Quantum Ergodicity for the uninitiated

Zeev Rudnick, TAU & IAS

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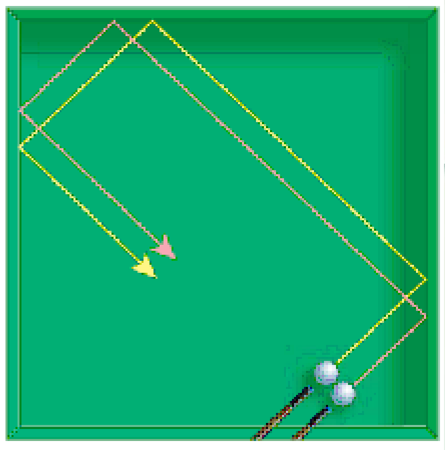
Plan

Formulation of Quantum Ergodicity: connects classical and quantum mechanics

Recent work on Quantum Ergodicity in configuration space:

- nonergodic - (pseudo-) integrable systems
- Small scale Quantum Ergodicity

Billiards as a model system



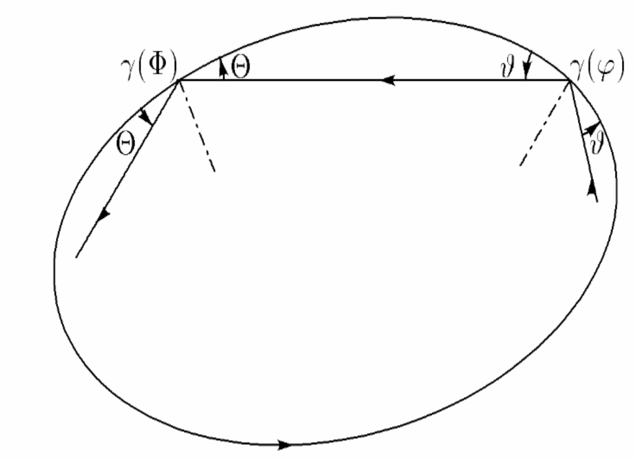
planar billiards: motion in planar domain B (configuration space)

angle of reflection = angle of incidence

Phase space (description of a particle):

$S^*B=BxS^1$ = { **position** x of point & **direction** vector ξ of motion}

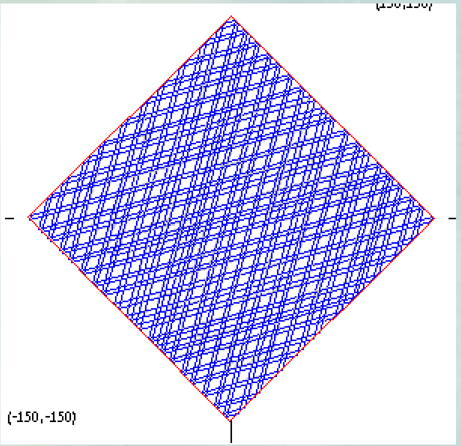
Reduced phase space

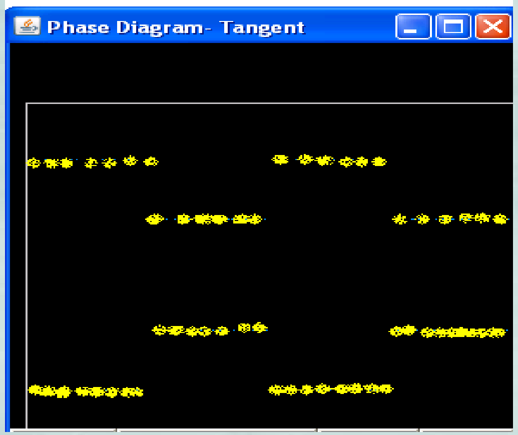


Reduced phase space (Birkhoff):

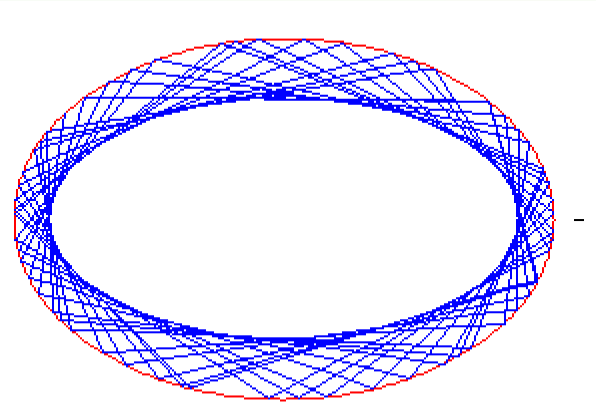
only impacts matter; parametrize using boundary coordinate φ and angle θ of trajectory with tangent

