The matching polytope has exponential extension complexity

Thomas Rothvoss

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Extended formulation

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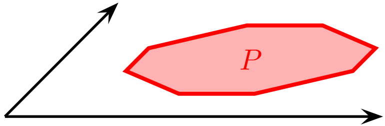
- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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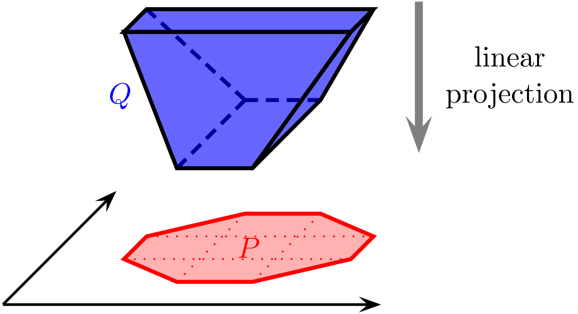

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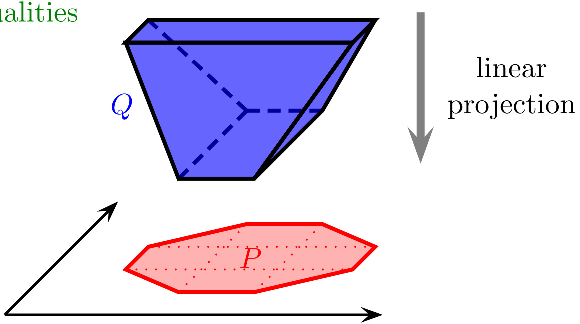

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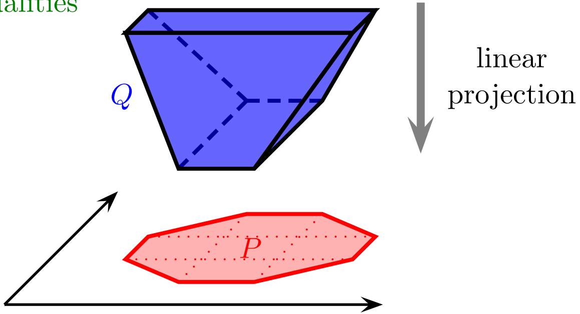

- **Extension complexity:**

$$\mathrm{xc}(P) := \min \left\{ \#\text{facets of } Q \;\middle|\; \begin{array}{c} Q \text{ polyhedron} \\ p \text{ linear map} \\ p(Q) = P \end{array} \right\}$$

What's known?

Compact formulations:

- Spanning Tree Polytope [Kipp Martin '91]
- Perfect Matching in planar graphs [Barahona '93]
- Perfect Matching in bounded genus graphs [Gerards '91]
- $O(n \log n)$-size for Permutahedron [Goemans '10] ($\rightarrow$ **tight**)
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Here: When is the extension complexity **super polynomial**?

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Only **NP**-hard polytopes!!

What about poly-time problems?

Perfect matching polytope

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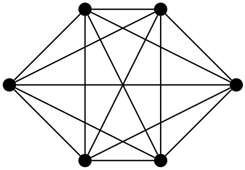
$G = (V, E)$
(complete)



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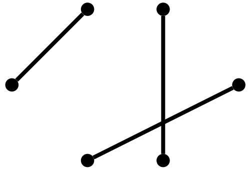

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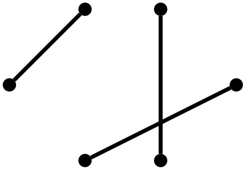

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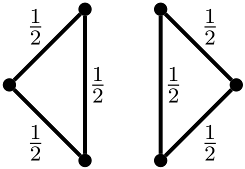

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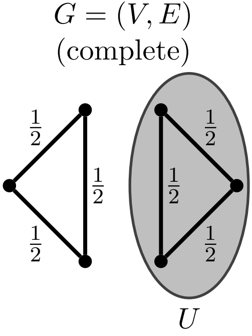

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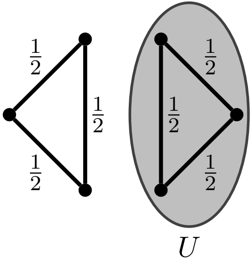

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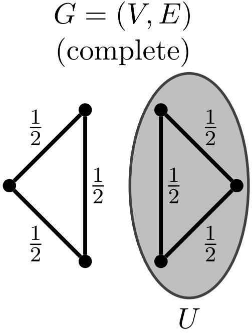

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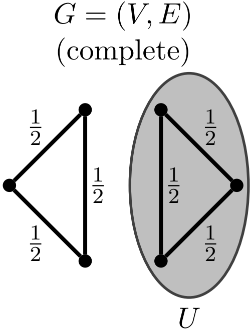

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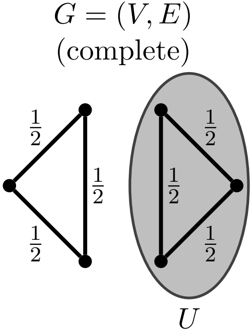

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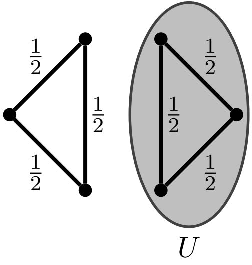

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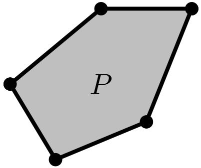
Theorem (R.13)

xc(perfect matching polytope) $\geq 2^{\Omega(n)}$.

- Previously known: $\text{xc}(P) \geq \Omega(n^2)$

Slack-matrix

Write: $P = \mathrm{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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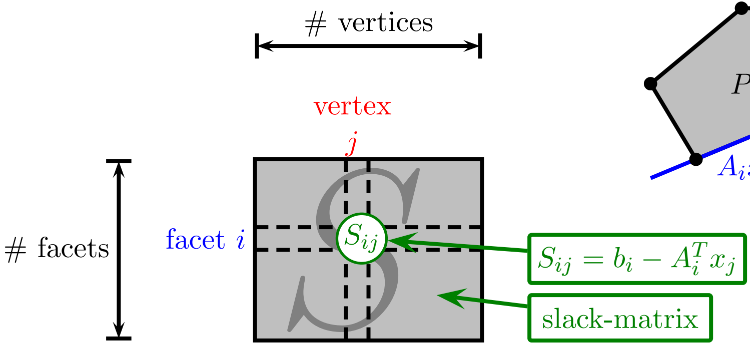

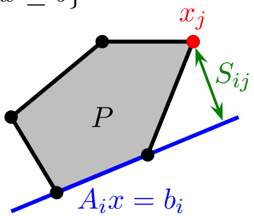

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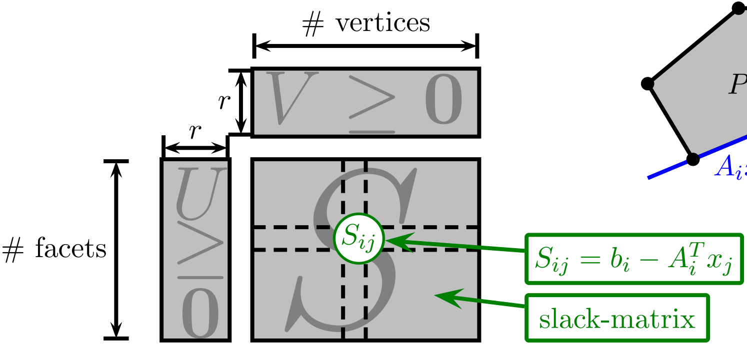

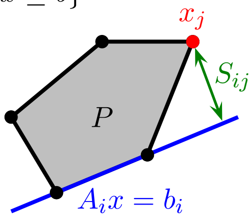

Non-negative rank:

$$\mathrm{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Yannakakis' Theorem

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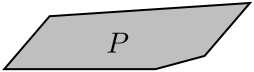
If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.



