Kinetic transport in quasicrystals

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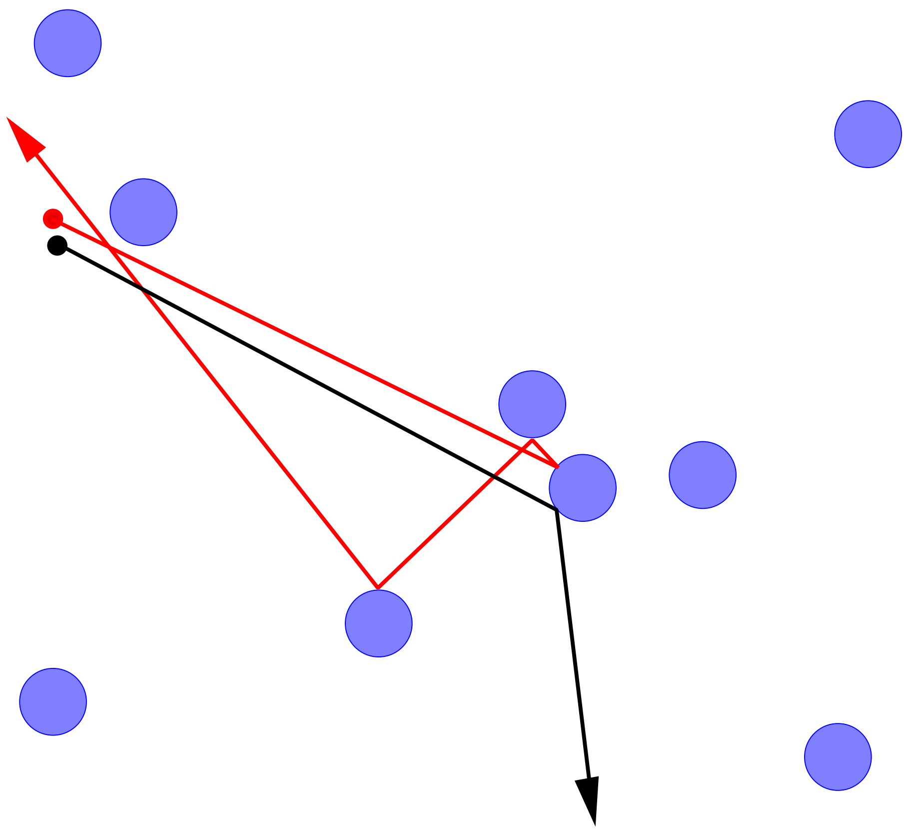
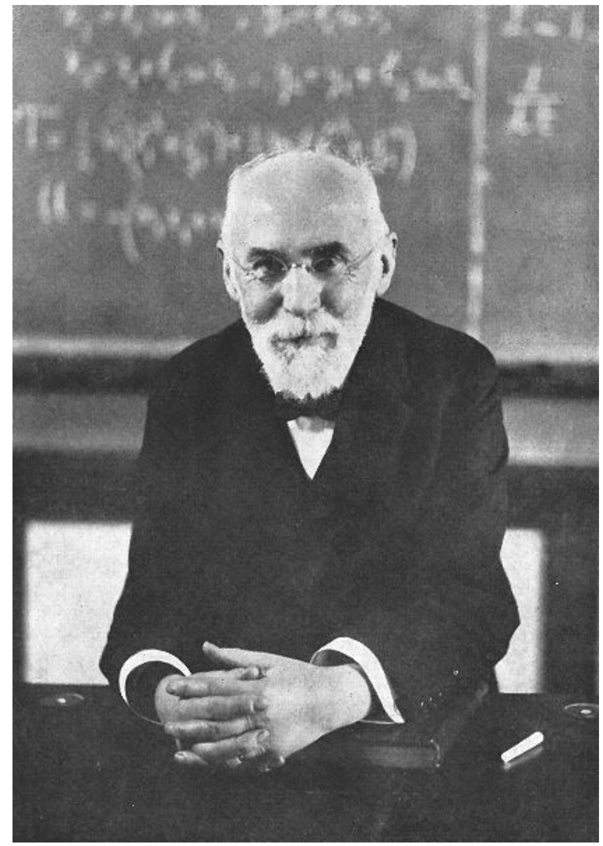
The Lorentz gas



Arch. Neerl. (1905)



Hendrik Lorentz (1853-1928)

The Boltzmann-Grad limit

- Consider the dynamics in the limit of small scatterer radius ρ

- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = "microscopic" phase space coordinate at time t

- A dimensional argument shows that, in the limit $\rho \to 0$, the mean free path length (i.e., the average time between consecutive collisions) scales like $\rho^{-(d-1)}$ (= 1/total scattering cross section)

- We thus re-define position and time and use the "macroscopic" coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t)\big)$$

The linear Boltzmann equation

- Time evolution of initial data $(\boldsymbol{Q},\boldsymbol{V})$:

$$(\boldsymbol{Q}(t),\boldsymbol{V}(t)) = \Phi_\rho^t(\boldsymbol{Q},\boldsymbol{V})$$

- Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f_t^{(\rho)}(\boldsymbol{Q},\boldsymbol{V}) := f\Big(\Phi_\rho^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that $f_t^{(\rho)}$ is governed, as $\rho \to 0$, by the linear Boltzmann equation:

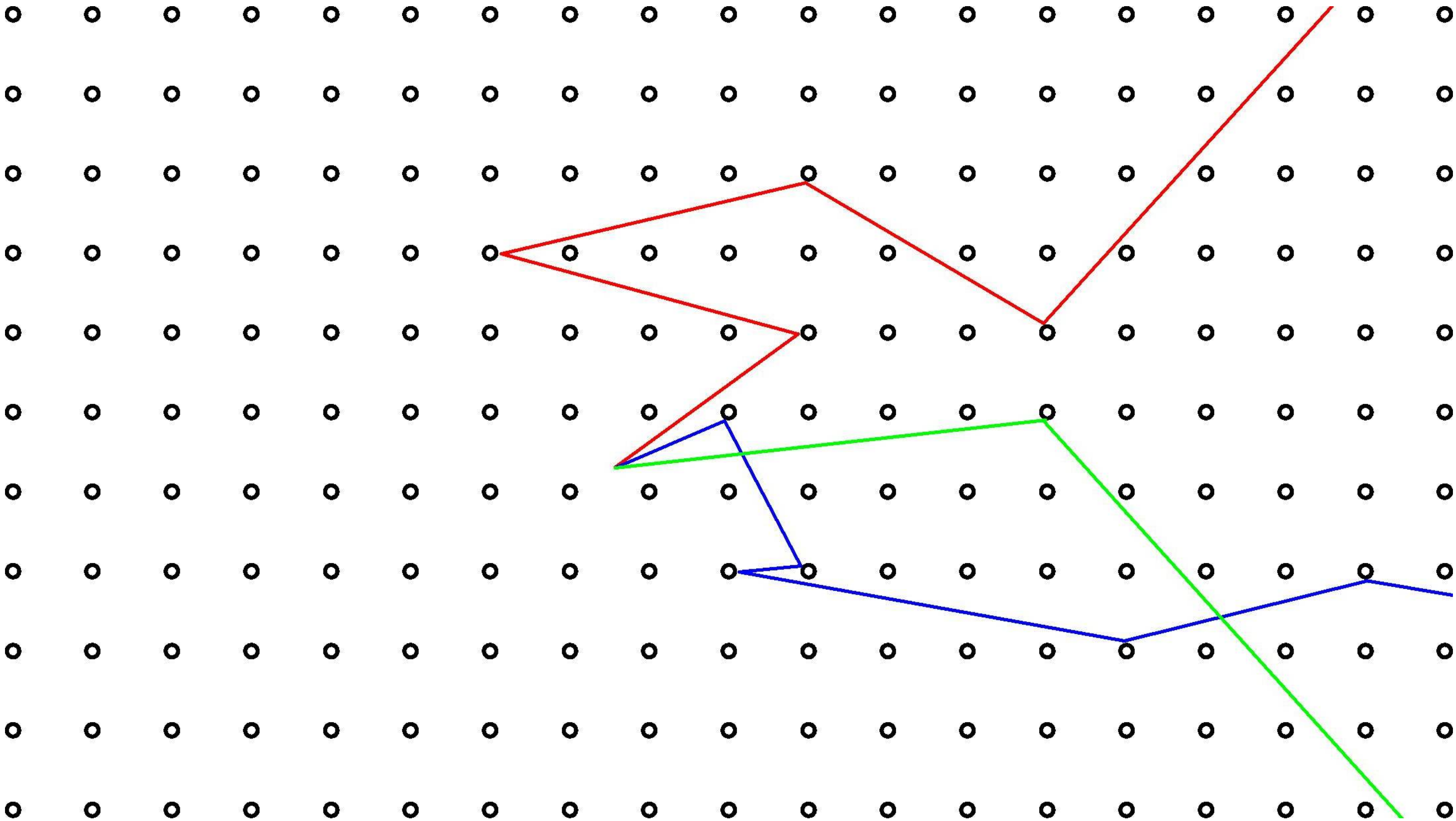
$$\left[\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q},\boldsymbol{V}) = \int_{\mathrm{S}_1^{d-1}} \Big[f_t(\boldsymbol{Q},\boldsymbol{V}_0) - f_t(\boldsymbol{Q},\boldsymbol{V})\Big]\sigma(\boldsymbol{V}_0,\boldsymbol{V})\,d\boldsymbol{V}_0$$