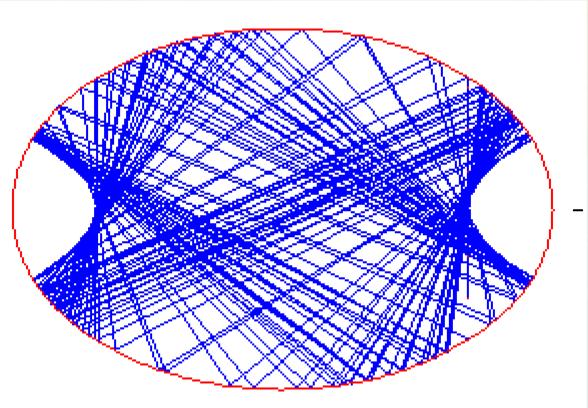
Billiard map $(\varphi,\theta)\rightarrow(\Phi,\Theta)$

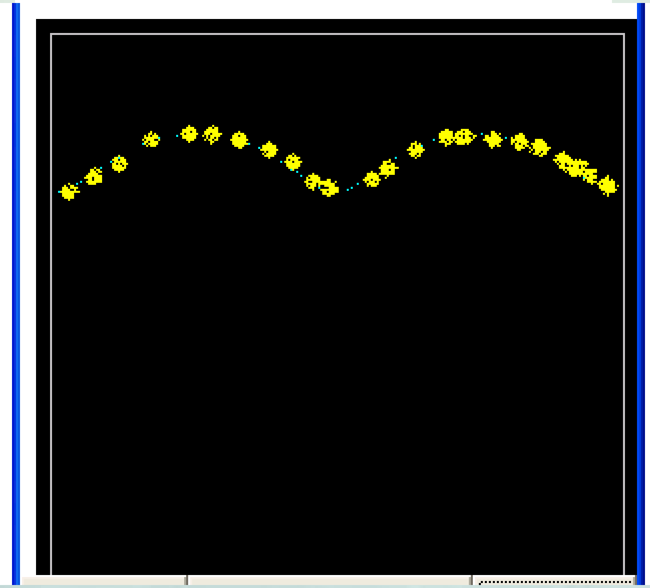


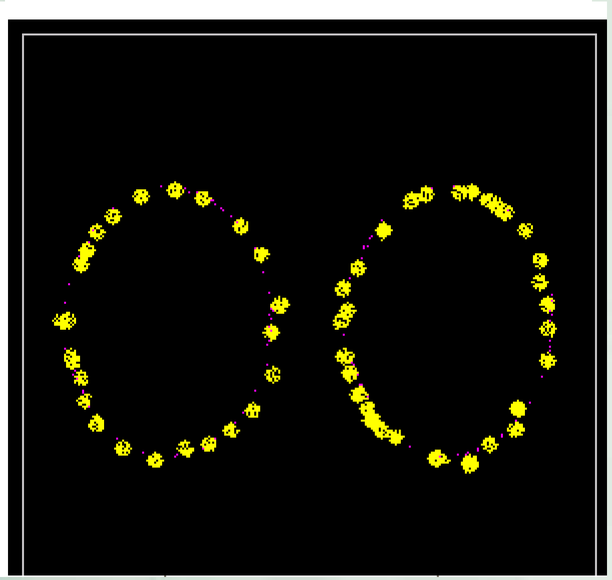


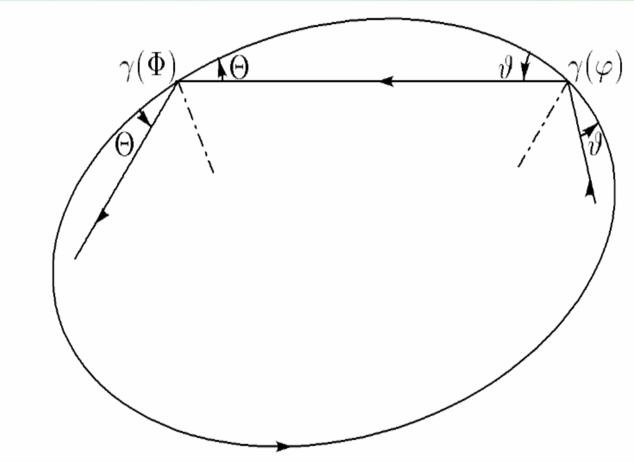
Billiards in an ellipse









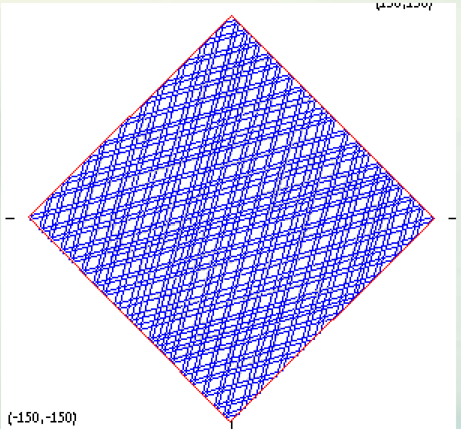


