Semitoric families

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Rutgers University

joint with Y. Le Floch
(work with J. Alonso, S. Hohloch, D.M. Kane, and Á. Pelayo will also be mentioned)

IAS/Princeton Symplectic geometry and dynamics seminar

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Symplectic manifolds and integrable systems

- Let (M, ω) be a symplectic manifold and $f \colon M \to \mathbb{R}$.
- Denote by $\mathcal{X}_f$ the *Hamiltonian vector field of* f, which satisfies

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- *Fixed point* or *rank zero point* is $p \in M$ such that $dF(p) = 0$.

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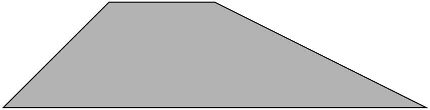
- $F \colon M \to \mathbb{R}^n$.
- Atiyah, Guillemin-Sternberg (1982) showed $F(M)$ is the convex hull of the images of the fixed points.

Toric integrable systems: the classification



Theorem (Delzant, 1988)

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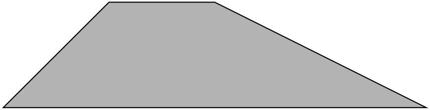

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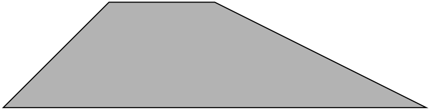

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- $\{\text{toric systems}\} \overset{1-1}{\longleftrightarrow} \{\text{Delzant polytopes}\}$.
- System can be recovered by symplectic reduction on $\mathbb{C}^d$.

Semitoric integrable systems: definition

Definition (Vũ Ngọc, 2007)

A *semitoric integrable system* is a triple $(M, \omega, F = (J, H))$ where (M, ω) is a 4-dimensional symplectic manifold and

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- *non-degenerate*, analogue of Morse theory
- *Simple* = at most one focus-focus point in each level set of J.

Semitoric integrable systems: fibers

Points in semitoric systems:

- regular points;
- rank one: elliptic-regular points;
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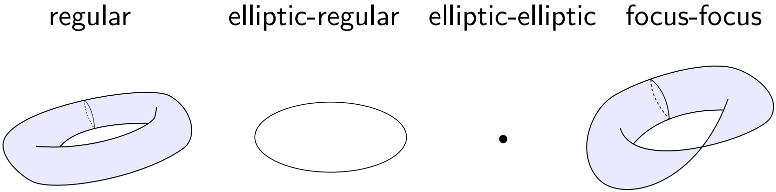

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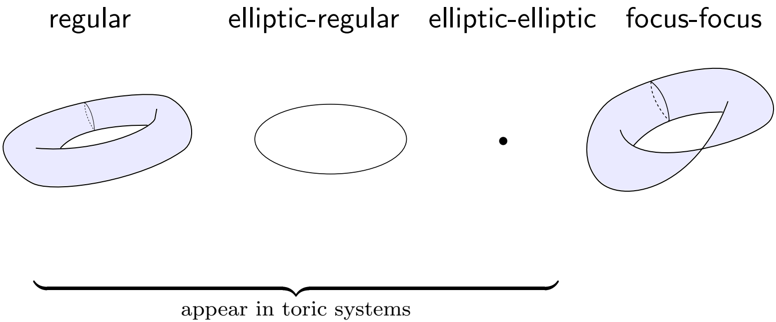

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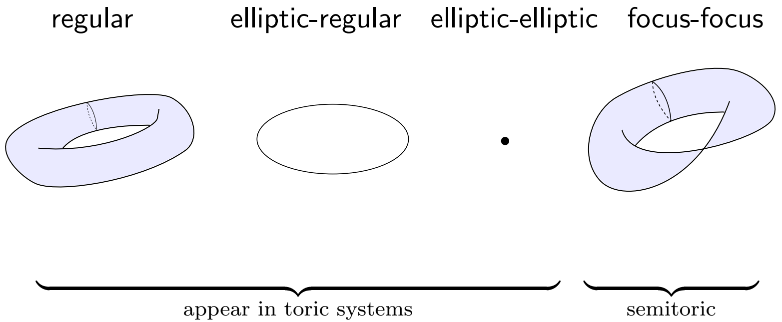
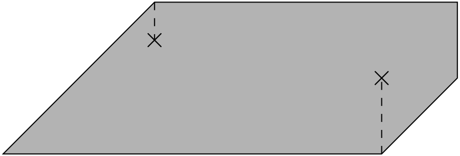


Semitoric integrable systems: the “complete invariant”

- Associated to each semitoric integrable system is a polygon with marked interior points.



Semitoric integrable systems: the "complete invariant"

- Associated to each semitoric integrable system is a polygon with marked interior points.

- Each interior point is labeled with an integer and a Taylor series in two variables.

Semitoric integrable systems: classification

The five invariants:

(1) the number of focus-focus points invariant;

(2) the semitoric polygon invariant;

(3) the height invariant;

(4) the Taylor series invariant;

(5) the twisting index invariant;

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Theorem (Pelayo-Vũ Ngọc classification (2009, 2011))

1. *Two simple semitoric systems are isomorphic if and only if they have the same invariants (1)-(5);*
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Construction in toric and semitoric cases

- Toric integrable systems can be recovered from the polytope by performing specific reductions on $\mathbb{C}^d$.
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Goal

Given specified semitoric polygon invariant try to find a simple system with that invariant (forgetting about the other invariants).

Semitoric invariants: 1. Number of focus-focus points

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- Their number is the first invariant.