where the collision kernel $\sigma(\boldsymbol{V}_0,\boldsymbol{V})$ is the cross section of the individual scatterer. E.g.: $\sigma(\boldsymbol{V}_0,\boldsymbol{V}) = \frac{1}{4}\|\boldsymbol{V}_0-\boldsymbol{V}\|^{3-d}$ for specular reflection at a hard sphere

The linear Boltzmann equation—rigorous proofs

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterers
- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations and potentials
- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration
- Quantum: Spohn (J Stat Phys 1977): Gaussian random potentials, weak coupling limit & small times; Erdös and Yau (Contemp Math 1998, Comm Pure Appl Math 2000): General random potentials, weak coupling limit; Eng and Erdös (Rev Math Phys 2005): Low density limit

The periodic Lorentz gas



The Boltzmann-Grad limit

- *Recall:* We are interested in the dynamics in the limit of small scatterer radius

- $(\boldsymbol{q}(t), \boldsymbol{v}(t))$ = "microscopic" phase space coordinate at time t

- Re-define position and time and use the "macroscopic" coordinates

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = (\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t))$$

A limiting random process

A cloud of particles with initial density $f(\boldsymbol{Q},\boldsymbol{V})$ evolves in time t to

$$f_t^{(\rho)}(\boldsymbol{Q},\boldsymbol{V}) = [L_\rho^t f](\boldsymbol{Q},\boldsymbol{V}) = f\Big(\Phi_\rho^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big).$$

Theorem A [JM & Strömbergsson, Annals of Math 2011].
For every $t > 0$ there exists a linear operator $L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$, such that for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and any set $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Liouville measure zero,

$$\lim_{\rho\to 0} \int_{\mathcal{A}} [L_\rho^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} [L^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V}.$$

The operator L^t thus describes the macroscopic diffusion of the Lorentz gas in the Boltzmann-Grad limit $\rho \to 0$.

Note: The family $\{L^t\}_{t\geq 0}$ does *not* form a semigroup.

A generalization of the linear Boltzmann equation

Consider extended phase space coordinates $(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$:

$$(\boldsymbol{Q}, \boldsymbol{V}) \in \mathrm{T}^1(\mathbb{R}^d) \text{ — usual position and momentum}$$
$$\xi \in \mathbb{R}_+ \text{ — flight time until the next scatterer}$$
$$\boldsymbol{V}_+ \in \mathrm{S}_1^{d-1} \text{ — velocity after the next hit}$$

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}_0, 0, \boldsymbol{V}) p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) d\boldsymbol{V}_0$$

with a new collision kernel $p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$, which can be expressed as a product of the scattering cross section of an individual scatterer and a ceratin transition probability for hitting a given point the next scatterer after time ξ. We obtain the original particle density via

$$f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_0^\infty \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) \, d\boldsymbol{V}_+ \, d\xi.$$

Why "a generalization" of the linear Boltzmann equation?

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \int_{\mathsf{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}_0, 0, \boldsymbol{V}) p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) d\boldsymbol{V}_0$$

Substituting in the above the transition density for the random (rather than periodic) scatterer configuration

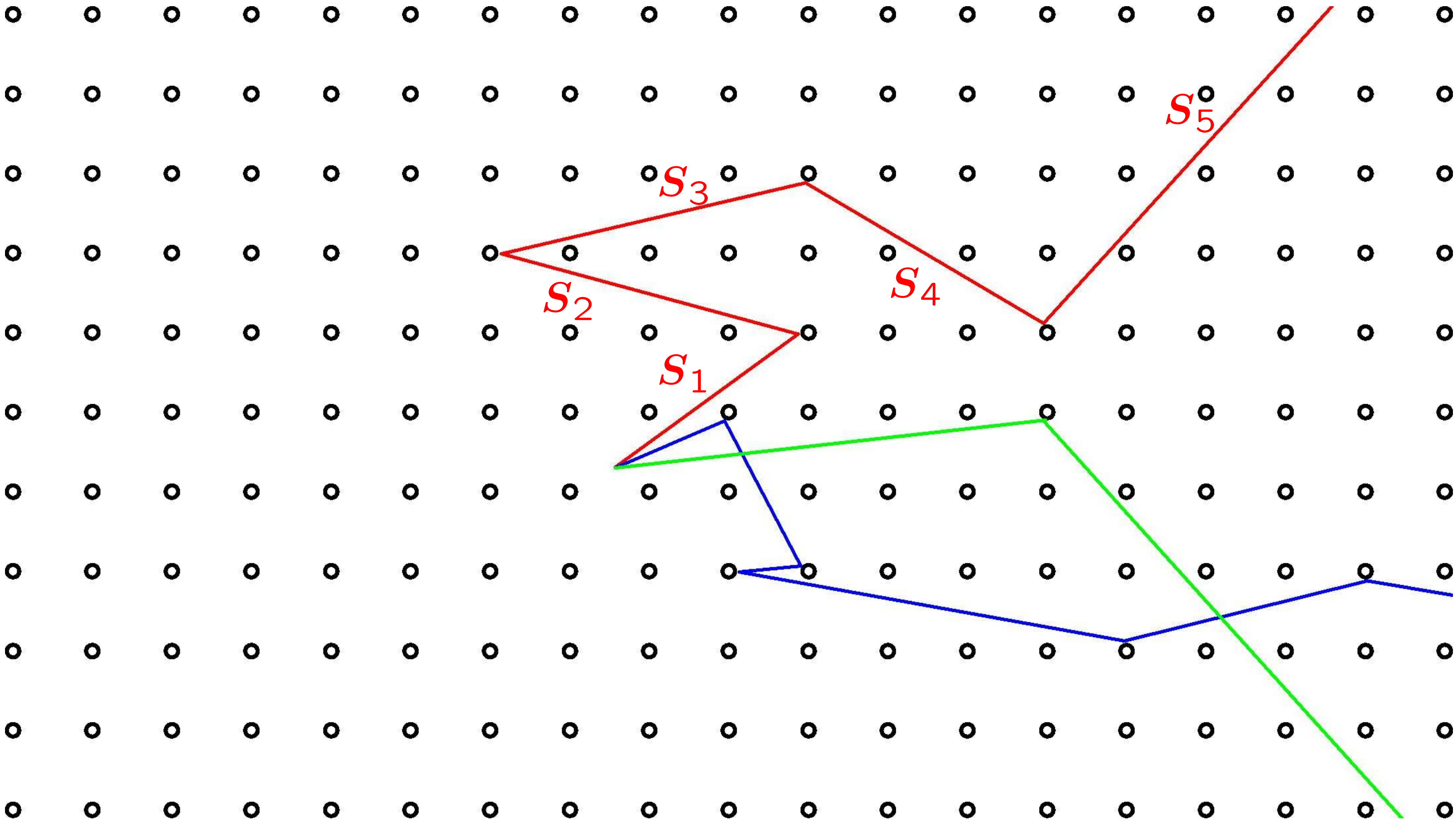
$$f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = g_t(\boldsymbol{Q}, \boldsymbol{V}) \sigma(\boldsymbol{V}, \boldsymbol{V}_+) \mathrm{e}^{-\overline{\sigma}\xi}, \qquad \overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1}),$$

$$p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \sigma(\boldsymbol{V}, \boldsymbol{V}_+) \mathrm{e}^{-\overline{\sigma}\xi}$$

yields the classical linear Boltzmann equation for $g_t(\boldsymbol{Q}, \boldsymbol{V})$.