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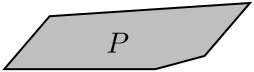

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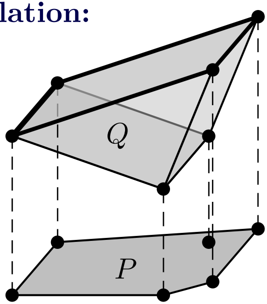

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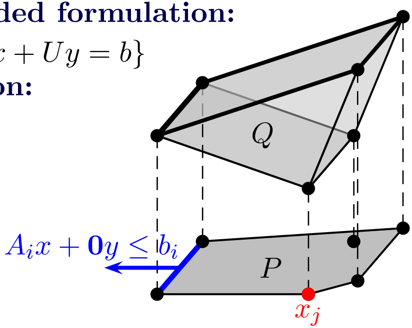

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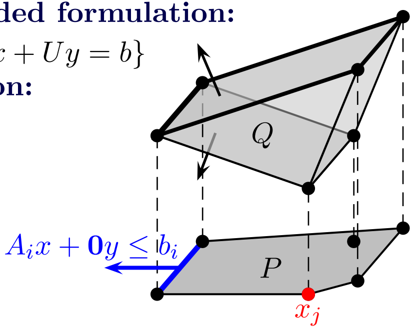

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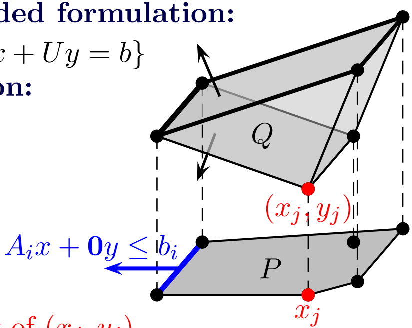

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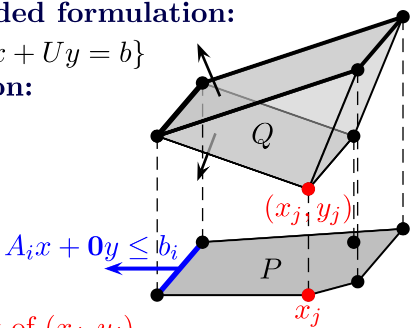

$$\langle u(i), v(j) \rangle = \underbrace{u(i)^T d}_{=b_i} - \underbrace{u(i)B}_{=A_i} x_j - \underbrace{u(i)C}_{=\mathbf{0}} y_j = S_{ij}$$

Rectangle covering lower bound

Observation

$\mathrm{rk}_+(S) \geq$ rectangle-covering-number(S).

Rectangle covering lower bound

$$V = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 10 & 3 & 5 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 4 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{pmatrix} S$$

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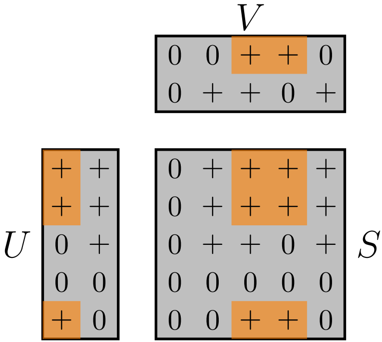

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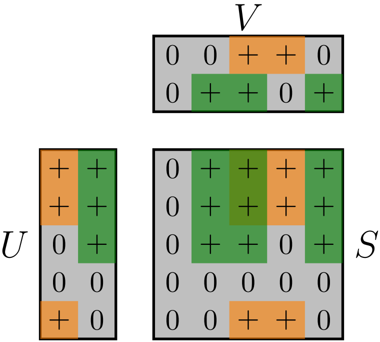

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Rectangle covering for matching

- Recall $S_{U,M} = |\delta(U) \cap M| - 1$

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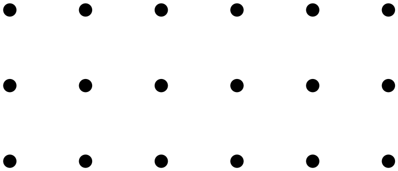
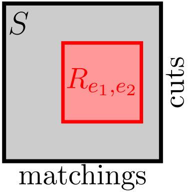
Rect-cov-num(matching polytope) $\leq O(n^4)$.





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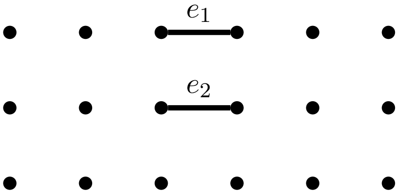

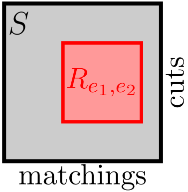

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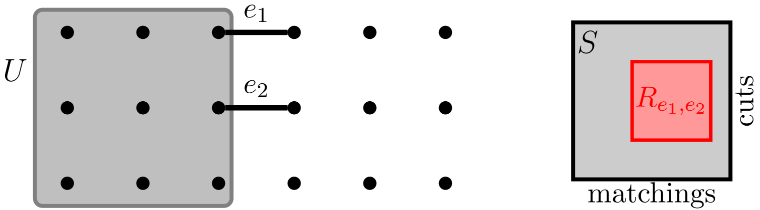

- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\}$

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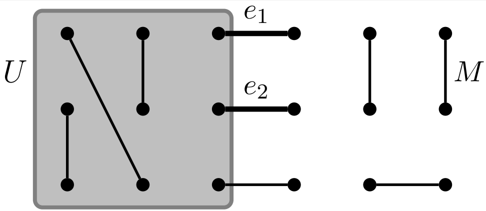

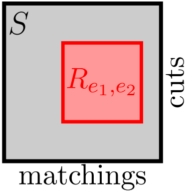

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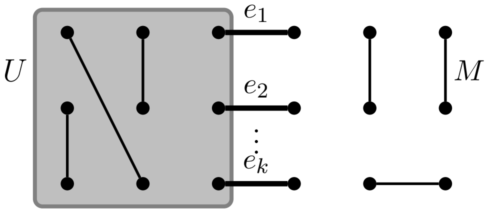

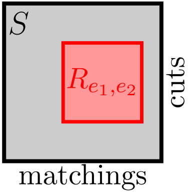

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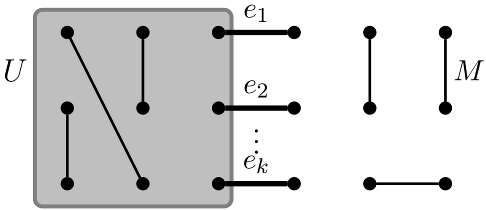

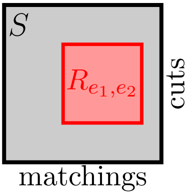

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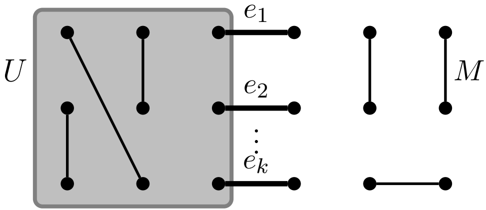

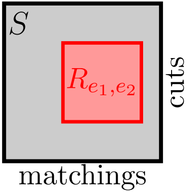

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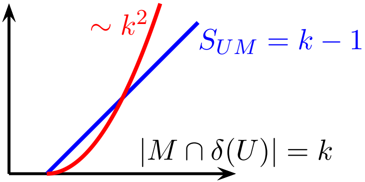

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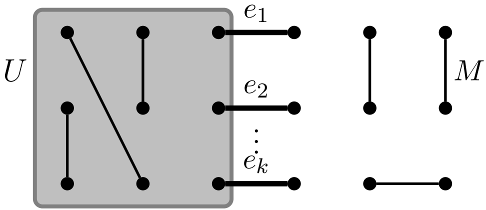

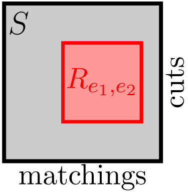

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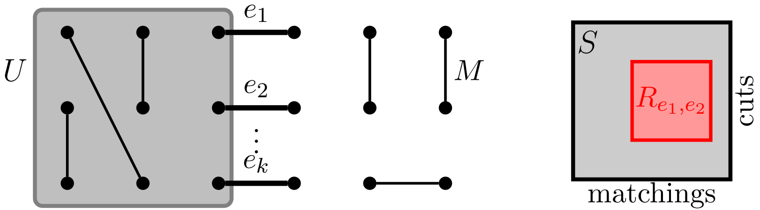

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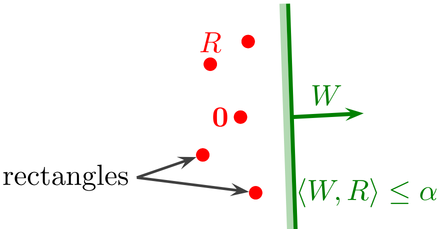

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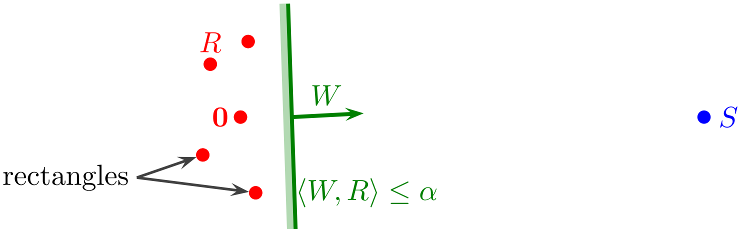