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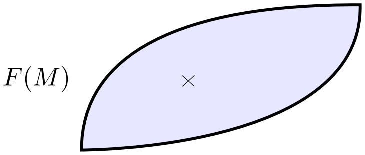

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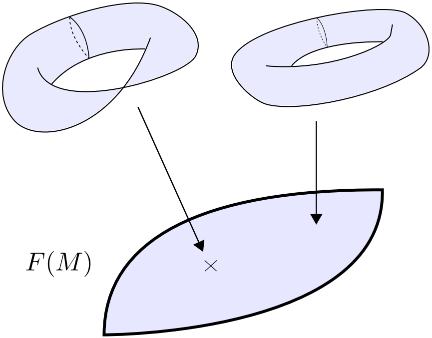

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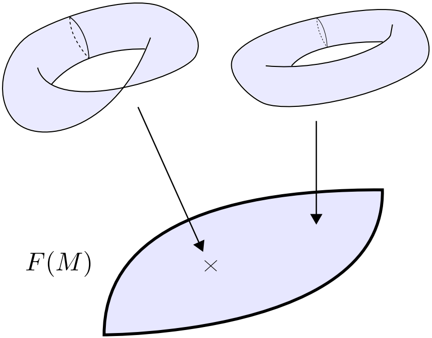

- Torus fibration $\to$ integral affine structure on $F(M) \subset \mathbb{R}^2$.
  - NOT equal to integral affine structure inherited from $\mathbb{R}^2$.

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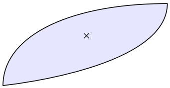

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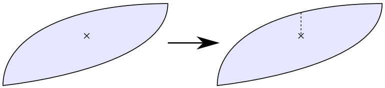

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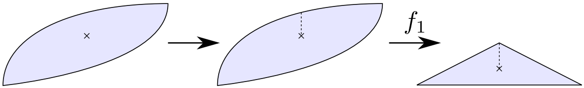

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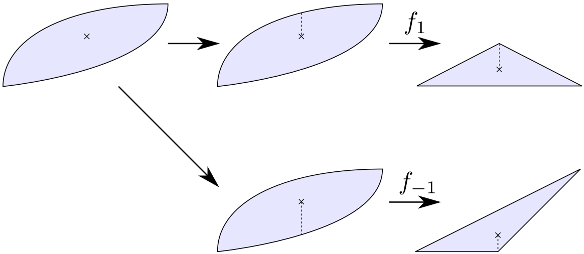

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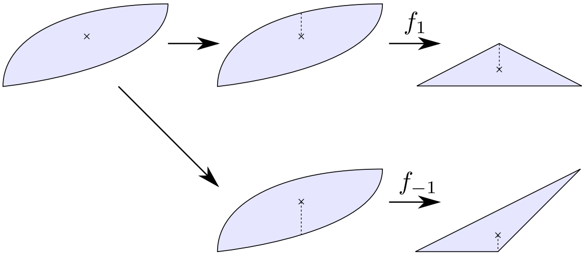

- *Semitoric polygon invariant*: Family of polygons.

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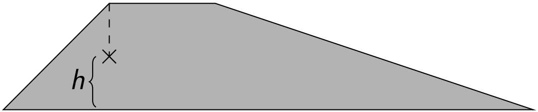

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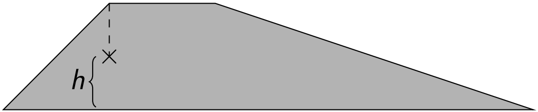

- Note: $f_\varepsilon \circ F$ is a toric momentum map away from the cuts.

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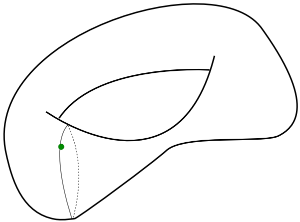

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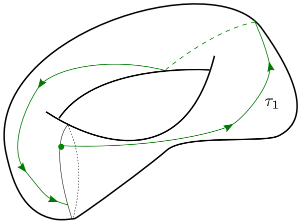

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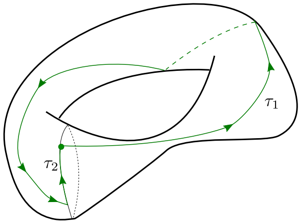

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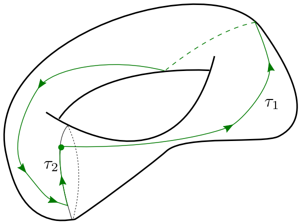

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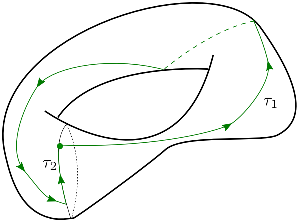

- Use τ_1 and τ_2 to specify a Taylor series in two variables.
- **Notice:** this construction only sees where the trajectory "lands" - it can't detect a Dehn twist.

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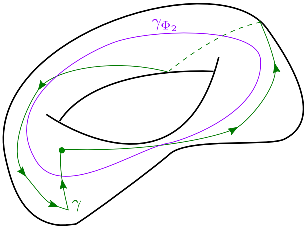

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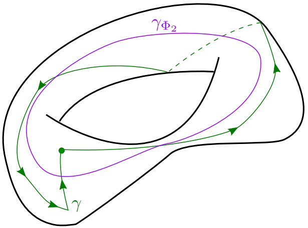

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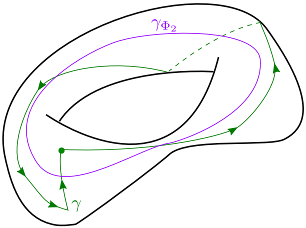

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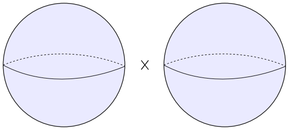
(related to upcoming work with J. Alonso and S. Hohloch, k originally defined by Pelayo-Vũ Ngọc via $\Phi = T^k\nu$)

Example: Coupled angular momenta

[Sadovskií and Zĥilinskií, 1999]



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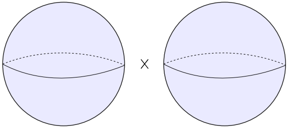

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$$\begin{cases} J = R_1 z_1 + R_2 z_2 \\ H_t = (1-t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2) \end{cases}$$

for $t \in [0, 1]$ and $R_1 < R_2$.