The key theorem:

Joint distribution of path segments



Joint distribution for path segments

The following theorem proves the existence of a Markov process that describes the dynamics of the Lorentz gas in the Boltzmann-Grad limit.

Theorem B [JM & Strömbergsson, Annals of Math 2011]. Fix an a.c. Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. Then, for each $n \in \mathbb{N}$ there exists a probability density $\Psi_{n,\Lambda}$ on $\mathbb{R}^{nd}$ such that, for any set $\mathcal{A} \subset \mathbb{R}^{nd}$ with boundary of Lebesgue measure zero,

$$\lim_{\rho\to 0} \Lambda\Big(\Big\{(\boldsymbol{Q}_0, \boldsymbol{V}_0) \in \mathrm{T}^1(\mathbb{R}^d) : (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n) \in \mathcal{A}\Big\}\Big) = \int_{\mathcal{A}} \Psi_{n,\Lambda}(\boldsymbol{S}_1', \ldots, \boldsymbol{S}_n')\, d\boldsymbol{S}_1' \cdots d\boldsymbol{S}_n',$$

and, for $n \geq 3$,

$$\Psi_{n,\Lambda}(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n) = \Psi_{2,\Lambda}(\boldsymbol{S}_1, \boldsymbol{S}_2) \prod_{j=3}^{n} \Psi(\boldsymbol{S}_{j-2}, \boldsymbol{S}_{j-1}, \boldsymbol{S}_j),$$

where Ψ is a continuous probability density independent of Λ (and the lattice).

Theorem A follows from Theorem B by standard probabilistic arguments.

First step: The distribution of free path lengths

References

- Polya (Arch Math Phys 1918): “Visibility in a forest” $(d = 2)$
- Dahlquist (Nonlinearity 1997); Boca, Cobeli, Zaharescu (CMP 2000); Caglioti, Golse (CMP 2003); Boca, Gologan, Zaharescu (CMP 2003); Boca, Zaharescu (CMP 2007): Limit distributions for the free path lengths for various sets of initial data $(d = 2)$
- Dumas, Dumas, Golse (J Stat Phys 1997): Asymptotics of mean free path lengths $(d \geq 2)$
- Bourgain, Golse, Wennberg (CMP 1998); Golse, Wennberg (CMP 2000): bounds on possible weak limits $(d \geq 2)$
- Boca & Gologan (Annales I Fourier 2009), Boca (NY J Math 2010): honeycomb lattice
- JM & Strömbergsson (Annals of Math 2010, 2011, GAFA 2011): proof of limit distribution and tail estimates in arbitrary dimension

Lattices

- $\mathcal{L} \subset \mathbb{R}^d$—euclidean lattice of covolume one

- recall $\mathcal{L} = \mathbb{Z}^d M$ for some $M \in \mathsf{SL}(d,\mathbb{R})$, therefore the homogeneous space $X_1 = \mathsf{SL}(d,\mathbb{Z})\backslash \mathsf{SL}(d,\mathbb{R})$ parametrizes the space of lattices of covolume one

- μ_1—right-$\mathsf{SL}(d,\mathbb{R})$ invariant prob measure on X_1 (Haar)

Affine lattices

- $\mathsf{ASL}(d,\mathbb{R}) = \mathsf{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^d$—the semidirect product group with multiplication law

$$(M,\boldsymbol{x})(M',\boldsymbol{x}') = (MM', \boldsymbol{x}M' + \boldsymbol{x}').$$

  An action of $\mathsf{ASL}(d,\mathbb{R})$ on $\mathbb{R}^d$ can be defined as

$$\boldsymbol{y} \mapsto \boldsymbol{y}(M,\boldsymbol{x}) := \boldsymbol{y}M + \boldsymbol{x}.$$

- the space of affine lattices is then represented by $X = \mathsf{ASL}(d,\mathbb{Z})\backslash \mathsf{ASL}(d,\mathbb{R})$ where $\mathsf{ASL}(d,\mathbb{Z}) = \mathsf{SL}(d,\mathbb{Z}) \ltimes \mathbb{Z}^d$, i.e.,

$$\mathcal{L} = (\mathbb{Z}^d + \boldsymbol{\alpha})M = \mathbb{Z}^d(1,\boldsymbol{\alpha})(M,\boldsymbol{0})$$

- μ—right-$\mathsf{ASL}(d,\mathbb{R})$ invariant prob measure on X

Let us denote by $\tau_1 = \tau(\boldsymbol{q}, \boldsymbol{v})$ the free path length corresponding to the initial condition $(\boldsymbol{q}, \boldsymbol{v})$.

Theorem C [JM & Strömbergsson, Annals of Math 2010]. Fix a lattice $\mathcal{L}_0$ and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{L}_0,\boldsymbol{q}}(\xi) := \lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

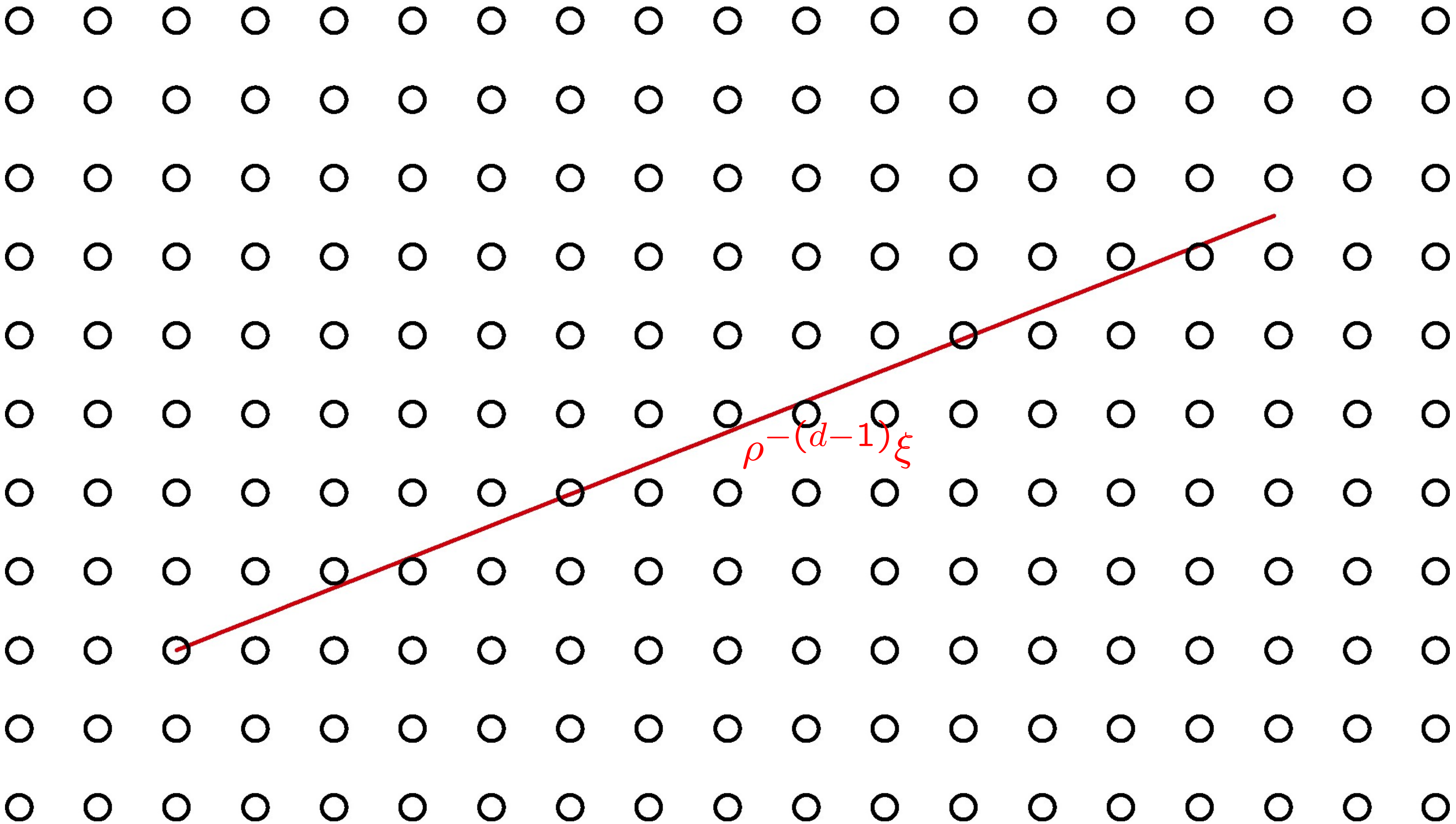
exists, is continuous in ξ and independent of λ. Furthermore