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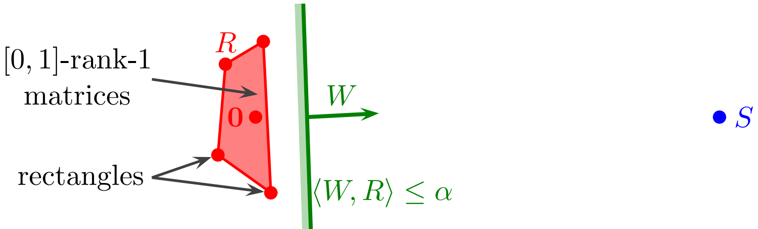
- **Proof:** Write $S = \sum_{i=1}^r R_i$ with $\text{rk}_+(R_i) = 1$. Then

$$\langle W, S\rangle = \sum_{i=1}^r \|R_i\|_\infty \cdot \underbrace{\left\langle W, \frac{R_i}{\|R_i\|_\infty}\right\rangle}_{\leq \alpha} \leq \alpha \cdot \sum_{i=1}^r \underbrace{\|R_i\|_\infty}_{\leq \|S\|_\infty} \leq \alpha \cdot r \cdot \|S\|_\infty .$$



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Lemma

For k large, any rectangle R has $\langle W, R\rangle \leq 2^{-\Omega(n)}$.

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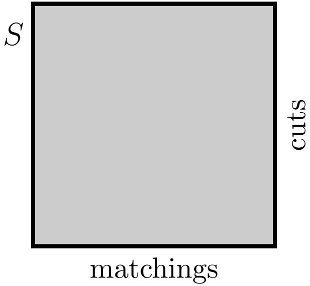

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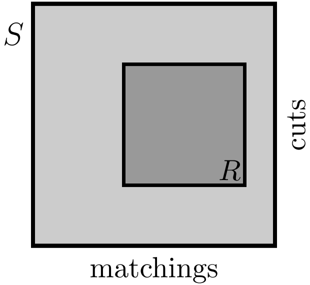

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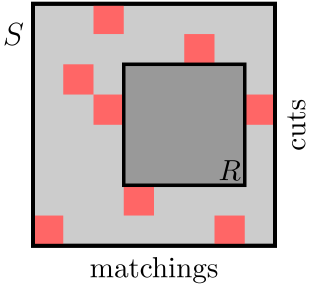

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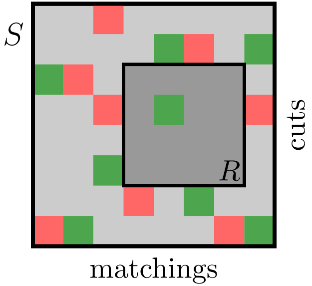

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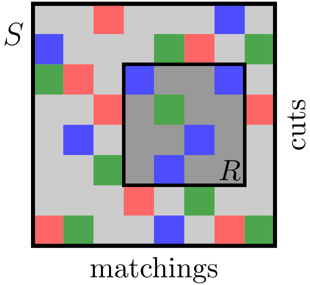

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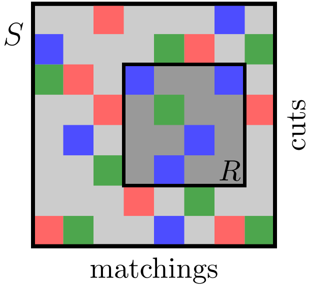

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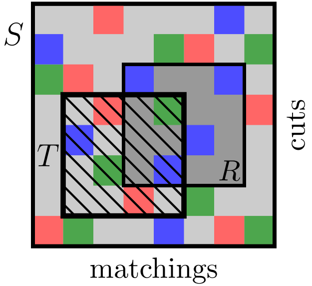

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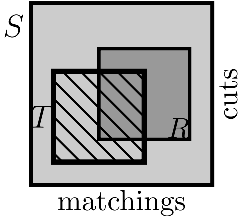

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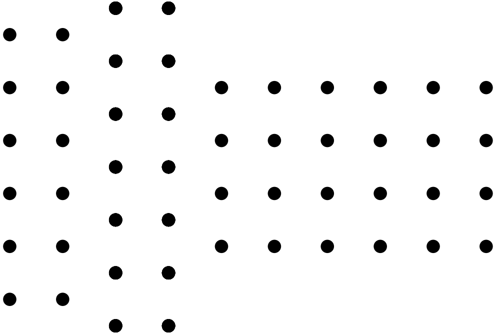

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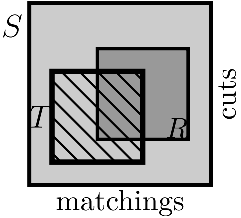

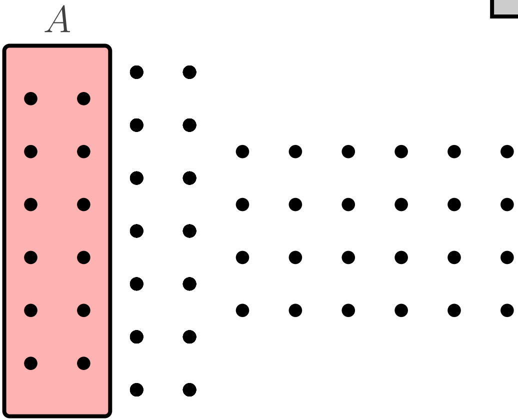

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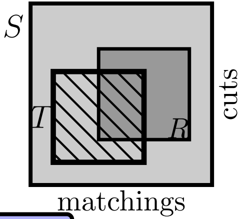

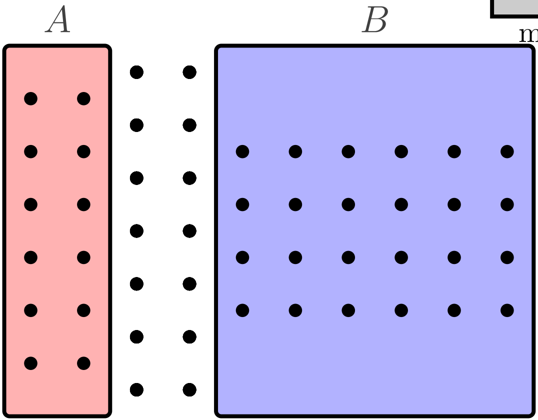

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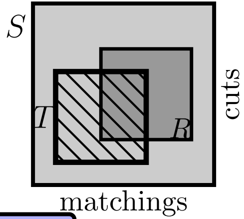

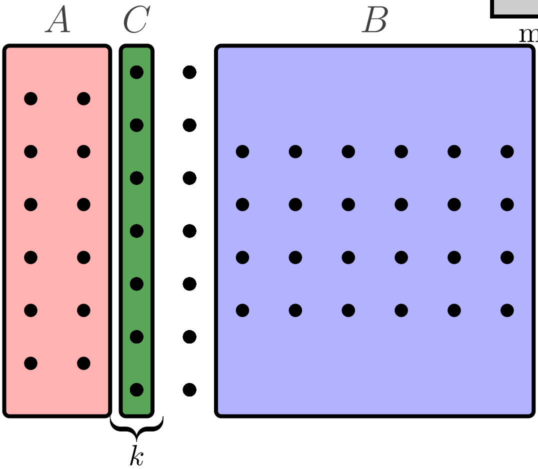

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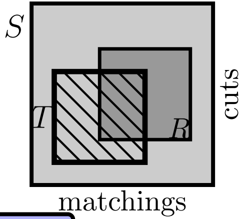

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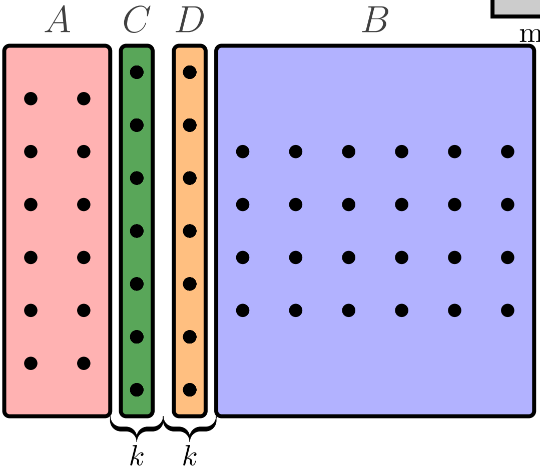