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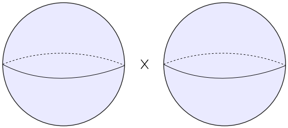

- $M = \mathbb{S}^2 \times \mathbb{S}^2, \quad \omega = R_1\omega_1 \oplus R_2\omega_2$
- coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$

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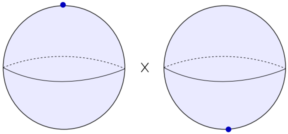

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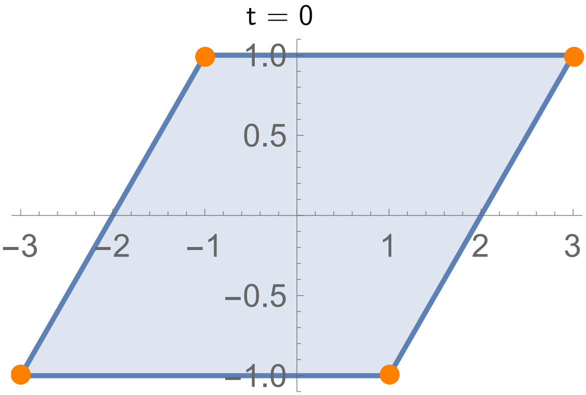
Coupled angular momenta: moment map image



Semitoric with zero focus-focus points
(figure made in Mathematica)

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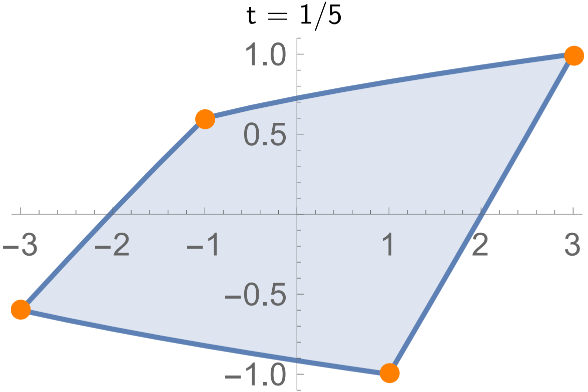

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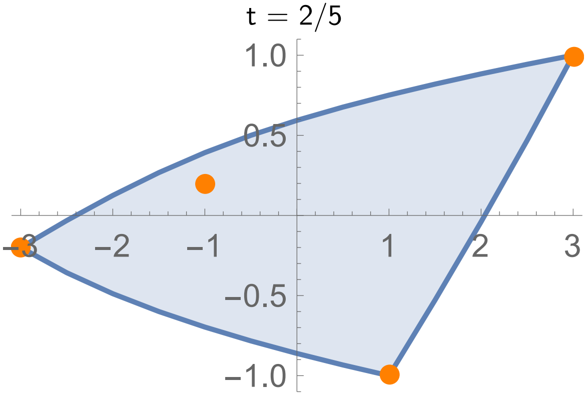

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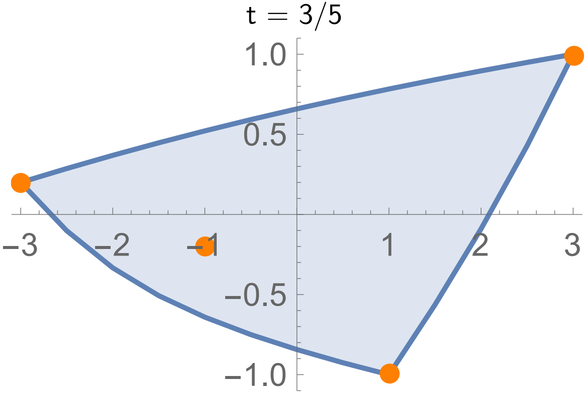

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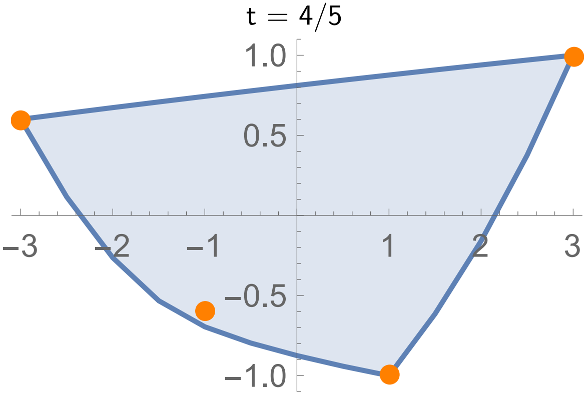

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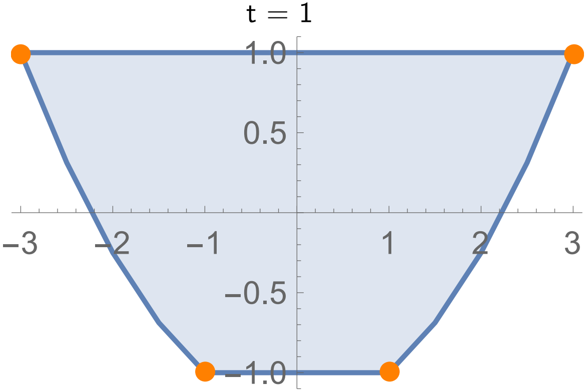

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Coupled angular momenta: semitoric polygon

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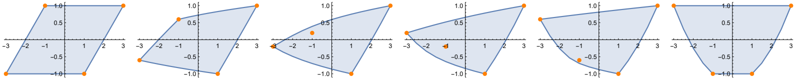

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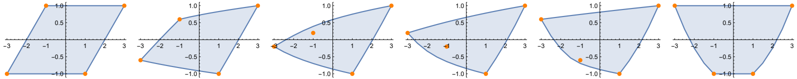

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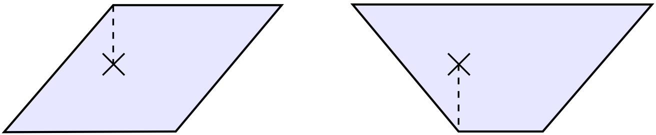

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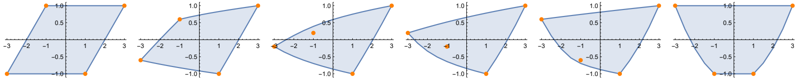

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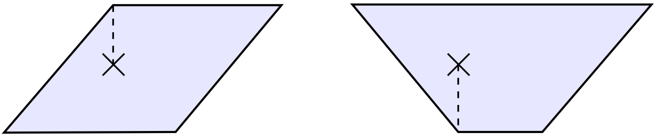

Idea

Interpolate between systems related to semitoric polygons to find desired semitoric system.

