Spectrum and abnormals in sub-Riemannian geometry

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Sub-Riemannian (sR) geometry

Sub-Riemannian (sR) geometry is the study of metric distributions $(X, E \subset TX, g^E)$ inside the tangent space.
Subbundle E is assumed to be *bracket-generating*.
Peculiar phenomena (Hausdorff dimension & abnormal geodesics..) arise.

References:

- R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*. 2002.
- M. Gromov, *Carnot-Carathéodory spaces seen from within*, 1996, (in Bellaïche & Risler, *Sub-Riemannian geometry*)
- A. Agrachev, D. Barilari, U. Boscain, *A Comprehensive Introduction to Sub-Riemannian Geometry*, 2019.

Bracket-generating distributions

$E \subset TX$ bracket generating: $C^{\infty}(E)$ generates $C^{\infty}(TX)$ under Lie bracket $[,]$.

Examples:
1. Contact case: $E^{2m} = \ker\alpha \subset TX^{2m+1}$; $\text{rank } d\alpha|_E = 2m$.
Normal form (Darboux): $\alpha = dz - \sum_{j=1}^{m} y_j dx_j$; $E = \mathbb{R}\left[\partial_{y_j}, \partial_{x_j} + y_j\partial_z\right]$
Generation (1 step): $\left[\partial_{y_j}, \partial_{x_j} + y_j\partial_z\right] = \partial_z$

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2. Quasi-contact case: $E^{2m+1} = \ker\alpha \subset TX^{2m+2}$; rank $d\alpha|_E = 2m$.
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3. Martinet case: $E^2 = \ker\alpha \subset TX^3$, $\underbrace{Z^2}_{\text{union of surfaces}} = \{\alpha \wedge d\alpha = 0\} \subset X$ with $TZ \pitchfork E$.
Normal form: $\alpha = dz - y^2 dx$, $E = \mathbb{R}\left[\partial_y, \partial_x + y^2\partial_z\right]$
Generation (2 step): $\left[\partial_y, \partial_x + y^2\partial_z\right] = 2y\partial_z$, $\left[\partial_y, \left[\partial_y, \partial_x + y^2\partial_z\right]\right] = 2\partial_z$

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4. Engel case: $E^2 \subset TX^4$ stable
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5. Goursat case: $E^2 \subset TX^n$ (... some general definition ...)
Normal form (eg.): $E = \mathbb{R}\left[\partial_{x_2}, \partial_{x_1} + x_2\partial_{x_3} + \ldots + x_{n-1}\partial_{x_n}\right]$
Generation (n-2 step): $\left[\partial_{x_2}, \partial_{x_1} + x_2\partial_{x_3} + \ldots + x_{n-1}\partial_{x_n}\right] = \partial_{x_3}$,
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and many more..

Flag, metric & dimension

Canonical flag:

$$\underbrace{E_0}_{=\{0\}} \subset \underbrace{E_1}_{=E} \subset \ldots \subset \underbrace{E_j}_{\substack{\text{span of}\\ j\text{th brackets}}} \subset \ldots \subset \underbrace{E_{r(x)}}_{=TX}$$

Step$= r(x)$, Growth vector $= k^E(x) = \left(\dim E_0, \dim E_1, \ldots, \dim E_{r(x)}\right)$.
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Theorem (Chow-Rashevsky '37)

E bracket-generating $\implies$ any two $x_1, x_2 \in X$ connected by horizontal path γ s.t. $\dot{\gamma}(t) \in E_{\gamma(t)}$ a.e.

Idea: $e^{tX} e^{tY} e^{-tX} e^{-tY} = e^{t^2[X,Y]} + O(t)$

(X, d^E) is a metric space with $d^E = \inf_{\gamma \text{ horizontal}} \int_0^1 dt \, |\dot{\gamma}(t)|$.