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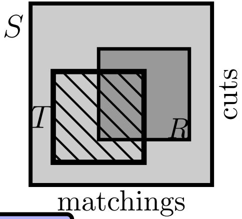

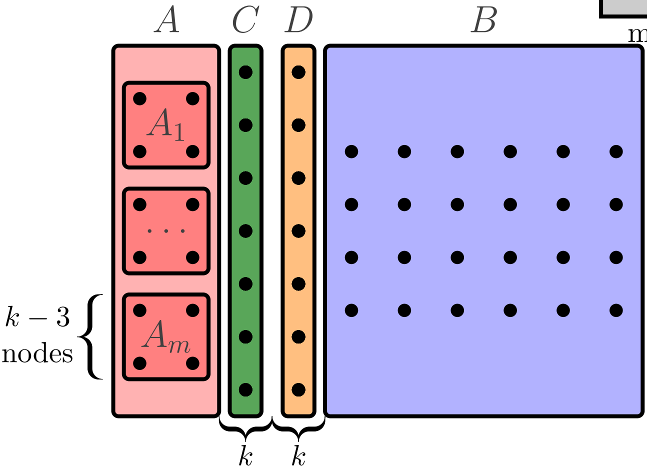

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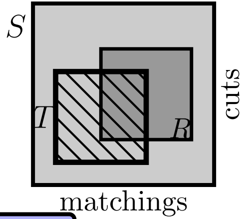

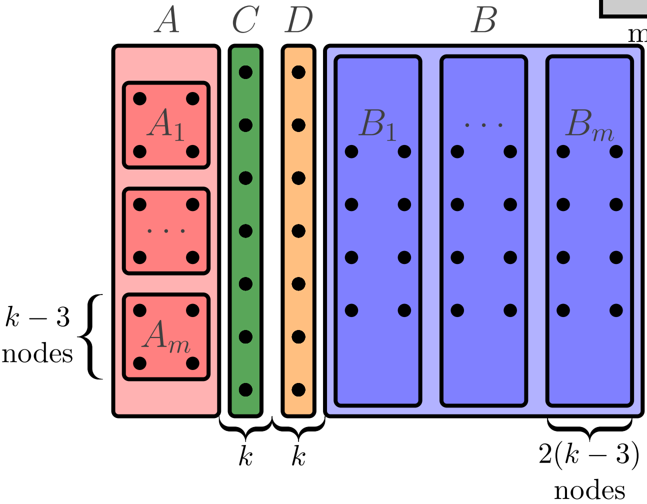

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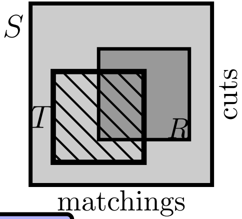

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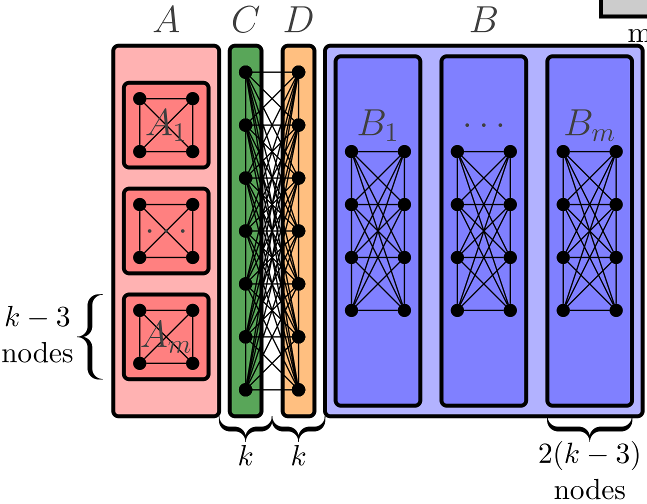

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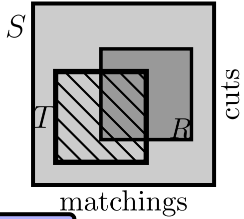

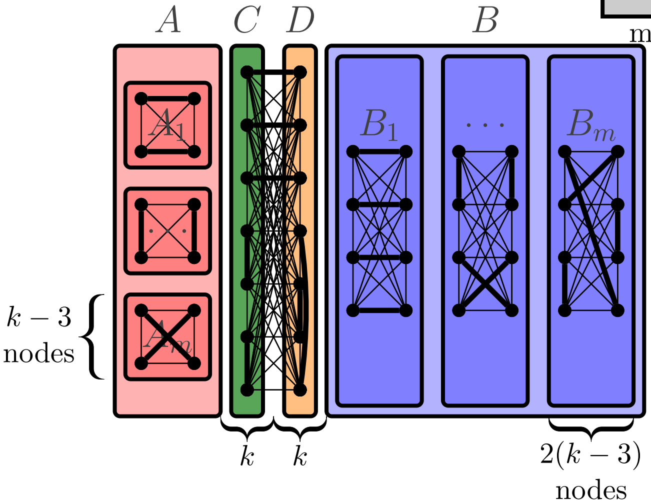

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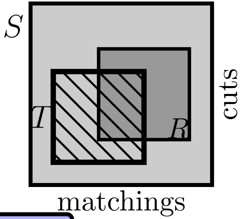
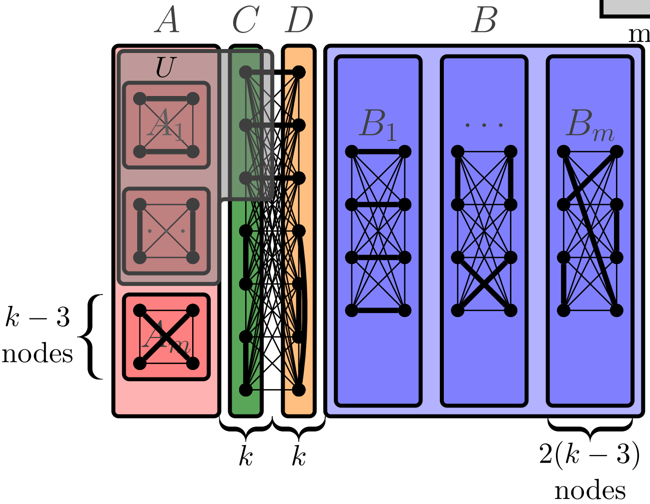
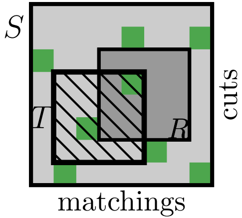
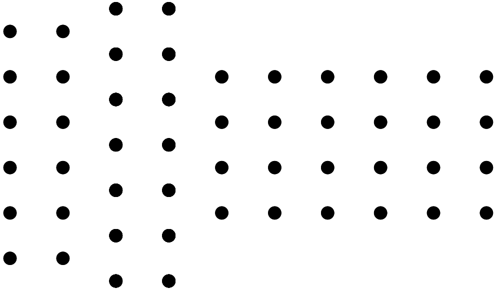




Rewriting $\mu_3(R)$

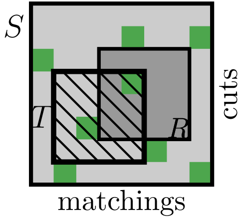


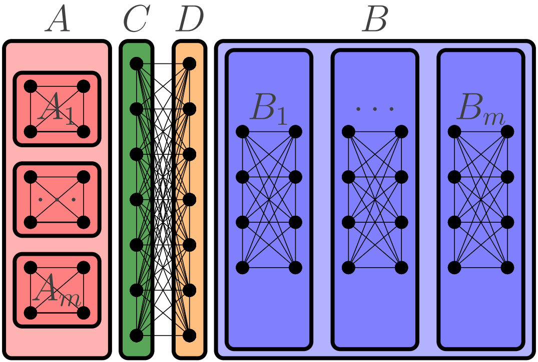


Randomly generate $(U, M) \sim Q_3$:

$$\mu_3(R) =$$

Rewriting $\mu_3(R)$



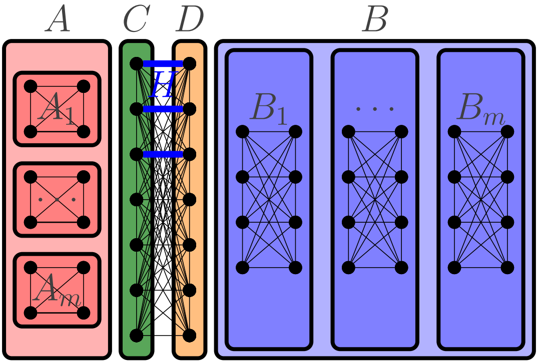


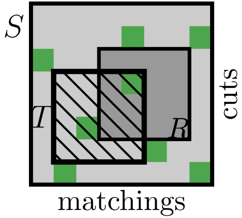
Randomly generate $(U, M) \sim Q_3$:

1. Choose T

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\Big[\qquad \Big]$$

Rewriting $\mu_3(R)$





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1. Choose T
2. Choose 3 edges $H \subseteq C \times D$

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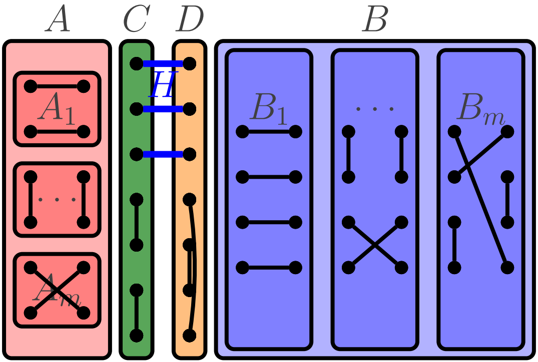
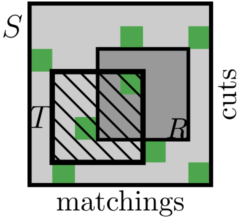
Rewriting $\mu_3(R)$





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3. Choose $M \supseteq H$ (not cutting any other edge in $C \times D$)

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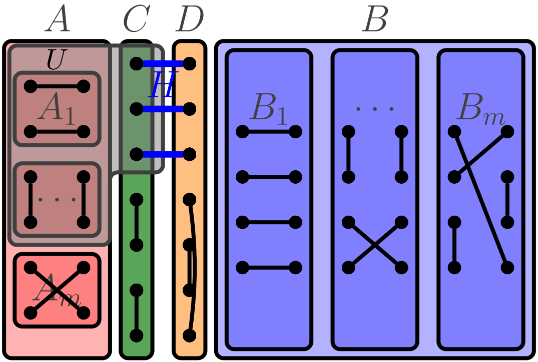
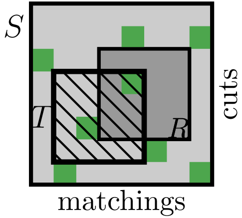
Rewriting $\mu_3(R)$





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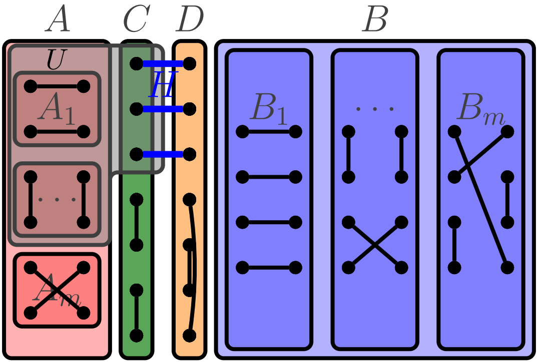
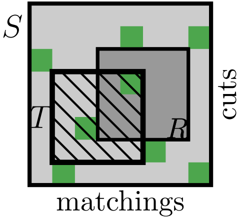
4. Choose U cutting H (not cutting any A_i)

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U, M) \in R \mid T, H]\right]\right]$$

Rewriting $\mu_3(R)$





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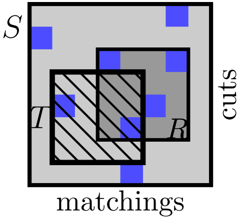
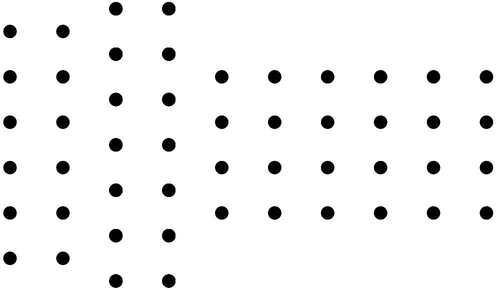
Rewriting $\mu_k(R)$

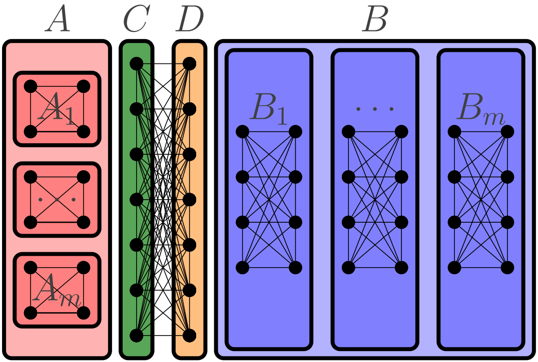


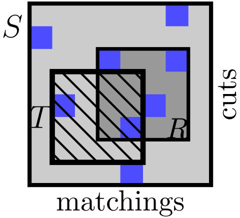


Randomly generate $(U, M) \sim Q_k$:

$\mu_k(\mathcal{R}) =$

Rewriting $\mu_k(R)$





Randomly generate $(U, M) \sim Q_k$:

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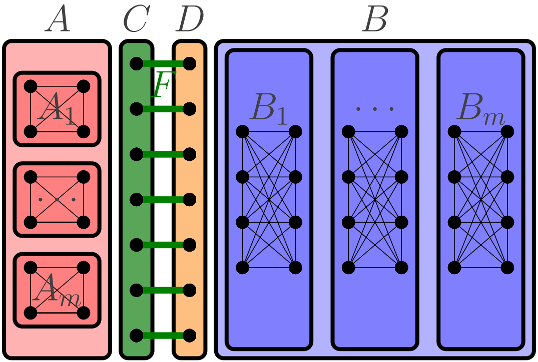
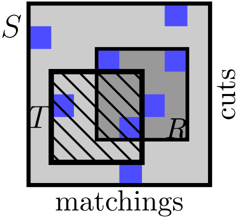
$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T} \Big[\qquad\qquad \Big]$$

Rewriting $\mu_k(R)$





Randomly generate $(U, M) \sim Q_k$:

1. Choose T
2. Choose k edges $F \subseteq C \times D$

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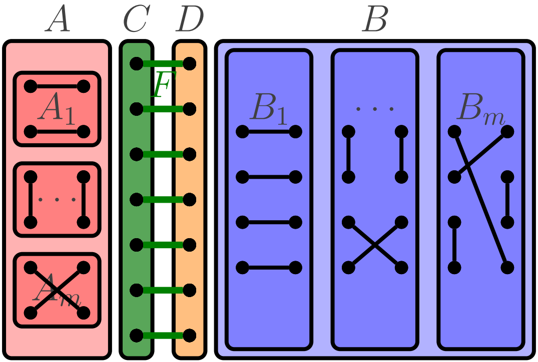
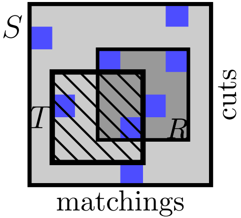
Rewriting $\mu_k(R)$





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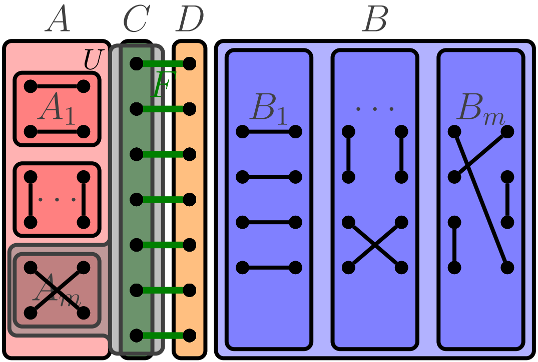
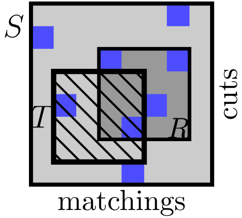
$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|F|=k} \left[\Pr[M \in \mathcal{R} \mid T, H] \qquad \right]\right]$$

Rewriting $\mu_k(R)$





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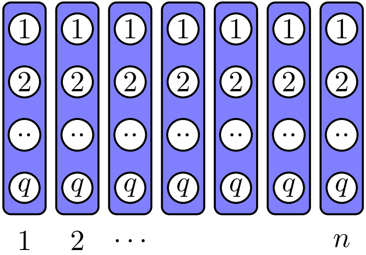
$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|F|=k}\left[\Pr[M \in \mathcal{R} \mid T, H] \cdot \Pr[U \in R \mid T, H]\right]\right]$$

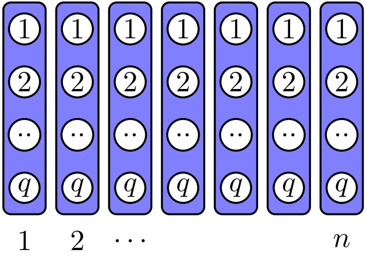
Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.



Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
- Draw $x \sim X$.



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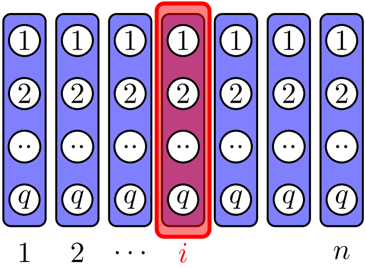

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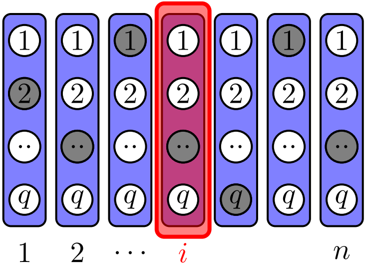

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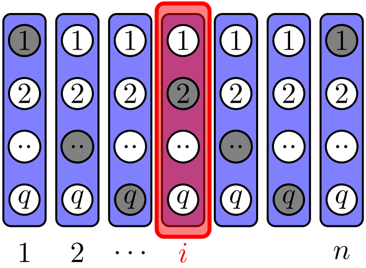

Pseudorandom-behaviour of large sets

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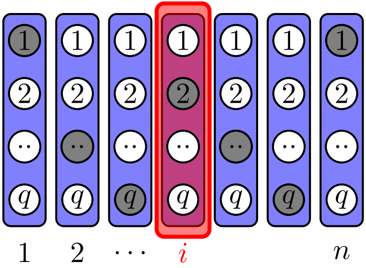
Lemma

$|X|$ **large** $\Rightarrow$ for most indices x_i is **approx. uniform**



Pseudorandom-behaviour of large sets

- Consider vectors $X \subseteq [q]^n$.
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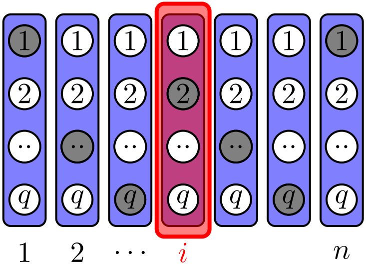
Lemma

εn **biased** indices $\Rightarrow \frac{|X|}{q^n} \leq 2^{-\Omega(n)}$.



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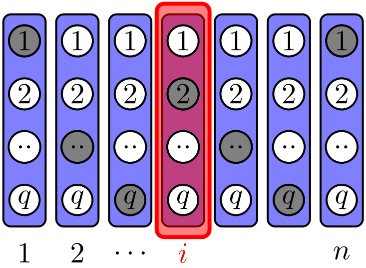
$$\log_2(|X|) = H(x)$$



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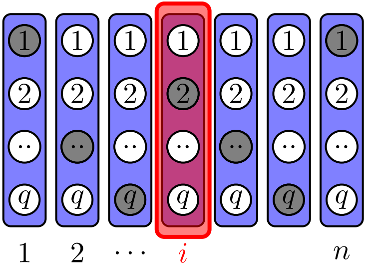
$$\log_2(|X|) = H(x) \leq \sum_{i=1}^{n} H(x_i)$$



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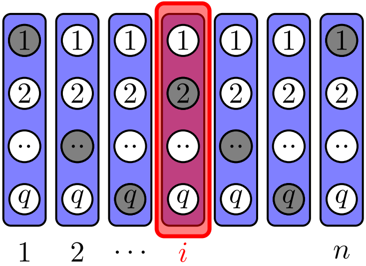
$$\log_2(|X|) = H(x) \leq \sum_{i \text{ biased}} H(x_i) + \sum_{i \text{ unbiased}} H(x_i)$$



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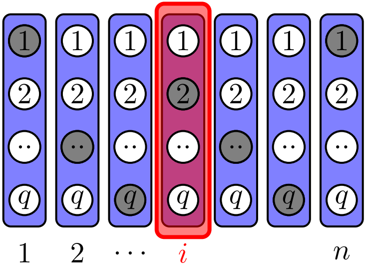

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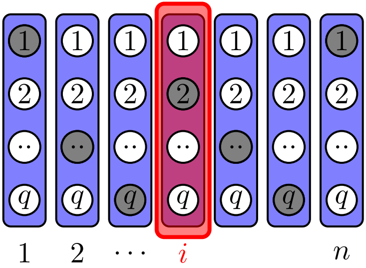

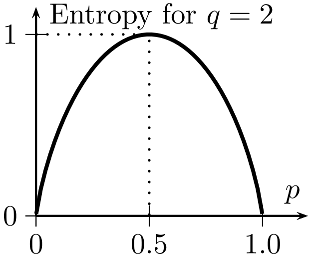

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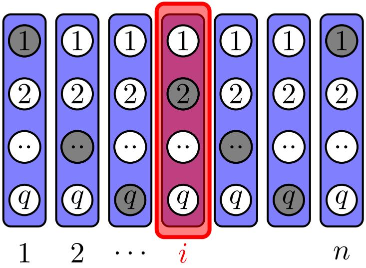

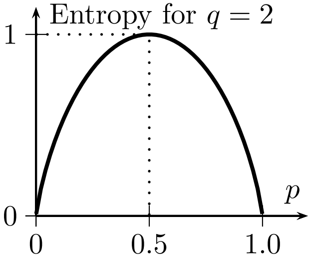

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Corollary

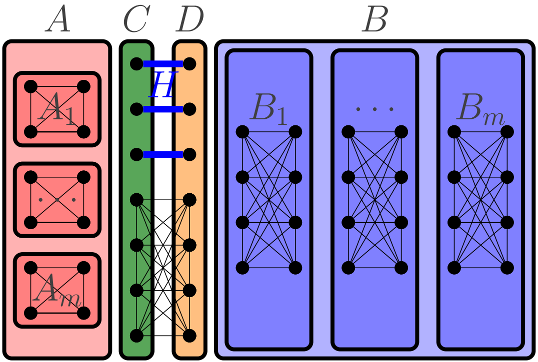
If X large, then for most i

$$\Pr_{x \sim [q]^n}[x \in X] \approx \Pr_{x \sim [q]^n}[x \in X \mid x_i = j]$$

M-good

Definition

(T, H) **M-good** if $M \sim \{M \in R \mid H \subseteq M \subseteq E(T)\}$ is ε-uniform on $(C \cup D) \setminus V(H)$.



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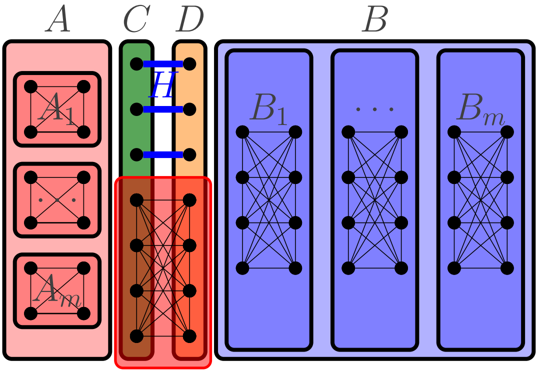

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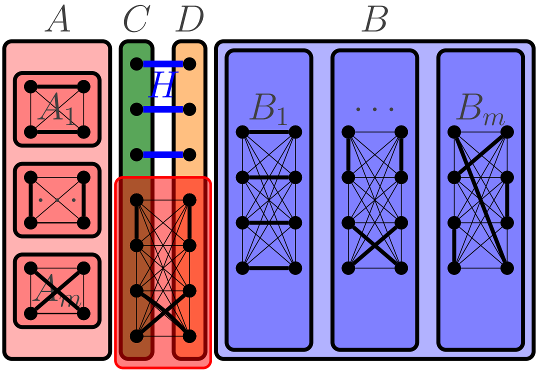
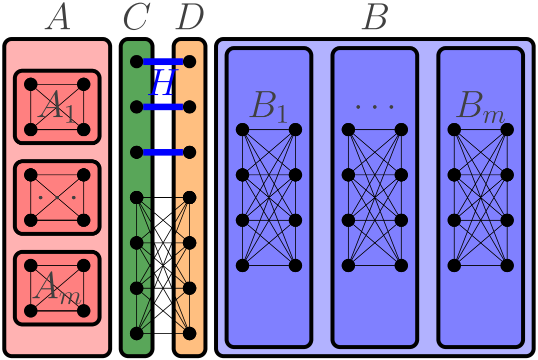