Semitoric families: definition

Definition (Le Floch-P.)

A *fixed-$\mathbb{S}^1$ family* is a family of integrable systems (M, ω, F_t), $0 \leq t \leq 1$, where

- $\dim(M) = 4$;
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It is a *semitoric family* if additionally it is semitoric for all but finitely many values of t (called the *degenerate times*).

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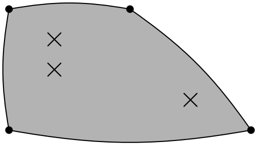

Semitoric transition families

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A *semitoric transition family* with *transition point p* is a semitoric family with exactly two degenerate times t^-, t^+ such that $0 < t^- < t^+ < 1$ and

- p is of elliptic-elliptic type for $t < t^-$;
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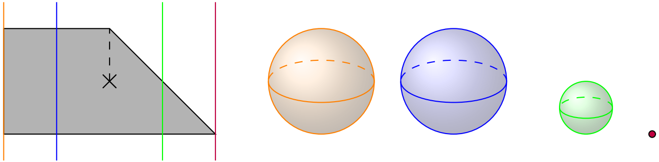

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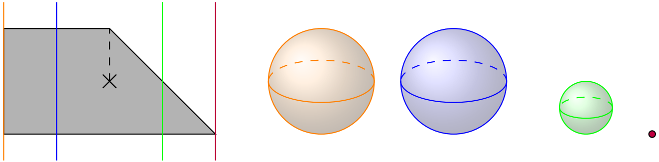
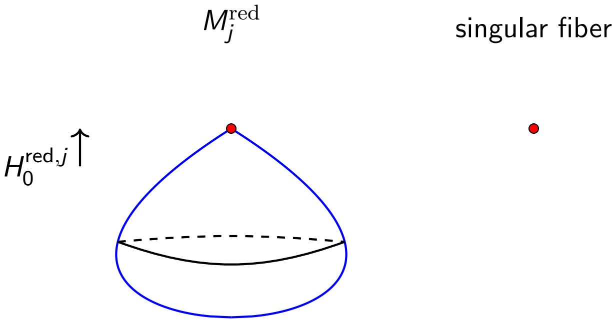


- If $dJ_j = 0$ get a 'teardrop' or 'pinched sphere' orbifold.

Coupled angular momentum: reduction



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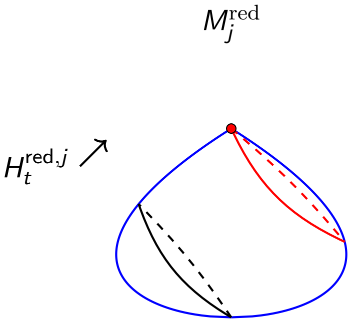

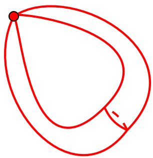

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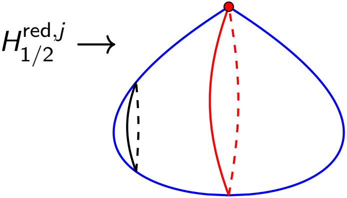
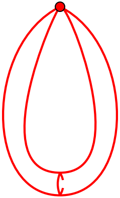


singular fiber



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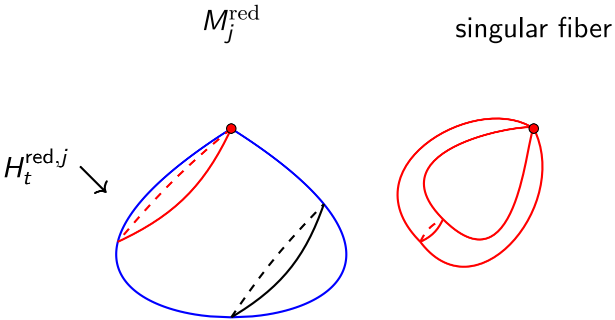

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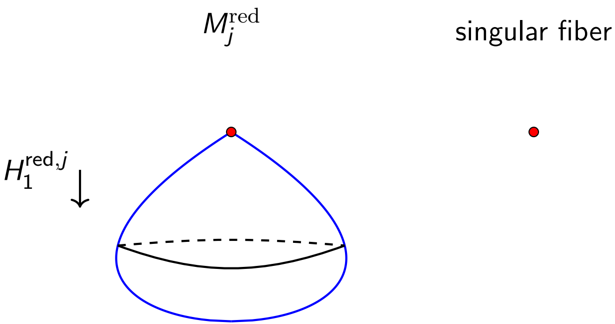

Polygons in a semitoric family

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Lemma (Le Floch-P.)

Let $(M, \omega, (J, H_t))$ be a semitoric transition family with transition point p. If the weights of the J-action at p are ± 1 then, roughly, the set of semitoric polygons for $t^- < t < t^+$ is the union of the ones for $t < t^-$ and $t > t^+$.

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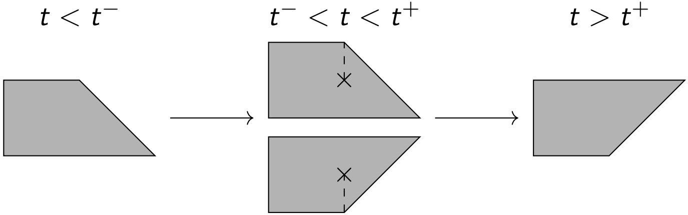

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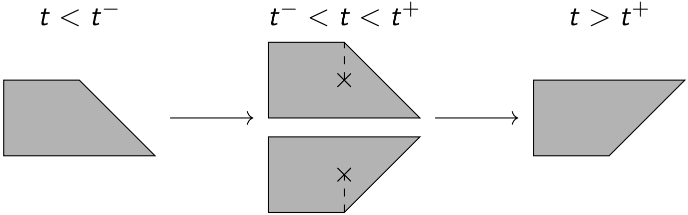
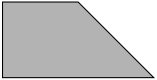


- Can help us find systems for $t = 0, 1$ to transition between.