$$F_{\mathcal{L}_0,\boldsymbol{q}}(\xi) = \begin{cases} F_{\mathbf{0}}(\xi) := \mu_1(\{\mathcal{L} \in X_1 : \mathcal{L} \cap \mathcal{Z}(\xi) = \emptyset\}) & \text{if } \boldsymbol{q} \in \mathcal{L}_0 \\ F(\xi) := \mu(\{(\mathcal{L} \in X : \mathcal{L} \cap \mathcal{Z}(\xi) = \emptyset\}) & \text{if } \boldsymbol{q} \notin \mathbb{Q}\mathcal{L}_0. \end{cases}$$

with the cylinder

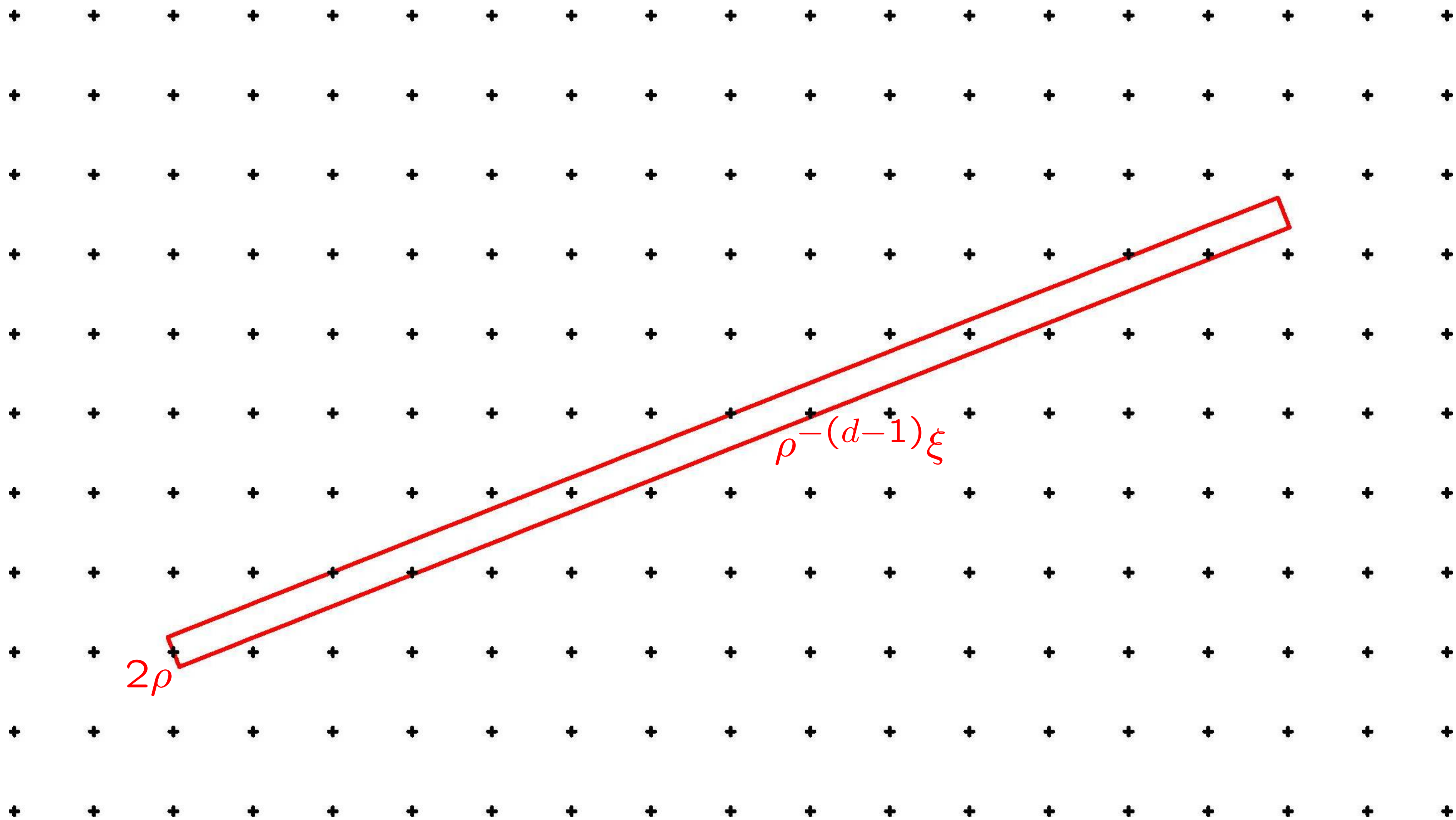
$$\mathcal{Z}(\xi) = \Big\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_1 < \xi, x_2^2 + \ldots + x_d^2 < 1\Big\}.$$

Idea of proof $(q = 0)$



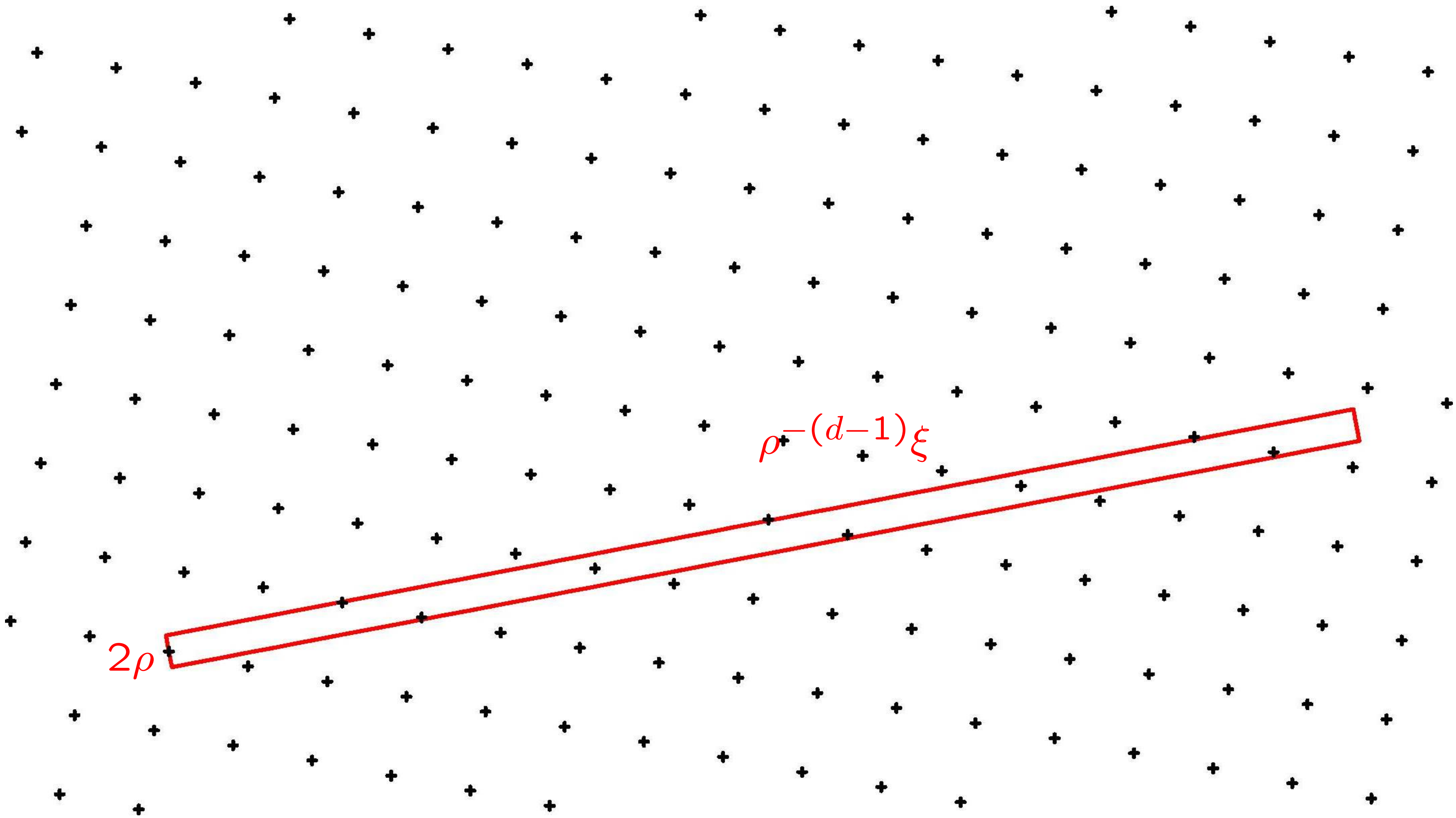
$= \lambda\Big(\Big\{ \boldsymbol{v} \in \mathsf{S}_1^{d-1} :$ no scatterer intersects ray$(\boldsymbol{v}, \rho^{-(d-1)}\xi)\Big\}\Big)$

