W \ll \sqrt{N}$ they are localized.

In terms of eigenvalues this means that the local eigenvalue statistics in the bulk of the spectrum changes from Poisson, for $W \ll \sqrt{N}$, to GUE for $W \gg \sqrt{N}$.

In 2d case the localization length is expected to be exponentially growing in W, and for $d \geq 3$, it is macroscopic, i.e. the system is delocalized.

At the present time only some upper and lower bounds are proved rigorously.

- Schenker (2009) $\ell \leq W^8$;
- Erdös, Knowles (2010): $\ell \gg W^{7/6}$;
- Erdös, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$.

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory.

The main objects of the local regime are k-point correlation functions R_k $(k = 1, 2, \ldots)$, which can be defined by the equalities:

$$E\left\{\sum_{j_1\neq\ldots\neq j_k}\varphi_k(\lambda_{j_1}^{(N)},\ldots,\lambda_{j_k}^{(N)})\right\}$$
$$=\int_{\mathbb{R}^k}\varphi_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})R_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})d\lambda_1^{(N)}\ldots d\lambda_k^{(N)},$$

where $\varphi_k : \mathbb{R}^k \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments.

Universality conjecture in the bulk of the spectrum (hermitian case, deloc.eg.s.) (Dyson):

$$(n\rho(\lambda_0))^{-k}R_k(\{\lambda_0 + \xi_j/n\rho(\lambda_0)\}) \stackrel{n\to\infty}{\longrightarrow} \det\left\{\frac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^{k}.$$

Grassmann integration:

Anticommutation relations:

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0, \quad j,k = 1,\dots,n.$$

These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra $\mathfrak{A}$. We can also define functions of the Grassmann variables. Let χ be an element of $\mathfrak{A}$, i.e.

$$\chi = a + \sum_{j=1}^n (a_j\psi_j + b_j\overline{\psi}_j) + \sum_{j\neq k} (a_{j,k}\psi_j\psi_k + b_{j,k}\psi_j\overline{\psi}_k + c_{j,k}\overline{\psi}_j\overline{\psi}_k) + \dots .$$

For any analytical function f we mean by $f(\chi)$ the element of $\mathfrak{A}$ obtained by substituting $\chi - a$ in the Taylor series of f at the point a.

Examples:

$$\exp\{\mathrm{b}\,\overline{\psi}\psi\} = 1 + \mathrm{b}\,\overline{\psi}\psi + (\mathrm{b}\,\overline{\psi}\psi)^2/2 + \ldots = 1 + \mathrm{b}\,\overline{\psi}\psi,$$

$$\begin{aligned}&\exp\{\mathrm{a}_{11}\overline{\psi}_1\psi_1 + \mathrm{a}_{12}\overline{\psi}_1\psi_2 + \mathrm{a}_{21}\overline{\psi}_2\psi_1 + \mathrm{a}_{22}\overline{\psi}_2\psi_2\}\\ &= 1 + \mathrm{a}_{11}\overline{\psi}_1\psi_1 + \mathrm{a}_{12}\overline{\psi}_1\psi_2 + \mathrm{a}_{21}\overline{\psi}_2\psi_1\\ &+ \mathrm{a}_{22}\overline{\psi}_2\psi_2 + (\mathrm{a}_{11}\mathrm{a}_{22} - \mathrm{a}_{12}\mathrm{a}_{21})\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2.\end{aligned}$$

Integral (Berezin):

$$\int d\,\psi_j = \int d\,\overline{\psi}_j = 0, \quad \int \psi_j d\,\psi_j = \int \overline{\psi}_j d\,\overline{\psi}_j = 1.$$