The first Hirzebruch surface



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$$N = (1/2)\left(|u_1|^2 + |u_2|^2 + |u_3|^2, |u_3|^2 + |u_4|^2\right) \text{ at } (2,1).$$

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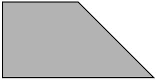

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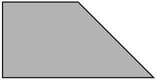

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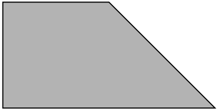

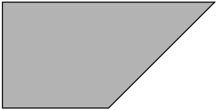

First example on W_1

Let $H_t = (1-t)H_0 + t(-H_0 + \gamma \mathrm{Re}(\overline{u}_1 u_3 \overline{u}_4))$.

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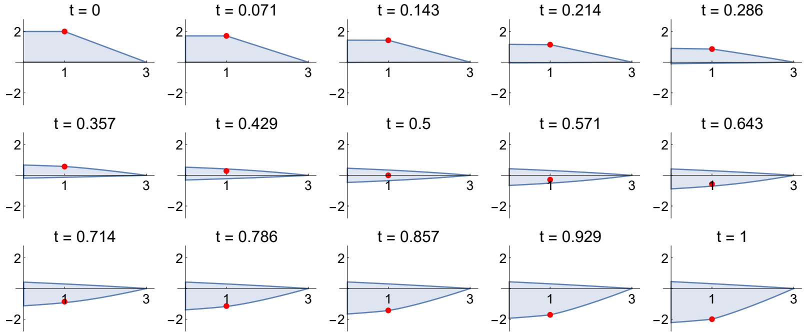

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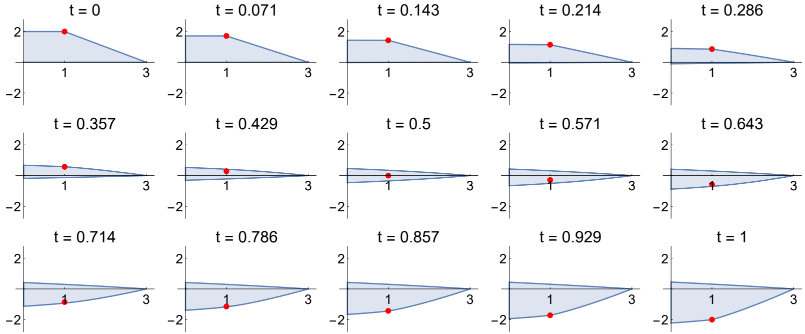

- Fixed points move on sphere $S = J^{-1}(0)$.

Second example on W_1

Let $H_t = (1-t)H_0 + t(-H_0 + \gamma J \mathrm{Re}(\overline{u}_1 u_3 \overline{u}_4) + 1)$

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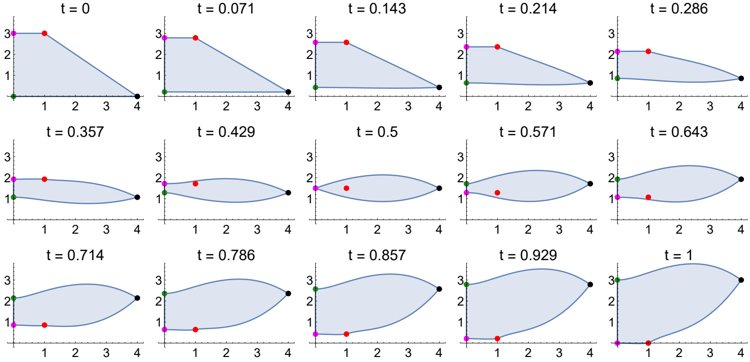

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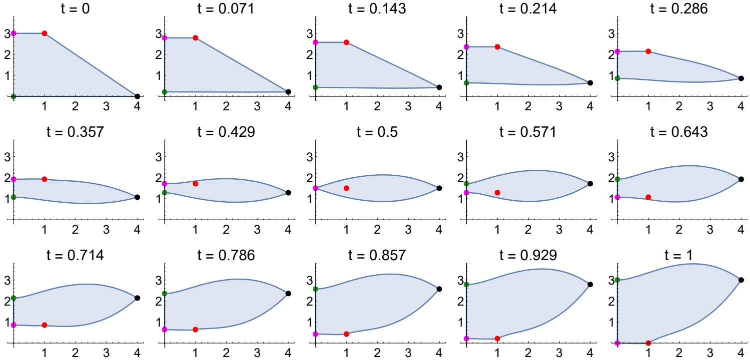

- Sphere $S = J^{-1}(0)$ 'collapses' at $t = 1/2$.

An example with hyperbolic points on W_1

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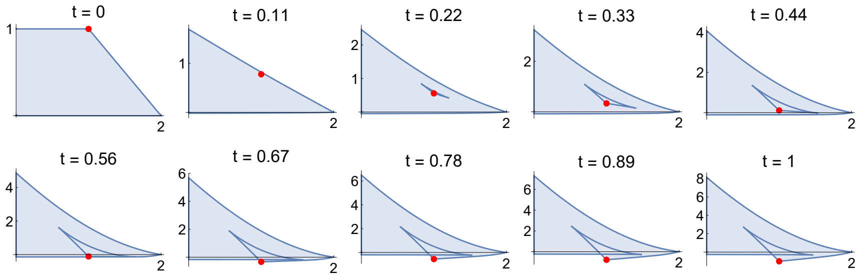

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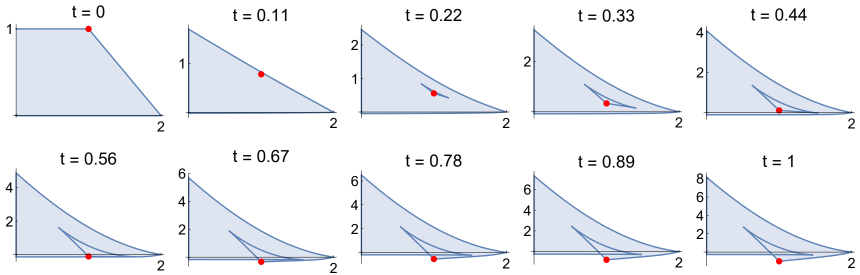

- Similar to Dullin-Pelayo (2016).