Idea of proof $(q = 0)$



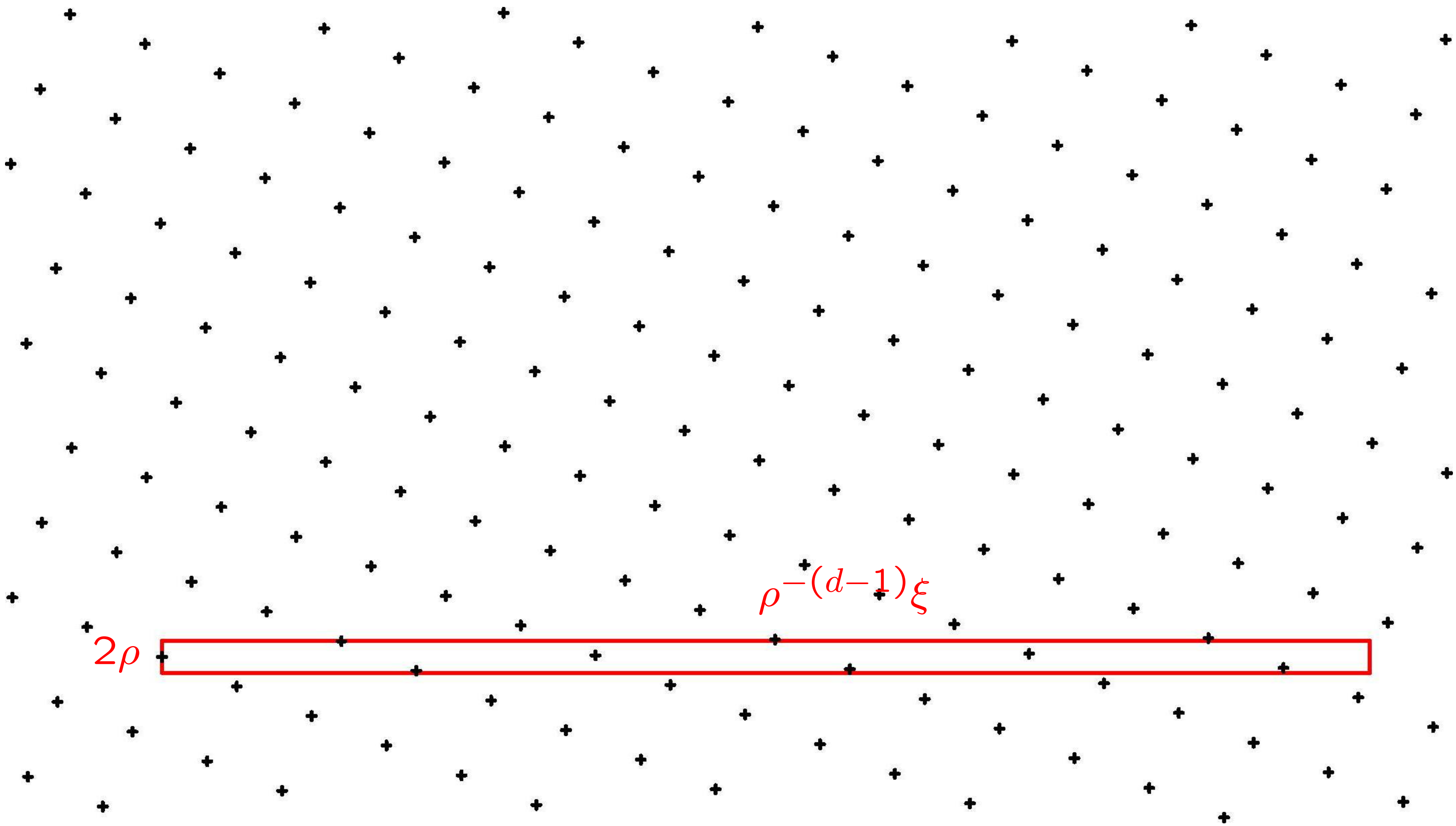
$$\approx \lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d \cap \mathcal{Z}(\boldsymbol{v}, \rho^{-(d-1)}\xi, \rho) = \emptyset\Big\}\Big)$$

Idea of proof ($q = 0$)



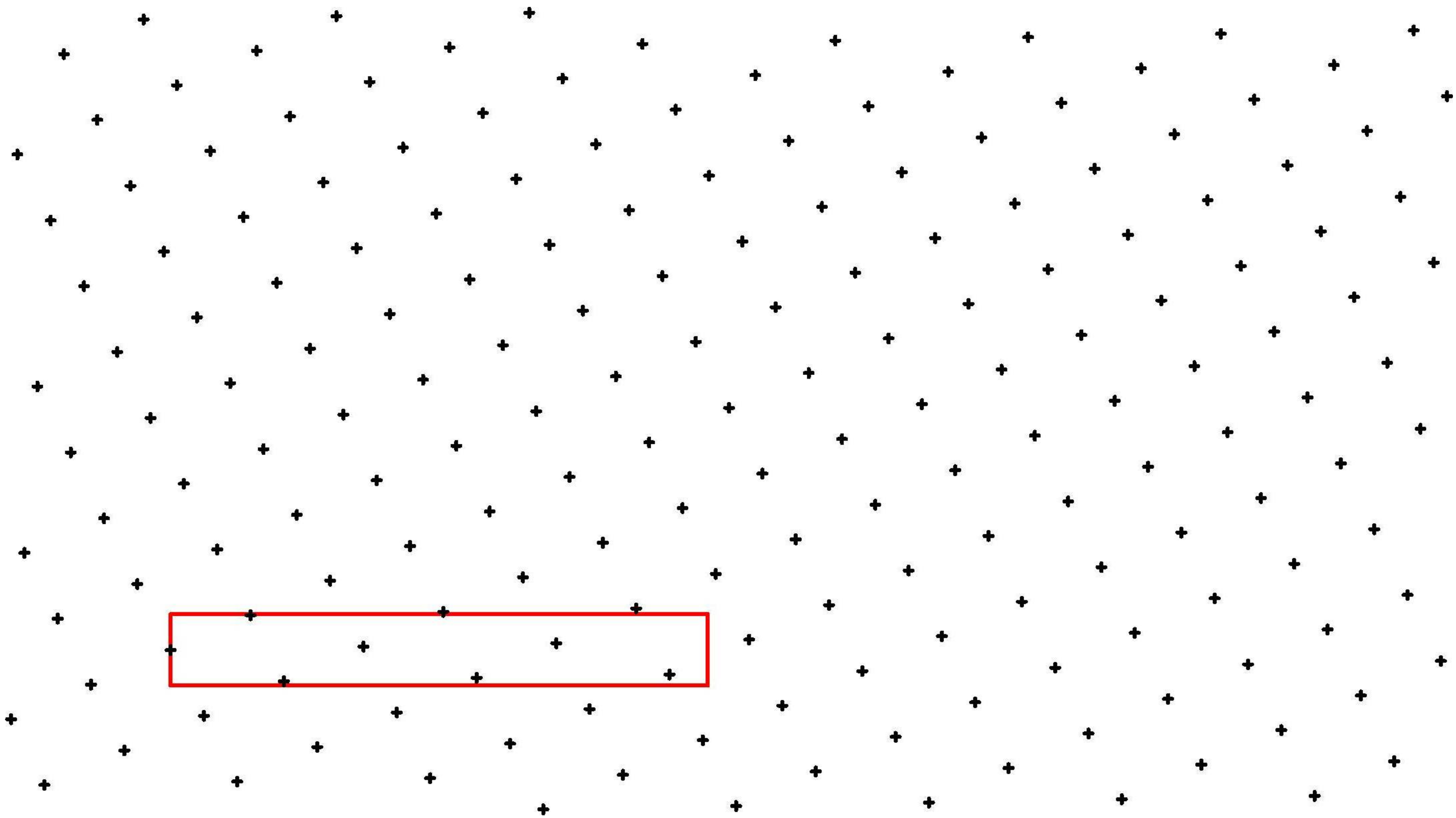
$\Big($Rotate by $K(v) \in \mathrm{SO}(d)$ such that $v \mapsto e_1\Big)$