U-good



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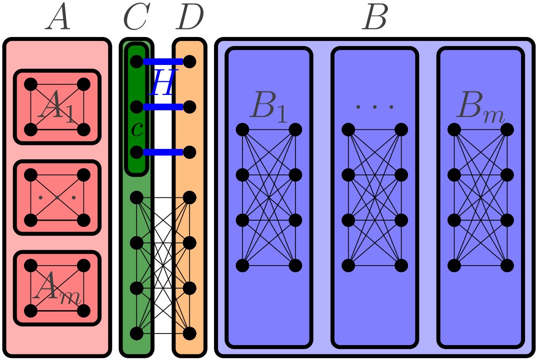

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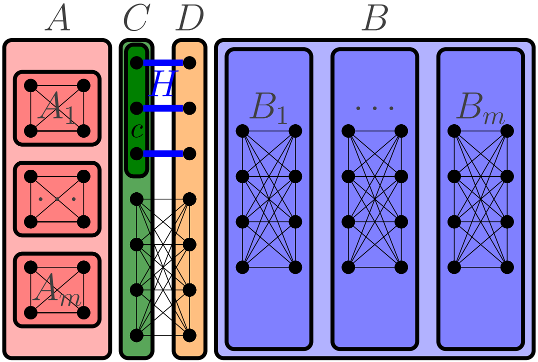

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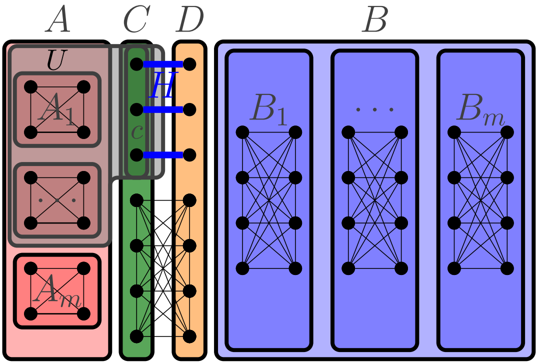

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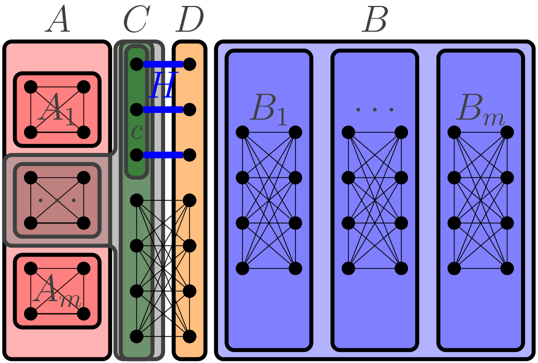

Splitting $\mu_3(R)$

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$$\mu_3(R) \quad = \quad \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U,M)\in R \mid T,H]\right]\right]$$

Splitting $\mu_3(R)$

$$
\begin{aligned}
\mu_3(R) \quad &= \quad \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U,M)\in R \mid T,H]\right]\right] \\
&\leq \quad \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\texttt{GOOD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right] \\
&\quad + \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[M\text{--}\texttt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right] \\
&\quad + \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[U\text{--}\texttt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right]
\end{aligned}
$$

Splitting $\mu_3(R)$

$$
\begin{aligned}
\mu_3(R) &= \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3} \left[\Pr[(U,M) \in R \mid T,H] \right] \right] \\
&\le \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3} \left[\mathtt{GOOD}(T,H) \cdot \Pr[(U,M) \in R \mid T,H] \right] \right] \\
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\end{aligned}
$$

$\le O(\frac{1}{k^2})\mu_k(R)$

Splitting $\mu_3(R)$

$$\mu_3(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U,M)\in R \mid T,H]\right]\right]$$

$$\le \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\texttt{GOOD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right]$$

$$+ \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[M\!-\!\texttt{BAD}(T,H)\cdot\Pr[(U,M)\in R \mid T,H]\right]\right]$$

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$$\le O\left(\tfrac{1}{k^2}\right)\mu_k(R)$$

$$\le \varepsilon\cdot\mu_3(R) + 2^{-\Omega(n)}$$

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$$
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\mu_3(R) &= \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|H|=3}\left[\Pr[(U,M)\in R \mid T,H]\right]\right] \\
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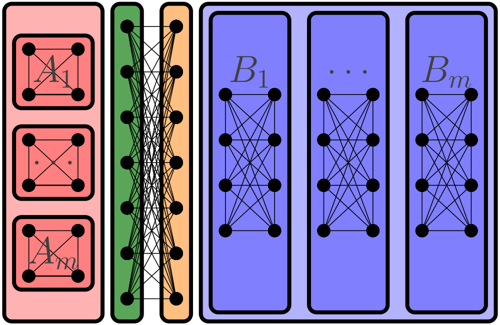
Contribution of good partitions

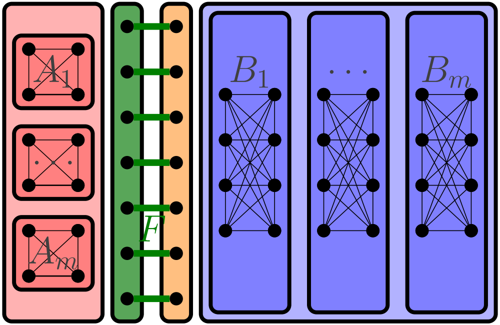
For T



Contribution of good partitions

For T and $F \subseteq C \times D$ with $|F| = k$ compare:

- Contribution to $\mu_k(R)$:



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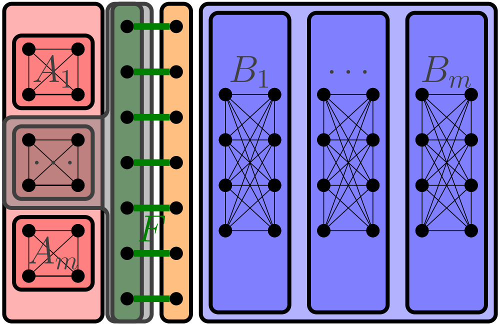

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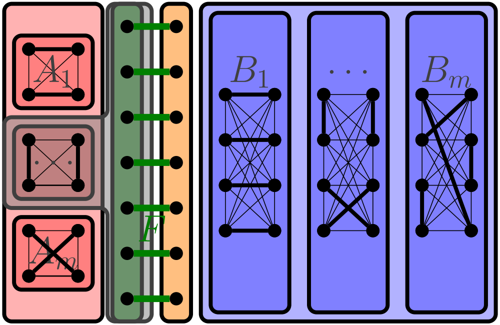

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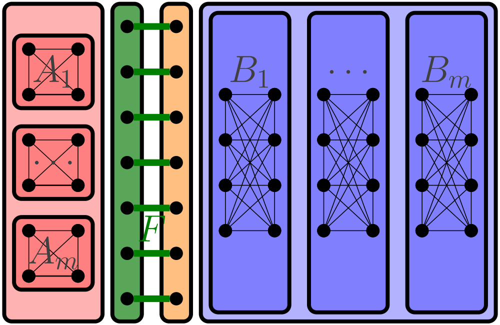

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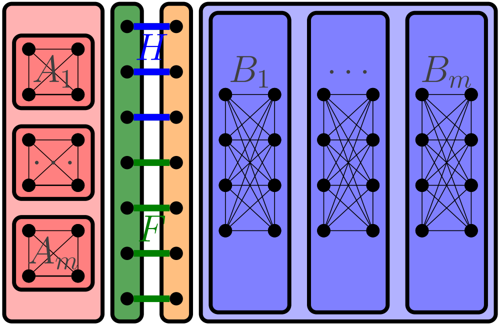

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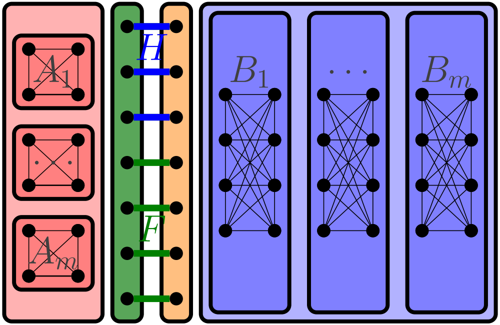