Minimal models

- Around an elliptic-elliptic point can perform a *blowup of toric type*, by blowing up with respect to $f_\varepsilon \circ F$ (blowing down is the inverse operation)

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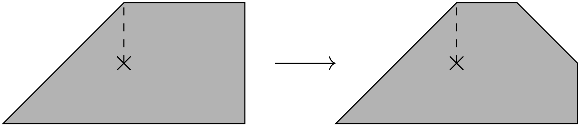
Find all compact semitoric systems which do not admit a blowdown (*minimal models*).

- Then all systems can be obtained from these by performing a sequence of blowups.

Minimal models: blowups and corner chops

- Blowups correspond to a corner chop of the semitoric polygon.



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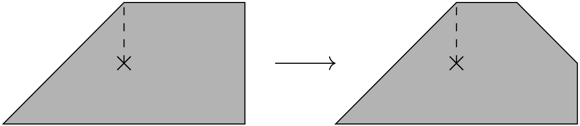

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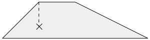

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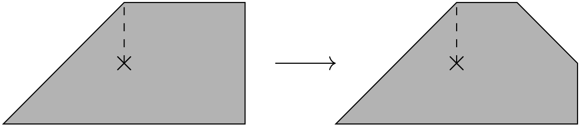

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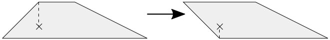

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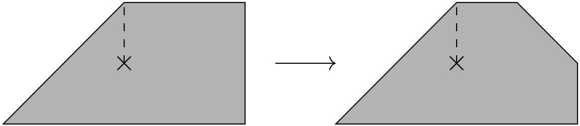

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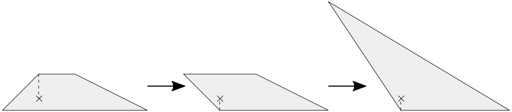

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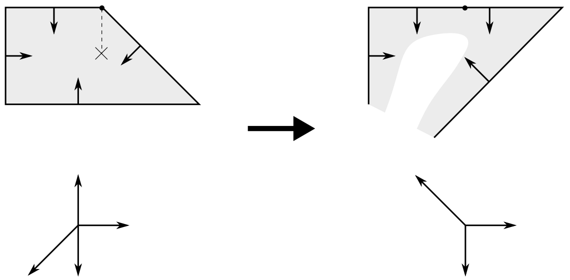

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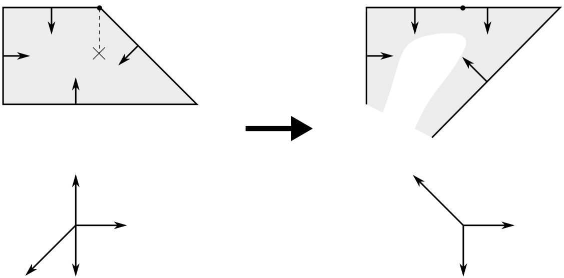
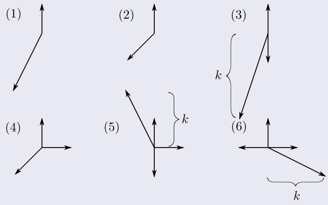


- Classify by lifting to "universal cover" of $\mathrm{SL}_2(\mathbb{Z})$.

Minimal models: minimal helices

Theorem (Kane-P.-Pelayo, 2016)

The minimal helices come in 7 families:



with (1) $c = 1$; (2) $c = 2$; (3) $k \neq 2$, $c = 1$;
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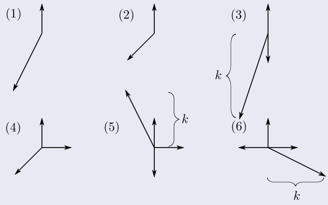

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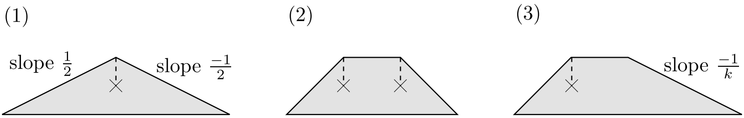
- $\mathbb{S}^1$-action on types (4)-(7) has a fixed sphere.

Minimal models: minimal polygons (1), (2), (3)

The polygons of the remaining minimal systems:



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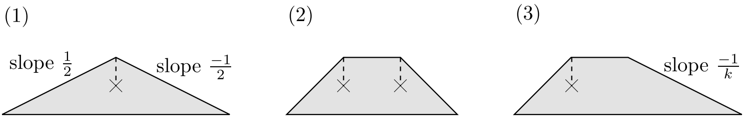

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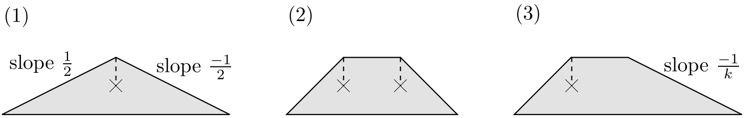

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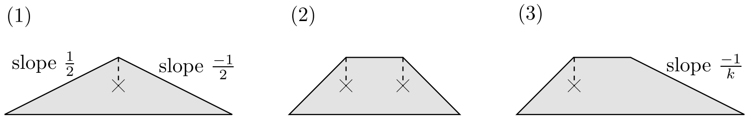

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3. Exhibit these examples in a semitoric family?