Thus, if

$$f(\psi_1, \dots, \psi_k) = p_0 + \sum_{j_1=1}^{k} p_{j_1}\psi_{j_1} + \sum_{j_1<j_2} p_{j_1 j_2}\psi_{j_1}\psi_{j_2} + \dots + p_{1,2,\dots,k}\psi_1 \dots \psi_k,$$

then

$$\int f(\psi_1, \dots, \psi_k) d\,\psi_k \dots d\,\psi_1 = p_{1,2,\dots,k}.$$

Gaussian integration in Grassmann variables:

$$\int \exp\Big\{-\sum_{j,k=1}^{n} A_{jk}\overline{\psi}_j\psi_k\Big\} \prod_{j=1}^{n} d\,\overline{\psi}_j d\,\psi_j = \det A. \tag{1}$$

Let

$$\mathrm{F} = \begin{pmatrix} \mathrm{a} & \sigma \\ \rho & \mathrm{b} \end{pmatrix},$$

where a and b > 0 are Hermitian complex $\mathrm{k} \times \mathrm{k}$ matrices and σ, ρ are $\mathrm{k} \times \mathrm{k}$ matrix of the anticommuting Grassmann variables, and let

$$\Phi = (\psi_1, \dots, \psi_\mathrm{k}, \mathrm{z}_1, \dots, \mathrm{z}_\mathrm{k})^\mathrm{t},$$

where $\{\psi_\mathrm{j}\}_{\mathrm{j}=1}^\mathrm{k}$ are independent Grassmann variables and $\{\mathrm{z}_\mathrm{j}\}_{\mathrm{j}=1}^\mathrm{k}$ are complex variables. Combining (1) with usual Gaussian integration we obtain

$$\int \exp\{-\Phi^+ \mathrm{F} \Phi\} \prod_{\mathrm{j}=1}^{\mathrm{k}} d\overline{\psi}_\mathrm{j}\, d\psi_\mathrm{j} \prod_{\mathrm{j}=1}^{\mathrm{k}} \frac{\mathrm{Re}\, \mathrm{z}_\mathrm{j}\, \mathrm{Im}\, \mathrm{z}_\mathrm{j}}{\pi} = \mathrm{sdet}\, \mathrm{F},$$

where

$$\mathrm{sdet}\, \mathrm{F} = \frac{\det (\mathrm{a} - \sigma\, \mathrm{b}^{-1}\, \rho)}{\det \mathrm{b}}.$$

To prove universality, we want to obtain the control of $\mathrm{E}\,|g_N(\lambda_0 + i\varepsilon)|^2$ for $\varepsilon \sim N^{-1}$, where $g_N(z) = N^{-1}\mathrm{Tr}\,(H_N - z)^{-1}$.

This means that we have to control

$$\mathrm{E}\left\{\frac{\partial^2}{\partial x \partial y}\frac{\det(H_N - \lambda_0 - i\varepsilon - x/N)\cdot\det(H_N - \lambda_0 + i\varepsilon - y/N)}{\det(H_N - \lambda_0 - i\varepsilon)\cdot\det(H_N - \lambda_0 + i\varepsilon)}\Bigg|_{x=y=0}\right\}.$$

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The second correlation function of the characteristic polynomials:

$$F_2(\Lambda) = \int \det(\lambda_1 - H_n) \det(\lambda_2 - H_n) P_n(d\, H_n),$$

where $\Lambda = \mathrm{diag}\,\{\lambda_1, \lambda_2\}$.

GUE

Asymptotic behavior of the 2k-point mixed moment for GUE:

$$(n\rho(\lambda_0))^{-k^2} F_{2k}\left(\Lambda_0 + \hat{\xi}/(n\rho(\lambda_0))\right)$$

$$= C_n \frac{\det\left\{\dfrac{\sin\pi(\xi_i - \xi_{j+k})}{\pi(\xi_i - \xi_{j+k})}\right\}_{i,j=1}^{k}}{\triangle(\xi_1,\ldots,\xi_k)\triangle(\xi_{k+1},\ldots,\xi_{2k})}(1+o(1)),$$

where

$$\triangle(\xi_1,\ldots,\xi_k) = \prod_{j<l}(\xi_j - \xi_l).$$

- Similar result for the hermitian β-invariant ensemble ($\beta = 2$) was proved by Brezin, Hikami (2000), Fyodorov, Strahov (2003).
- The same is valid for hermitian Wigner and general sample covariance matrices (Gotze, Kosters (2009-2010) for $k = 1$, TS for any k (2010-2011)).