Idea of proof ($q = 0$)



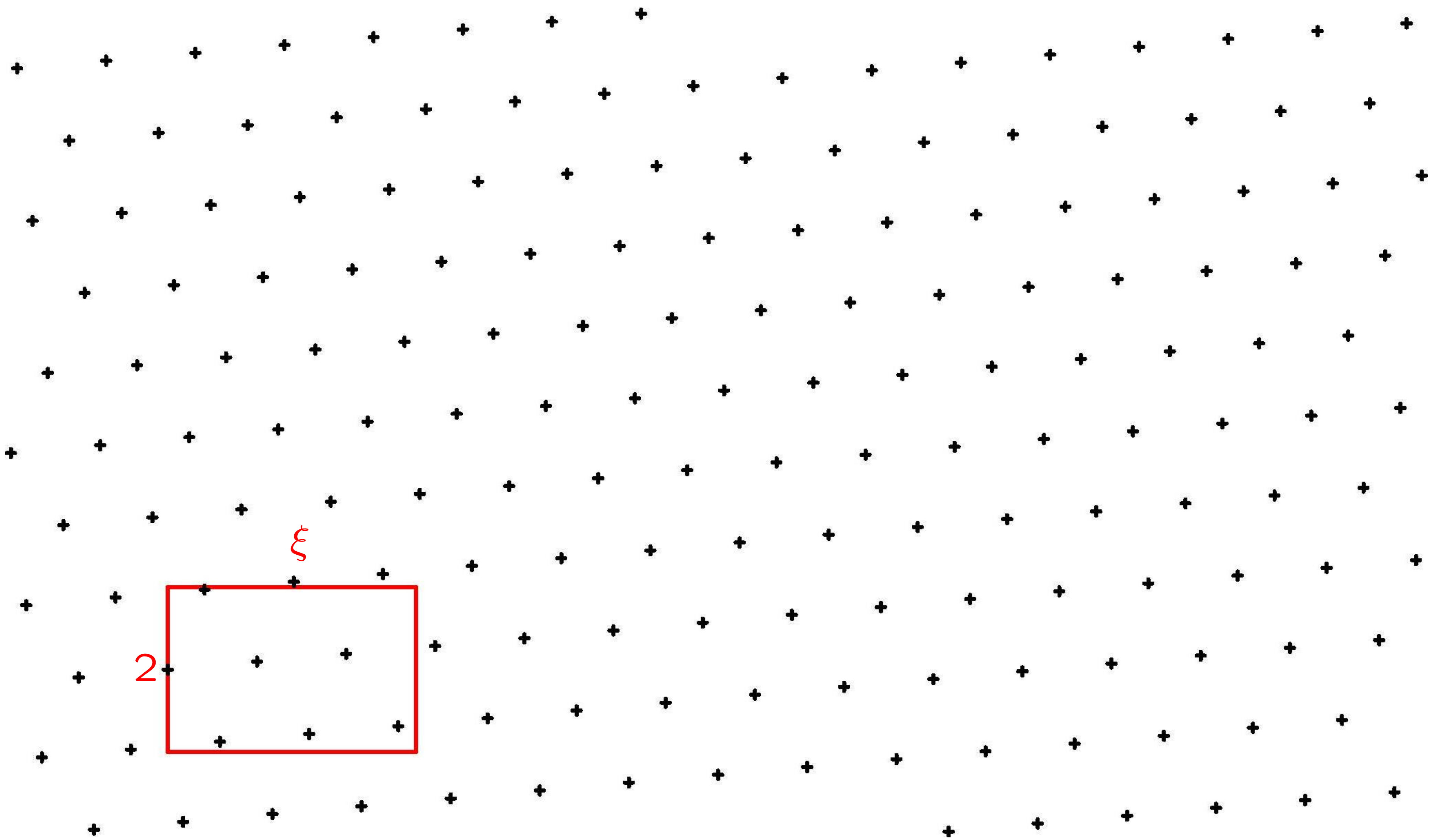
$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(\boldsymbol{v}) \cap \mathcal{Z}(\boldsymbol{e}_1, \rho^{-(d-1)}\xi, \rho) = \emptyset\Big\}\Big)$$

Idea of proof ($q = 0$)



$\left(\text{Apply } D_\rho = \mathrm{diag}(\rho^{d-1}, \rho^{-1}, \ldots, \rho^{-1}) \in \mathsf{SL}(n, \mathbb{R})\right)$

Idea of proof $(q = 0)$



$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(\boldsymbol{v}) D_\rho \cap \mathcal{Z}(\boldsymbol{e}_1, \xi, 1) = \emptyset\Big\}\Big)$$

The following Theorem shows that in the limit $\rho \to 0$ the lattice

$$\mathbb{Z}^d K(\boldsymbol{v}) \begin{pmatrix} \rho^{d-1} & \mathbf{0} \\ {}^t\mathbf{0} & \rho^{-1} 1 \end{pmatrix}$$

behaves like a random lattice with respect to Haar measure μ_1. Define a flow on $X_1 = \mathrm{SL}(n,\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{R})$ via right translation by

$$\Phi^t = \begin{pmatrix} \mathrm{e}^{-(d-1)t} & \mathbf{0} \\ {}^t\mathbf{0} & \mathrm{e}^t 1 \end{pmatrix}.$$

Theorem D. Fix any $M_0 \in \mathrm{SL}(n,\mathbb{R})$. Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then, for every bounded continuous function $f : X_1 \to \mathbb{R}$,

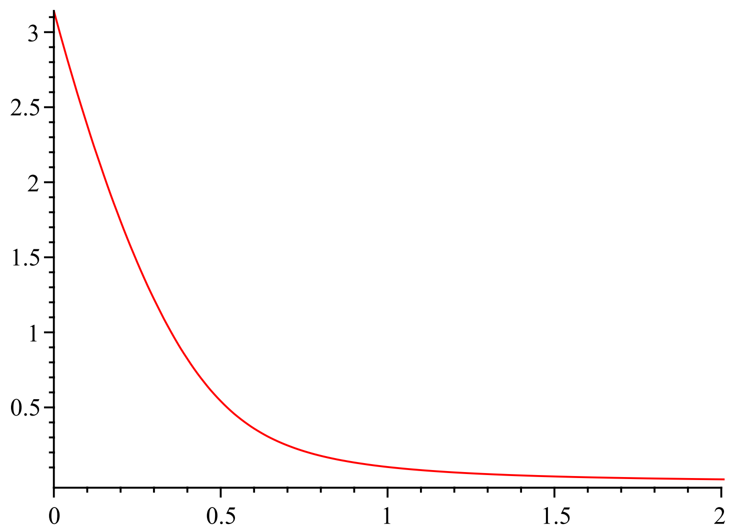
$$\lim_{t\to\infty} \int_{\mathrm{S}_1^{d-1}} f(M_0 K(\boldsymbol{v}) \Phi^t) d\lambda(\boldsymbol{v}) = \int_{X_1} f(M) d\mu_1(M).$$

Theorem D is a direct consequence of the mixing property for the flow Φ^t.

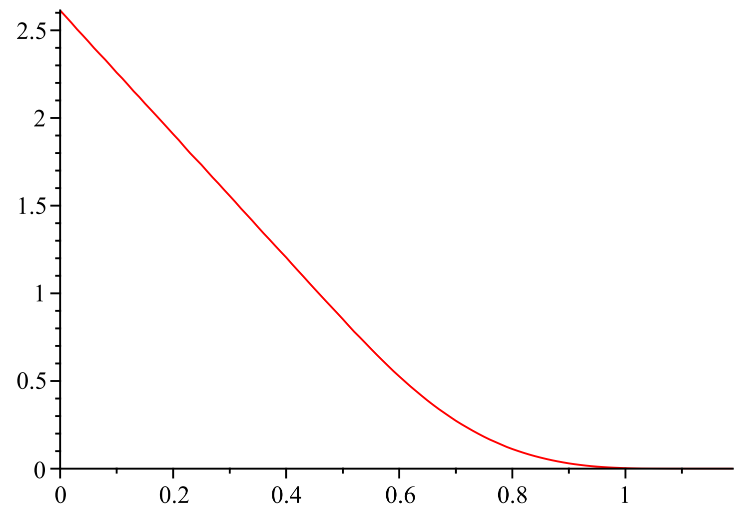
This concludes the proof of Theorem C when $\boldsymbol{q} \in \mathcal{L} = \mathbb{Z}^d M_0$.