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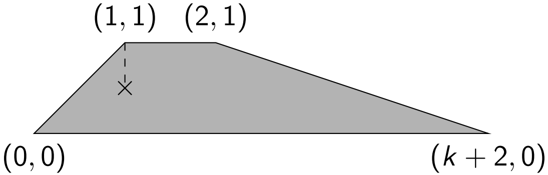

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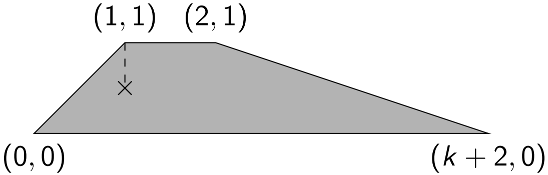

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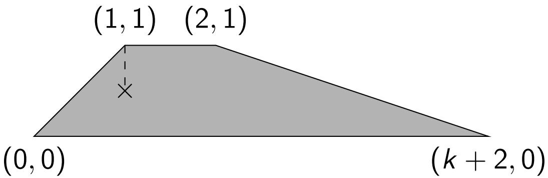

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Lemma (Le Floch-P.)

A blowup/down of a semitoric family is still a semitoric family.

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- Recall the *Hirzebruch surface with parameter n*, W_n:
  - Symplectic reduction of $\mathbb{C}^4$ by

$$N = (1/2)\left(|u_1|^2 + |u_2|^2 + n|u_3|^2, |u_3|^2 + |u_4|^2\right) \text{ at (n+1,1).}$$

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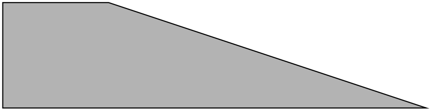

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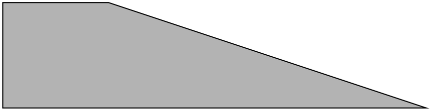

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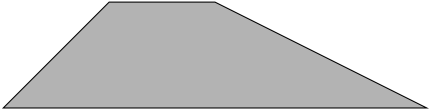

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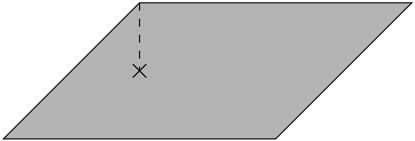
Finding a system of type (3)

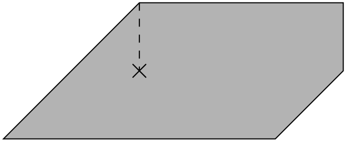
$M = W_0 \cong \mathbb{S}^2 \times \mathbb{S}^2$ (coupled angular momentum)



(minimal of type (3) with k=-1)

Finding a system of type (3)

$M = \mathrm{Blowup}(W_0)$



Finding a system of type (3)

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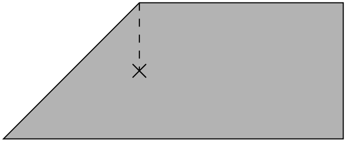

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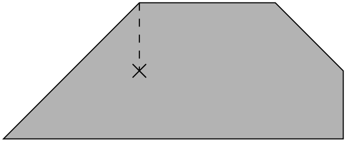

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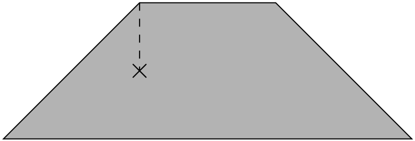

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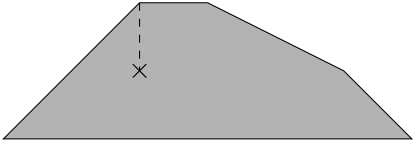
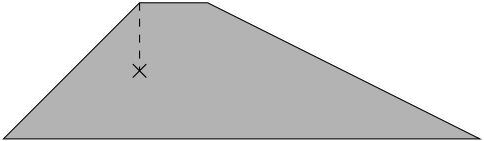


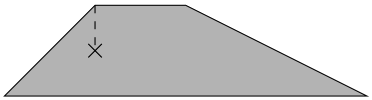
Finding a system of type (3)

$M = W_3$



(minimal of type (3) with k=2)

Existence of examples on Hirzebruch surfaces



Corollary (Le Floch-P.)

- *A semitoric system which is minimal of type (3) with parameter k exists on the $(k+1)^{\mathrm{st}}$ Hirzebruch surface.*
- *Moreover, that system is the $t = 1/2$ system in a semitoric family which can be obtained from the coupled angular momenta family on $\mathbb{S}^2 \times \mathbb{S}^2 \cong W_0$ by a sequence of alternating blowups and blowdowns.*

A system of type (2)

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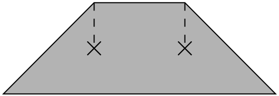

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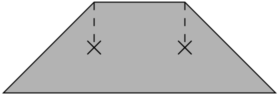
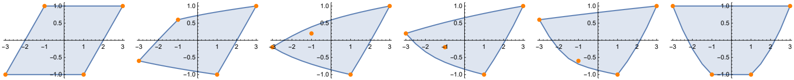


- Think about coupled angular momenta again:



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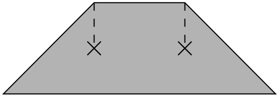

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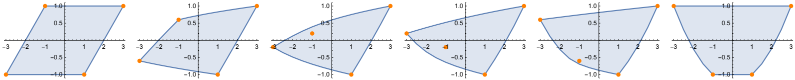

- The point NS passes through the interior and becomes focus-focus, can we do this with SN as well?

A system of type (2): 4 parameter family

$$\begin{cases} J & = R_1 z_1 + R_2 z_2 \\ H_{t_1,t_2,t_3,t_4} & = t_1 z_1 + t_2 z_2 + t_3(x_1 x_2 + y_1 y_2) + t_4 z_1 z_2 \end{cases}$$

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Theorem (Hohloch-P., 2018)

Let $R_1 = 1$ and $R_2 = 2$. Then $(J, H_{\frac{1}{4},\frac{1}{4},\frac{1}{2},0})$ is a semitoric integrable system with exactly two focus-focus points (and so is every system in an open neighborhood of these parameters).