GOE

In the case of real symmetric Gaussian matrices the behavior was found by Brezin, Hikami (2000) for $k = 1$ and by Borodin, Strahov (2006) for any k.

Asymptotic behavior of the 2k-point mixed moment for GOE:

$$(n\rho(\lambda_0))^{-k^2} F_{2k}\left(\Lambda_0 + \hat{\xi}/(n\rho(\lambda_0))\right)$$
$$= C_n \frac{\mathrm{Pf}\left\{-\dfrac{1}{\pi^2}\dfrac{d}{d\xi_i}\dfrac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^{2k}}{\triangle(\xi_1,\ldots,\xi_{2k})}(1+o(1)),$$

where $\triangle(\xi_1,\ldots,\xi_{2k})$ is the Vandermonde determinant of $\xi_1,\ldots,\xi_{2k}$. The same is valid for $k = 1$ for hermitian Wigner and general sample covariance matrices (Kosters (2009-2010)).

Results for the characteristic polynomials:

Let $D_2 = F_2(\lambda_0, \lambda_0)$.

Theorem (hermitian case)

For 1D Gaussian hermitian random band matrices with $W^2 = n^{1+\theta}$, $0 < \theta < 1$ we have

$$\lim_{n\to\infty} D_2^{-1} F_2\left(\lambda_0 + \frac{\xi}{2N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{2N\rho(\lambda_0)}\right) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

Theorem (real symmetric case)

For 1D Gaussian real symmetric random band matrices with $W^2 = n^{1+\theta}$, $0 < \theta < 1$ we have

$$\lim_{n\to\infty} D_2^{-1} F_2\left(\lambda_0 + \frac{\xi}{2N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{2N\rho(\lambda_0)}\right) = \frac{\sin(\pi\xi)}{\pi^3\xi^3} - \frac{\cos(\pi\xi)}{\pi^2\xi^2}.$$

Integral representation

$$F_2(\hat{\xi}) = C_N \int\limits_{\mathcal{H}_2^N} \exp\Big\{ -\frac{W^2}{2}\sum_{j=2}^{n} \mathrm{Tr}\,(X_j - X_{j-1})^2\Big\}\times$$

$$\exp\Big\{ -\frac{1}{2}\sum_{j=1}^{n} \mathrm{Tr}\,\Big(X_j + \frac{i\Lambda_0}{2} + \frac{i\hat{\xi}}{2N\rho(\lambda_0)}\Big)^2\Big\}\prod_{j=1}^{n} \det\big(X_j - i\Lambda_0/2\big)d\overline{X},$$

where $\{X_j\}$ are hermitian 2×2 matrices and $\hat{\xi} = \xi\sigma_3$.

Integral representation

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where $\{X_j\}$ are hermitian 2×2 matrices and $\hat\xi = \xi\sigma_3$.
Let us change the variables to $X_j = U_j^* A_j U_j$, where U_j is a unitary matrix and $A_j = \mathrm{diag}\,\{a_j, b_j\}$, $j = 1, \dots, n$. Then dX_j becomes

$$\frac{\pi}{2}(a_j - b_j)^2 da_j\, db_j d\mu(U_j),$$

where $d\mu(U_j)$ is the Haar measure on the unitary group $U(2)$.