The generalization of Theorem D required for the full proof of Theorem C uses Ratner's classification of ergodic measures invariant under a unipotent flow. We exploit a close variant of a theorem by N. Shah (Proc. Ind. Acad. Sci. 1996) on the uniform distribution of translates of unipotent orbits.

Limiting densities for $d = 3$



$\Phi(\xi) = -\frac{d}{d\xi}F(\xi)$



$\Phi_{\mathbf{0}}(\xi) = -\frac{d}{d\xi}F_{\mathbf{0}}(\xi)$

For random scatterer configuarions:

$\Phi(\xi) = \overline{\sigma}\mathrm{e}^{-\overline{\sigma}\xi}$, $\Phi_{\mathbf{0}}(\xi) = \overline{\sigma}\mathrm{e}^{-\overline{\sigma}\xi}$ with $\overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1})$

Tail asymptotics [JM & Strömbergsson, GAFA 2011].

$$\Phi(\xi) = \frac{\pi^{\frac{d-1}{2}}}{2^d d\, \Gamma(\frac{d+3}{2})\, \zeta(d)}\, \xi^{-2} + O\Big(\xi^{-2-\frac{2}{d}}\Big) \qquad \text{as } \xi \to \infty$$

$$\Phi_0(\xi) = 0 \qquad \text{for } \xi \text{ sufficiently large}$$

$$\Phi(\xi) = \overline{\sigma} - \frac{\overline{\sigma}^2}{\zeta(d)}\, \xi + O\Big(\xi^2\Big) \qquad \text{as } \xi \to 0$$

$$\Phi_0(\xi) = \frac{\overline{\sigma}}{\zeta(d)} + O(\xi) \qquad \text{as } \xi \to 0.$$

with $\overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1}) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}$.

Quasicrystals

Cut and project

- $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$, π and π_{int} orthogonal projections onto $\mathbb{R}^d$, $\mathbb{R}^m$

- $\mathcal{L} \subset \mathbb{R}^n$ an (in general) affine lattice of full rank

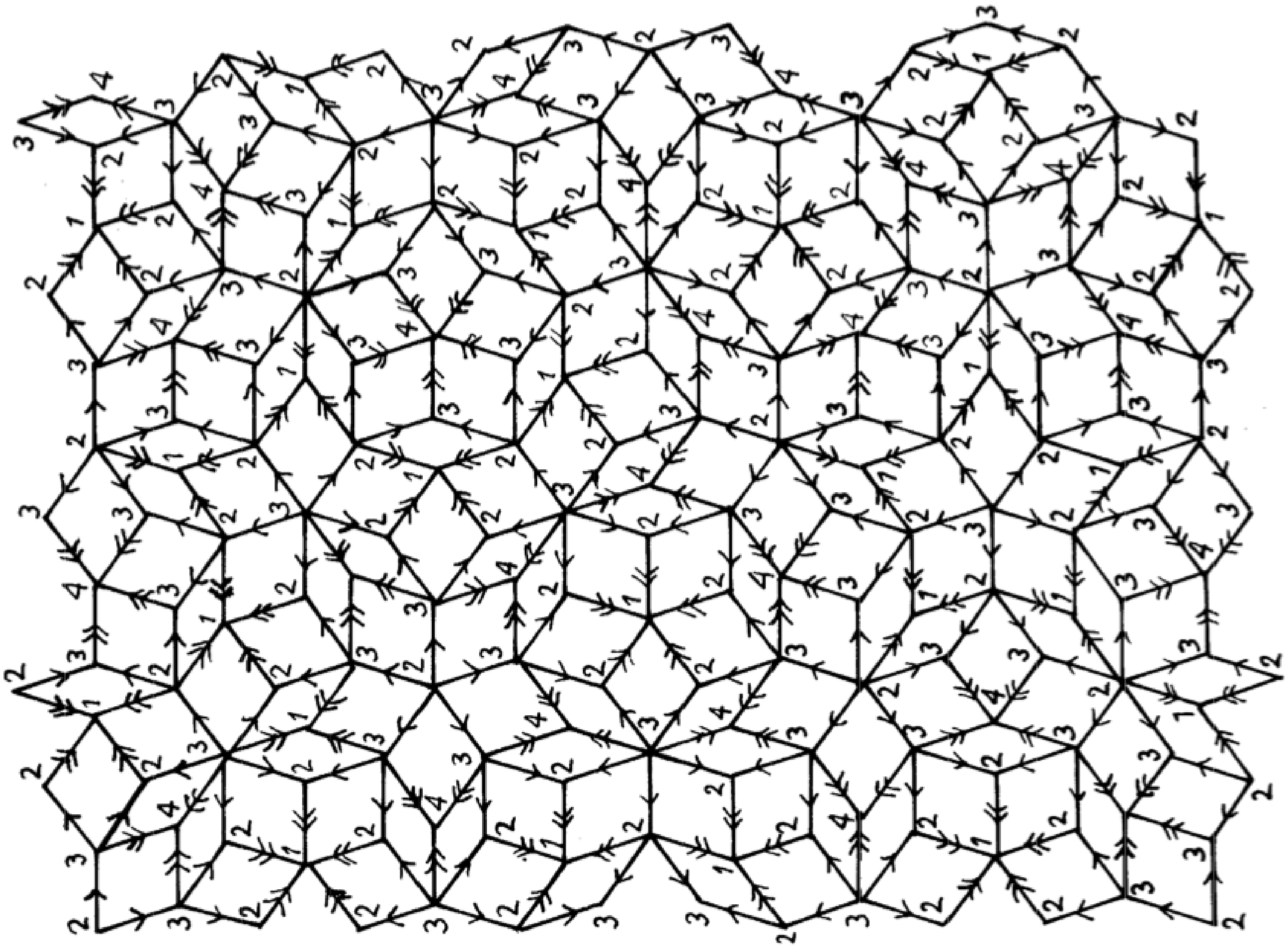
- $\mathcal{A} := \overline{\pi_{\mathsf{int}}(\mathcal{L})}$ is an abelian subgroup of $\mathbb{R}^m$, with Haar measure $\mu_{\mathcal{A}}$

- $\mathcal{W} \subset \mathcal{A}$ a “regular window set”
  (i.e. bounded with non-empty interior, $\mu_{\mathcal{A}}(\partial\mathcal{W}) = 0$)

- $\mathcal{P}(\mathcal{W}, \mathcal{L}) = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{L},\ \pi_{\mathsf{int}}(\boldsymbol{y}) \in \mathcal{W}\} \subset \mathbb{R}^d$
  is called a “regular cut-and-project set”

- $\mathcal{P}(\mathcal{W}, \mathcal{L})$ defines the locations of scatterers in our quasicrystal

Example: The Penrose tiling



(from: de Bruijn, Kon Nederl Akad Wetensch Proc Ser A, 1981)

Density

We have the following well known facts:

- $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is a Delone set, i.e., uniformly discrete and relatively dense in $\mathbb{R}^d$

- For any bounded $\mathcal{D} \subset \mathbb{R}^d$ with boundary of Lebesgue measure zero,

$$\lim_{T \to \infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = c_{\mathcal{L}} \, \mathrm{vol}(\mathcal{D}) \mu_{\mathcal{A}}(\mathcal{W})$$

(the constant $c_{\mathcal{L}}$ is explicit)

Recall $\tau_1 = \tau(\boldsymbol{q}, \boldsymbol{v})$ denotes the free path length corresponding to the initial condition $(\boldsymbol{q}, \boldsymbol{v})$.

Theorem E [JM & Strömbergsson, 2013]. Fix a regular cut-and-project set $\mathcal{P}_0$ and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) := \lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

exists, is continuous in ξ and independent of λ.

In analogy with Theorem C and the space of lattices, we will express $F_{\mathcal{P}_0,\boldsymbol{q}}(\xi)$ in terms of a random variable in the space of quasicrystals.