A system of type (2): rewriting the system

Let

$$\begin{cases} H_{0,0} & = x_1 x_2 + y_1 y_2 + z_1 z_2 \\ H_{1,0} & = z_1 \\ H_{0,1} & = z_2 \\ H_{1,1} & = x_1 x_2 + y_1 y_2 - z_1 z_2 \end{cases}$$

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A system of type (2): rewriting the system

Let

$$\begin{cases} H_{0,0} & = x_1 x_2 + y_1 y_2 + z_1 z_2 \\ H_{1,0} & = z_1 \\ H_{0,1} & = z_2 \\ H_{1,1} & = x_1 x_2 + y_1 y_2 - z_1 z_2 \end{cases}$$

and

$$H_{s_1,s_2} = (1-s_2)\Big((1-s_1)H_{0,0} + s_1 H_{1,0}\Big) + s_2\Big((1-s_1)H_{0,1} + s_1 H_{1,1}\Big).$$

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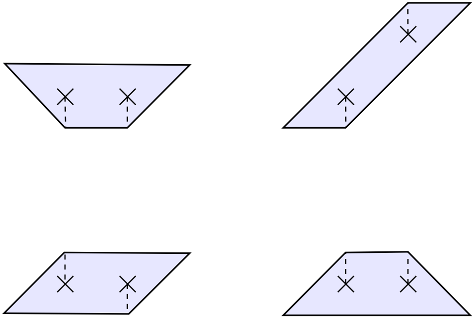
Then

$$(J, H_{\frac{1}{4},\frac{1}{4},\frac{1}{2},0}) = (J, H_{\frac{1}{2},\frac{1}{2}})$$

A system of type (2): the semitoric polygons

The semitoric polygons for $(J, H_{\frac{1}{2},\frac{1}{2}})$ (minimal polygons of type (2)):



A system of type (2): the momentum map image

Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$

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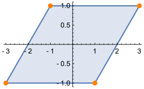
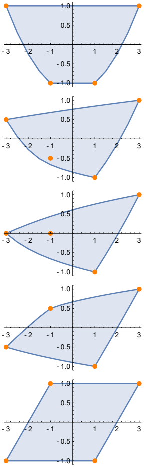
Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$



A system of type (2): the momentum map image

Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$



A system of type (2): the momentum map image

Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$

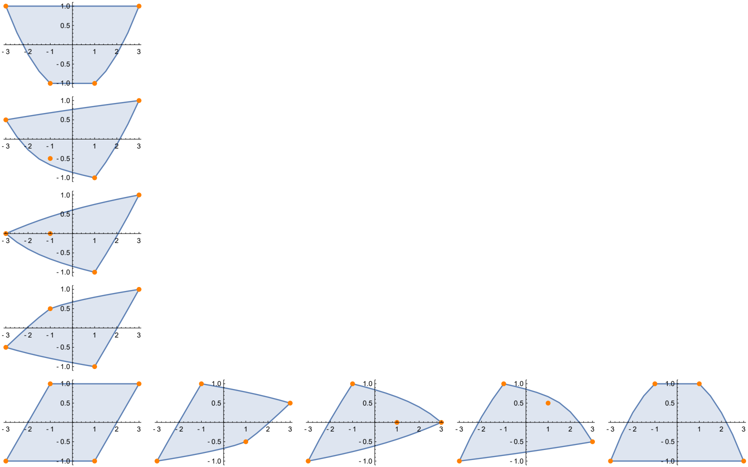


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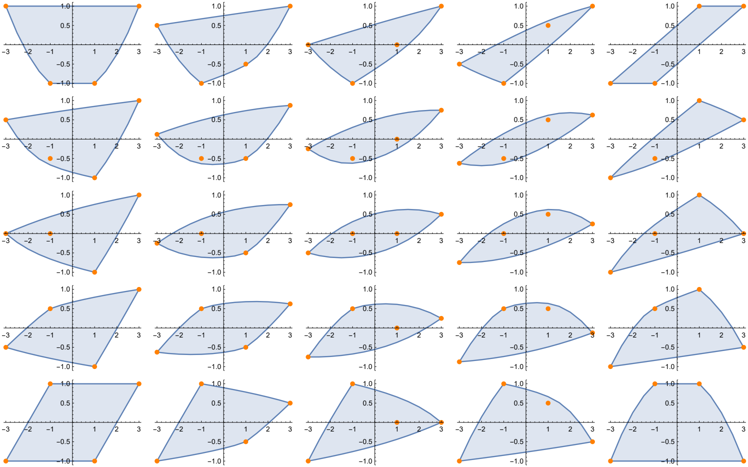
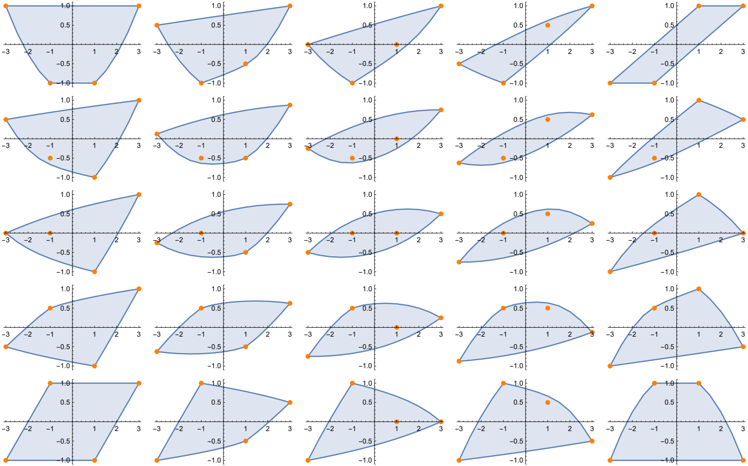
Image of (J, H_{s_1,s_2}) for $s_1, s_2 \in [0,1]$



End

Thanks for listening!



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