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$$\underbrace{Q(x)}_{\text{Hausdorff dimension}} = \lim_{\varepsilon \to 0} \frac{\ln \mathrm{vol} B_\varepsilon(x)}{\ln \varepsilon} \overset{\text{Ball-Box}}{\underset{\text{thm}}{=}} \sum_{j=1}^{r(x)} j \left[k_j(x) - k_{j-1}(x)\right] > n$$

A metric ball

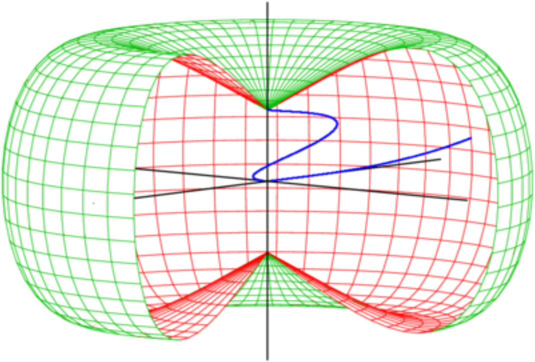

Figure: A metric ball in $\mathbb{H}^3$: $|x|^2 + |y|^2 + |z| \approx 1$

Popp's volume

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O. Popp (Montgomery 2002) defined one in the equiregular case $\left(\substack{E_0 \subset E_1 \subset \ldots \subset E_r \\ \text{vector bundles}}\right)$:

Surjective bracketing map (defines pushforward metric /volume element on latter)	$B_j : E^{\otimes j} \to E_j / E_{j-1}$
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μ_{Popp} not smooth in general.

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Not all minimizers obtained this way!

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$Z = \{\alpha \wedge d\alpha = 0\} = \{y = 0\}$.
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Open question: Are abnormal minimizers smooth?

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Well understood in cases:

- Contact case: none.
- Quasi contact case: Integral curves of $L^E := \ker d\alpha|_E$ (topological)
- Martinet case: Integral curves of $\ker \alpha|_Z =: L^E \to Z$ (topological)

sR Laplacian

Let $(X, E \subset TX, g^E)$ sR manifold.
Choose an auxiliary density μ to define

$$\text{sR Laplacian}: \quad \Delta_{g^E,\mu} := \left(\nabla^{g^E}\right)^*_\mu \circ \nabla^{g^E}$$

where $g^E\left(\nabla^{g^E} f, U\right) = U(f)$, $\forall U \in C^\infty(E)$ is sR-gradient.

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Characteristic variety: $\Sigma = \left\{\sigma\left(\Delta_{g^E,\mu}\right) = H^E = 0\right\} = E^\perp$.

(Hormander '67) E bracket generating $\implies$ $\Delta_{g^E,\mu}$ is hypoelliptic
(Rothschild & Stein '76) $\|f\|^2_{H^{1/r}} \lesssim \left\langle \Delta_{g^E,\mu} f, f\right\rangle + \|f\|^2_{L^2}$ where $r = \max_{x \in X} r(x)$.

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Discrete spectrum (φ_j, λ_j); $\Delta_{g^E,\mu}\varphi_j = \lambda_j\varphi_j$, on a compact manifold.
Spectral asymptotics questions: Weyl law, trace formula, propagation, ergodicity ...
(mostly open)

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Theorem (Ben Arous 1989, Léandre 1992...)

There exist $a_j(x) \in C^\infty(X)$, $j = 0, 1, \ldots$,

$$e^{-t\Delta_{g^E,\mu}}(x,x) \sim t^{-Q(x)/2}\left[\sum_{j=0}^{\infty} a_j(x)\, t^j\right].$$

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Theorem (Métivier '76)

If E *is equiregular*

$$N(\lambda) \sim \frac{\lambda^{Q/2}}{\Gamma(Q/2+1)} \int_X a_0.$$

Colin de Verdière-Hillairet-Trélat (singular Weyl law, ongoing)
Martinet eg.: $N(\lambda) \sim c\lambda^2 \ln \lambda$.

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$$N(\lambda) \sim \lambda^2 \left(\int \mu_{Popp} \right) + O\left(\lambda^{3/2}\right).$$

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Semiclassical analog: trace of magnetic Dirac operator S. '17 (APDE).

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(X^4, E^3) 4D quasi-contact.