Spaces of quasicrystals (I)

- Set $G = \mathrm{ASL}(n,\mathbb{R})$, $\Gamma = \mathrm{ASL}(n,\mathbb{Z})$.

- For $g \in G$, define an embedding of $\mathrm{SL}(d,\mathbb{R})$ in G by the map

$$\varphi_g : \mathrm{SL}(d,\mathbb{R}) \to G, \quad A \mapsto g\left(\begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix}, \mathbf{0}\right) g^{-1}.$$

- It follows from Ratner's theorems that there exists a closed connected subgroup H_g ofc G such that
  - $\Gamma \cap H_g$ is a lattice in H_g
  - $\varphi_g(\mathrm{SL}(d,\mathbb{R})) \subset H_g$
  - the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by $\Gamma\backslash\Gamma H_g$.

- Denote the unique right-H_g invariant probability measure on $\Gamma\backslash\Gamma H_g$ by μ_g.

Spaces of quasicrystals (II)

- Given an affine lattice $\mathcal{L} \subset \mathbb{R}^n$ as above, choose $\delta > 0$ and $g \in G$ so that $\mathcal{L} = \delta^{1/n}\mathbb{Z}^n g$. Then $\mathcal{A} = \overline{\pi_{\mathrm{int}}(\mathbb{Z}^n g)}$.

- Since $\Gamma\backslash\Gamma H_g$ is the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d,\mathbb{R}))$, it follows that $\overline{\pi_{\mathrm{int}}(\mathbb{Z}^n hg)} \subset \mathcal{A}$ for all $h \in H_g$, and $\overline{\pi_{\mathrm{int}}(\mathbb{Z}^n hg)} = \mathcal{A}$ for almost all $h \in H_g$.

Spaces of quasicrystals (III)

- For each fixed $\delta > 0$ and window $\mathcal{W} \subset \mathcal{A}$, the map from $\Gamma\backslash\Gamma H_g$ to the set of point sets in $\mathbb{R}^d$, $\Gamma\backslash\Gamma h \mapsto \mathcal{P}(\delta^{1/n}\mathbb{Z}^n hg, \mathcal{W})$ gives a natural parametrisation of a space of quasicrystals by $\Gamma\backslash\Gamma H_g$.

- Denote the image of this map by $\mathfrak{Q}_g = \mathfrak{Q}_g(\mathcal{W}, \delta)$, and define a probability measure on $\mathfrak{Q}_g$ as the push-forward of μ_g (for which we will use the same symbol). This defines a random point process in $\mathbb{R}^d$.

- In the topology inherited from $\Gamma\backslash\Gamma H_g$, the space $\mathfrak{Q}_g$ is the closure of the set $\{\mathcal{P}(\mathcal{L}h, \mathcal{W}) : h \in \mathrm{SL}(d, \mathbb{R})\}$, with the (in general) affine lattice $\mathcal{L} = \delta^{1/n}\mathbb{Z}^n g$.

Theorem F [JM & Strömbergsson, 2013].

For $\mathcal{P}_0 = \mathcal{P}(\mathcal{L}, \mathcal{W})$ and initial position $\boldsymbol{q}$, pick $g \in G$ and $\delta > 0$ such that $\mathcal{L} - (\boldsymbol{q}, 0) = \delta^{1/n}\mathbb{Z}^n g$. Then

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) = \mu_g(\{\mathcal{P} \in \mathfrak{Q}_g : \mathfrak{Z}_\xi \cap \mathcal{P} = \emptyset\}).$$

Examples

- If $\mathcal{P}_0 = \mathcal{P}(\mathcal{L}, \mathcal{W})$, then for almost every $\mathcal{L}$ in the space of lattices and $\boldsymbol{q} \in \mathcal{P}_0$, we have $H_g = \mathrm{SL}(n, \mathbb{R})$, $\Gamma \cap H_g = \mathrm{SL}(n, \mathbb{Z})$.

  $F_{\mathcal{P}_0,\boldsymbol{q}}(\xi)$ is independent of $\mathcal{P}_0$ and $\boldsymbol{q}$ and has compact support.

- If $\mathcal{P}_0 = \mathcal{P}(\mathcal{L}, \mathcal{W})$, then for almost every $\mathcal{L}$ in the space of lattices and almost every $\boldsymbol{q}$, we have $H_g = \mathrm{ASL}(n, \mathbb{R})$, $\Gamma \cap H_g = \mathrm{ASL}(n, \mathbb{Z})$.

  $F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) \asymp \xi^{-1}$ ($\xi \to \infty$) where the implied constants depend on n, $\mathcal{W}$, δ. Again $F_{\mathcal{P}_0,\boldsymbol{q}}(\xi)$ is independent of $\mathcal{P}_0$ and $\boldsymbol{q}$.

- If $\mathcal{P}_0$ is the Penrose quasicrystal and $\boldsymbol{q} \in \mathcal{P}_0$, we have $H_g = \mathrm{SL}(2, \mathbb{R})^2$, $\Gamma \cap H_g =$ a congruence subgroup of the Hilbert modular group $\mathrm{SL}(n, \mathcal{O}_K)$, with $\mathcal{O}_K$ the ring of integers of $K = \mathbb{Q}(\sqrt{5})$.

  $F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) \asymp$ work in progress ...

Other aperiodic scatterer configurations: unions of lattices

- Consider now scatterer locations at the point set

$$\mathcal{P}_0 = \bigcup_{i=1}^{N} \mathcal{L}_j, \qquad \mathcal{L}_i = \overline{n}_i^{-1/d}(\mathbb{Z}^d + \boldsymbol{\omega}_i) M_i$$

with $\boldsymbol{\omega}_i \in \mathbb{R}^d$, $M_i \in \mathrm{SL}(d,\mathbb{R})$ and $\overline{n}_i > 0$ such that $\overline{n}_1 + \ldots + \overline{n}_N = 1$

- Let $\mathcal{S}$ be the commensurator of $\mathrm{SL}(d,\mathbb{Z})$ in $\mathrm{SL}(d,\mathbb{R})$:

$$\mathcal{S} = \{(\det T)^{-1/d} T : T \in \mathrm{GL}(d,\mathbb{Q}),\ \det T > 0\}.$$

- We say that the matrices $M_1, \ldots, M_N \in \mathrm{SL}(n,\mathbb{R})$ *are pairwise incommensurable* if $M_i M_j^{-1} \notin \mathcal{S}$ for all $i \neq j$. A simple example is

$$M_i = \zeta^{-i/d} \begin{pmatrix} \zeta^i & 0 \\ 0 & 1_{d-1} \end{pmatrix}, \qquad i = 1, \ldots, N,$$

where ζ is any positive number such that $\zeta, \zeta^2, \ldots, \zeta^{N-1} \notin \mathbb{Q}$.

Free path lengths for unions of lattices

Theorem G [JM & Strömbergsson, 2013]. Let $\mathcal{P}_0$ as on the previous slide with $M_i \in \mathrm{SL}(d,\mathbb{R})$ pairwise incommensurable. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) := \lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

exists. If for instance $\boldsymbol{\omega}_i - \overline{n}_i^{1/d}\boldsymbol{q}M_i^{-1} \notin \mathbb{Q}^d$ for all i, then

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) = \prod_{i=1}^{N} F(\overline{n}_i\xi)$$

where $F(\xi)$ is the distribution of free path length corresponding to a single lattice and generic initial point (as in Theorem C).

Recall $F(\xi) \sim C\xi^{-1}$ for $\xi \to \infty$. Thus

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) \sim C^N \xi^{-N}.$$

Key step in the proof: equidistribution in products

...is as before an equidistribution theorem that follows from Ratner's measure classification (again via Shah's theorem):

Theorem H [JM & Strömbergsson, 2013]. Assume that $M_1, \ldots, M_N \in \mathrm{SL}(n, \mathbb{R})$ are pairwise incommensurable, and $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N \notin \mathbb{Q}^d$ (for simplicity). Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then

$$\lim_{t\to\infty} \int_{\mathrm{S}_1^{d-1}} f\Big((1_d, \boldsymbol{\alpha}_1)(M_1 K(\boldsymbol{v})\Phi^t, \mathbf{0}), \ldots, \ldots, (1_d, \boldsymbol{\alpha}_N)(M_N K(\boldsymbol{v})\Phi^t, \mathbf{0})\Big)\, d\lambda(\boldsymbol{v}) = \int_{X^N} f(g_1, \ldots, g_N)\, d\mu(g_1)\cdots d\mu(g_N).$$

Future work