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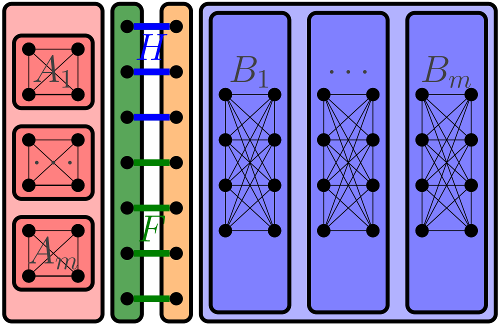

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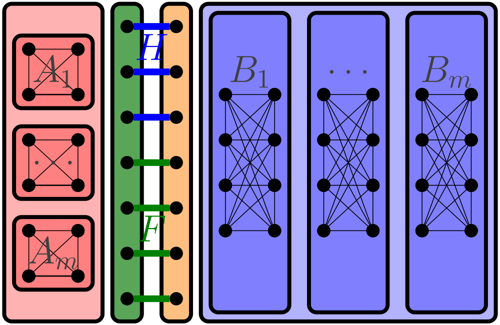

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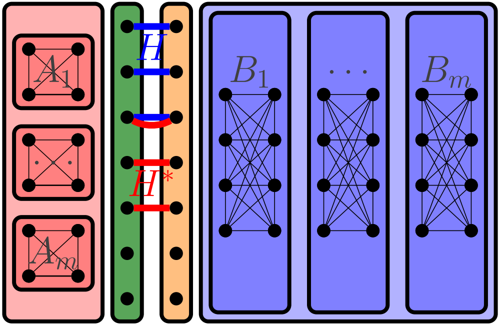

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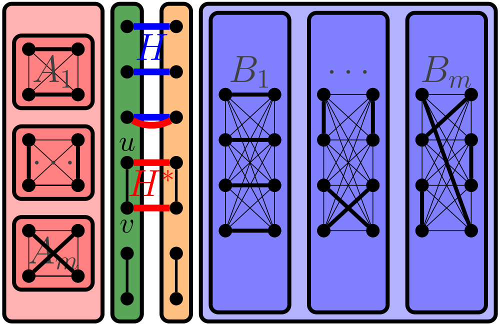

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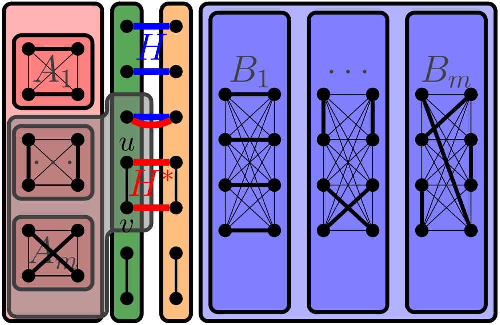

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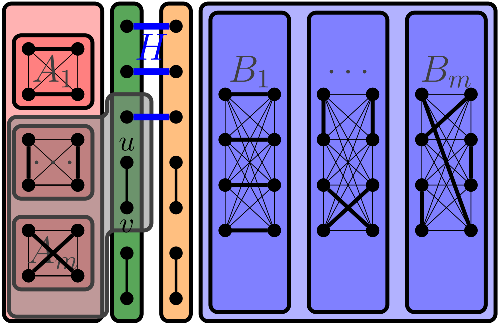
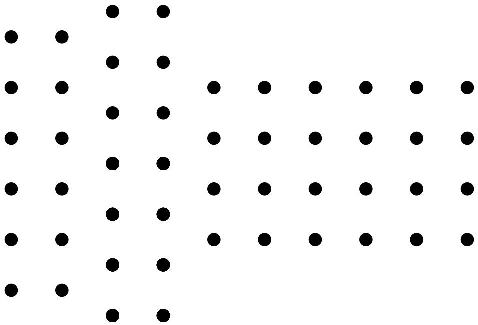
- (T,H) good $\Rightarrow \exists M : \{u,v\} \in M$
- (T,H^*) good $\Rightarrow \exists U : u, v \in U$
- $|\delta(U) \cap M| = 1$ Contradiction!



Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$



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Lemma

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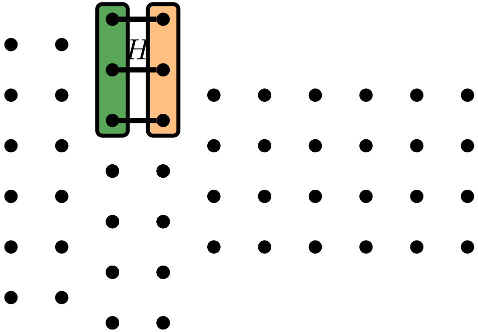
- Pick H



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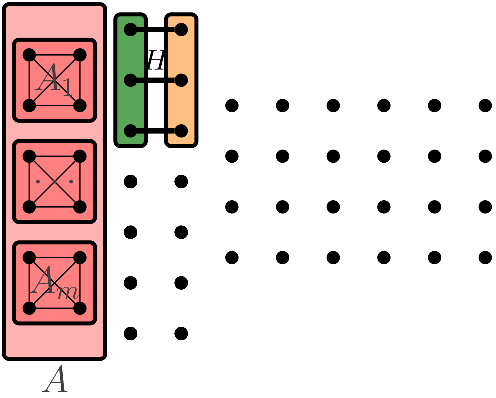
- Pick H, A



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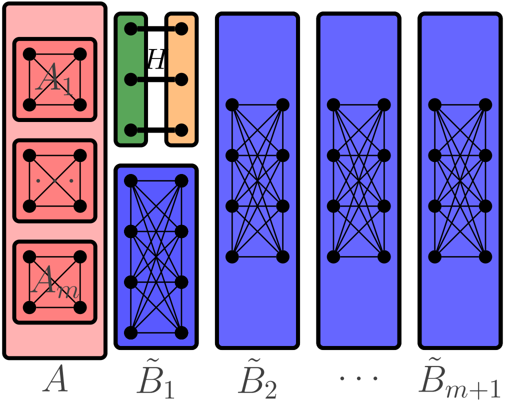
- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$.



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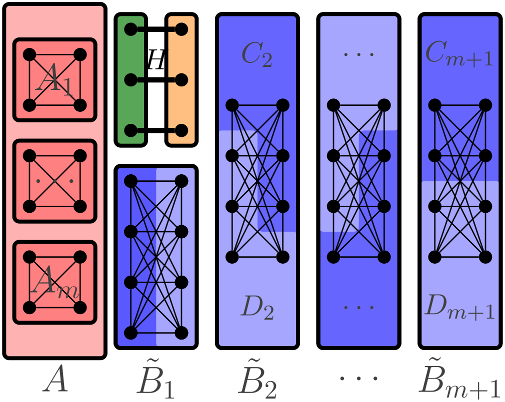

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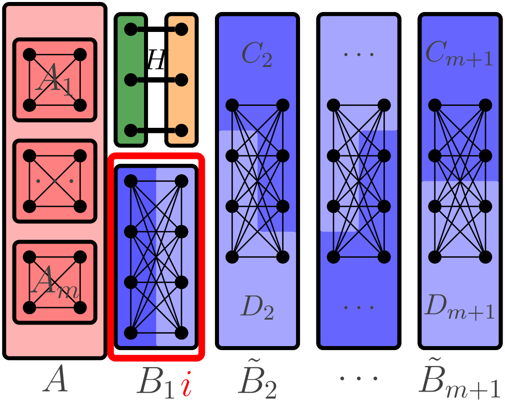
- Pick randomly $i \in \{1, \ldots, m\}$



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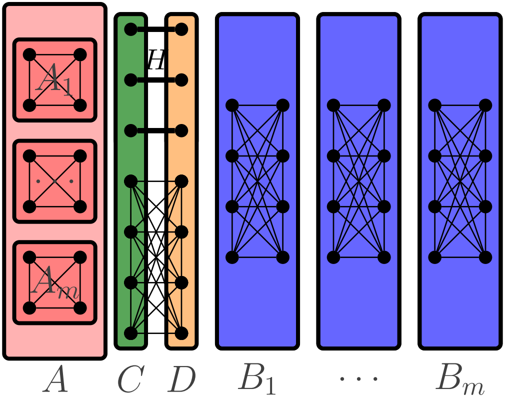
- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$. Split $\tilde{B}_i = C_i \dot{\cup} D_i$.
- Pick randomly $i \in \{1, \ldots, m\}$ and let $C := C_i$, $D := D_i$



Open problems

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Show that there is no small **SDP** representing the Correlation/TSP/matching polytope!

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Thanks for your attention