σ-model approximation

The expected saddle-points: $a_\pm = \pm\sqrt{1-\lambda_0^2/4} = \pm\pi\rho(\lambda_0)$.
Fix $a_j = a_+$, $b_j = a_-$ for each j. Then $X_j = U_j^* A_j U_j = \pi\rho(\lambda_0)U_j^*\sigma_3 U_j$, and the integral representation transforms to

σ-model:

$$\int \exp\left\{\pi^2\rho(\lambda_0)^2 W^2 \sum_{j=2}^{n}(S_j S_{j-1}-1) + \frac{i\pi\xi}{2N}\sum_{j=1}^{n} S_j\sigma_3\right\}\prod_{j=1}^{n} dS_j,$$

where $S_j = U_j^*\sigma_3 U_j$. In σ-model the result states that for $W^2 \gg N$

$$Z_n^{-1}\int e^{\pi^2\rho(\lambda_0)^2 W^2 \sum\limits_{j=2}^{n}(S_j S_{j-1}-1)+\frac{i\pi\xi}{2N}\sum\limits_{j=1}^{n} S_j\sigma_3}\prod_{j=1}^{n} dS_j \longrightarrow$$

$$\longrightarrow \int e^{i\pi\xi S_0\sigma_3/2}dS_0 = \frac{\sin(\pi\xi)}{\pi\xi}.$$

Result for the full model:

Theorem

Let $\Lambda = [1, n]^d \cap \mathbb{Z}^d$ be a periodic box, H_N be the BBM, $N = W|\Lambda|$, and let the number of sites $|\Lambda|$ be fixed. Then

$$(N\rho(\lambda_0))^{-2} R_2 \left(\lambda_0 + \frac{\xi_1}{\rho(\lambda_0)\, N}, \lambda_0 + \frac{\xi_2}{\rho(\lambda_0)\, N} \right) \xrightarrow{w} 1 - \frac{\sin^2(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2},$$

as $W \to \infty$, for any $|\lambda_0| < \sqrt{2}$.

The condition $|\Lambda| < \infty$ is not necessary here. For example, for n which grows like the small power of W the proof can be repeated almost literally. Moreover, for $d = 1$ the same method is expected to work for $|\Lambda| \ll W^{1/2}$, i.e. $W \gg N^{2/3}$, but this requires more delicate techniques.

Integral representation

$$C_N \int \exp\Big\{ -\frac{\alpha W}{2} \sum_{j\sim j'} \mathrm{str}\,(F_j - F_{j'})^2 \Big\} \times$$

$$\exp\Big\{ \frac{W}{2} \sum_{j\in\Lambda} \mathrm{str}\,\Big(F_j + \frac{i\Lambda_0}{2} + \frac{i\widehat{\xi}}{N\rho(\lambda_0)}\Big)^2 \Big\} \prod_{j\in\Lambda} \mathrm{sdet}^{-W}(F_j - i\Lambda_0/2)\, d\overline{F},$$

where

$$F_j = \begin{pmatrix} U_j & \rho_j \\ \tau_j & B_j \end{pmatrix},$$

U_j is 2×2 unitary matrix, B_j is 2×2 positive Hermitian matrix times σ_3, and ρ, τ are 2×2 matrices whose entries are independent Grassmann variables.

Here

$$\mathrm{sdet}\, F_j = \frac{\det\,(U_j - \tau_j\, B_j^{-1}\, \rho_j)}{\det\; B_j}, \quad \mathrm{str}\, F_j = \mathrm{Tr}\, U_j - \mathrm{Tr}\, B_j.$$

$$
\begin{aligned}
& C_N \int \exp\Big\{ -\frac{\alpha W}{2}(\nabla U)^2 + \frac{W}{2}\sum_{j\in\Lambda} \mathrm{Tr}\,\Big(U_j + \frac{i\Lambda_0}{2} + \frac{i\hat{\xi}_1}{N\rho(\lambda_0)}\Big)^2\Big\} \\
& \times \exp\Big\{ \frac{\alpha W}{2}(\nabla B)^2 - \frac{W}{2}\sum_{j\in\Lambda} \mathrm{Tr}\,\Big(B_j + \frac{i\Lambda_0}{2} + \frac{i\hat{\xi}_2}{N\rho(\lambda_0)}\Big)^2\Big\} \\
& \times \Big(\det(\alpha\triangle + I + \{U_j^{-1} \otimes B_j^{-1}\}) + \text{other terms}\Big)
\end{aligned}
$$