Theorem (S. '19)

$$\textit{sing spt}\left(\textit{tr } e^{it\sqrt{\Delta_{g^E,\mu}}}\right) \subset \begin{array}{l}\{0\} \cup \{\textit{lengths of (normal) closed geodesics}\} \\ \quad \cup \left(-\infty, -T^E_{abnormal}\right] \cup \left[T^E_{abnormal}, \infty\right)\end{array}$$

$$N(\lambda) \sim \lambda^{5/2}\left(\int \mu_{Popp}\right) + O\left(\lambda^2\right).$$

Union of L^E closed curves is of measure zero :
$$N(\lambda) \sim \lambda^{5/2}\left(\int \mu_{Popp}\right) + o\left(\lambda^2\right).$$

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Theorem (S. '19)

If L^E *ergodic &* $L_Z \mu_{Popp} = 0$. *Then one has quantum ergodicity (QE).*

T^E =shortest closed period of L^E, Z =unit generator of L^E;
Integral curves of L^E correspond to abnormals.
Proofs based on $\sqrt{\Delta_{g^E,\mu}} \in \Psi^{1,-1}_{\text{cl}}(X, \Sigma)$ (exotic pseudodifferential calculus).

Circle bundles

Natural place for sR-structures: $\left(\underbrace{X^n}_{S^1 L}, \underbrace{E^{n-1}}_{HX \text{ horizontal}}, \underbrace{g^E}_{\pi^* g^{TY}} \right)$ with $\left(L, h^L, \nabla^L\right) \to \left(Y^{n-1}, g^{TY}\right)$ is a Hermitian line bundle with connection.
Equivalently consider sR structure invariant by a free and transversal S^1 action.

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Decomposition: $Y = \cup_{j=2}^{r} Y_j$; $Y_j = \{y | r_y = j\}$

Bochner Laplacian

Fourier modes: $$C^{\infty}(X) = \oplus_{k=-\infty}^{\infty} C^{\infty}\left(X, L^k\right)$$

$$\underbrace{\Delta_{g^E,\mu_X}}_{\text{sR Laplacian}} = \oplus_{k=-\infty}^{\infty} \underbrace{\Delta_k}_{\text{Bochner}}$$

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sR heat kernel expansion analogously gives $e^{-\frac{t}{k^{2/r}}\Delta_k}(y,y)$, $k \to \infty$.

Theorem (Marinescu-S. '18)

The first eigenfunction/eigenvalue (ψ_0^k, λ_0^k) *of the Bochner Laplacian* Δ_k *satisfy*

$$\lambda_0^k \sim Ck^{2/r}$$

$$\left|\psi_0^k(y)\right| = O\left(k^{-\infty}\right), \quad y \notin Y_r.$$

Generalizes:
R. Montgomery '95 ($\dim Y = 2$, $r = 3$), Helffer & Mohamed '96, Pan & Kwek '02, Helffer & Kordyukov '09, Bonnaillie-Noël, Hérau & Raymond ('16) ...

Bergman kernel

If Y cpx. Hermitian and L holomorphic semipositive

$$\text{Kodaira Laplacian:} \qquad \Box_k : \Omega^{0,*}\left(Y; L^k\right) \rightarrow \Omega^{0,*}\left(Y; L^{\otimes k}\right).$$

$$\text{Bergman kernel:} \qquad \Pi_k\left(y\right) := \sum_{j=1}^{\dim H^0\left(Y;L^k\right)} \underbrace{\left|s_j\left(y\right)\right|^2}_{\text{orth. basis}}$$

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$$\text{Bergman kernel:} \qquad \Pi_k(y) := \sum_{j=1}^{\dim H^0\left(Y; L^k\right)} \underbrace{\left|s_j(y)\right|^2}_{\text{orth. basis}}$$

Lichnerowicz + Mckean-Singer: $\mathrm{Spec}\left(\Box_k\right) \subset \{0\} \cup \left[c_1 k^{2/r} - c_2, \infty\right)$ in 2D

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Local index theory of Bismut-Lebeau '91, Dai-Liu-Ma '06, Ma-Marinescu '07 gives

Theorem (Marinescu-S. '18)

For $dimY = 2$ & R^L semi-positive of finite order

$$\Pi_k\left(y\right) \sim k^{2/r_y}\left[\sum_{j=0}^{N} c_j\left(y\right) k^{-j/r_y}\right]$$

where $r_y - 2 = ord\left(R^L_y\right)$.

Tian '91, Catlin '97, Zelditch '99 (positive case);
R. Berman '09, Hsiao-Marinescu '14 (on positive part in some cases).

Thank you.