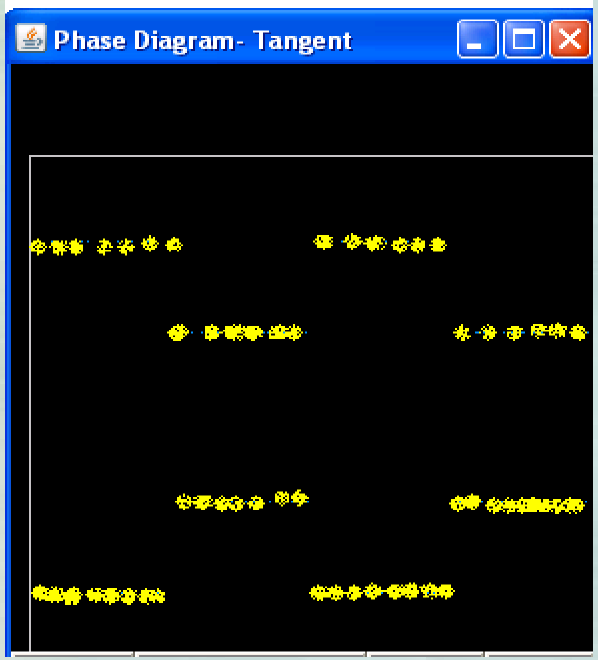
Billiard map $(\varphi,\theta)\rightarrow(\Phi,\Theta)$

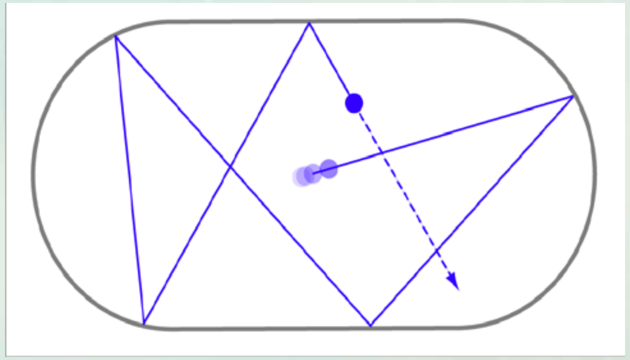
The Billiards Simulation
Bryn Mawr

Regular vs. chaotic motion

Square billiard

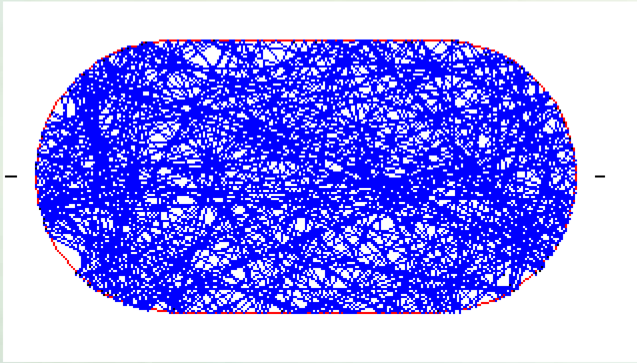


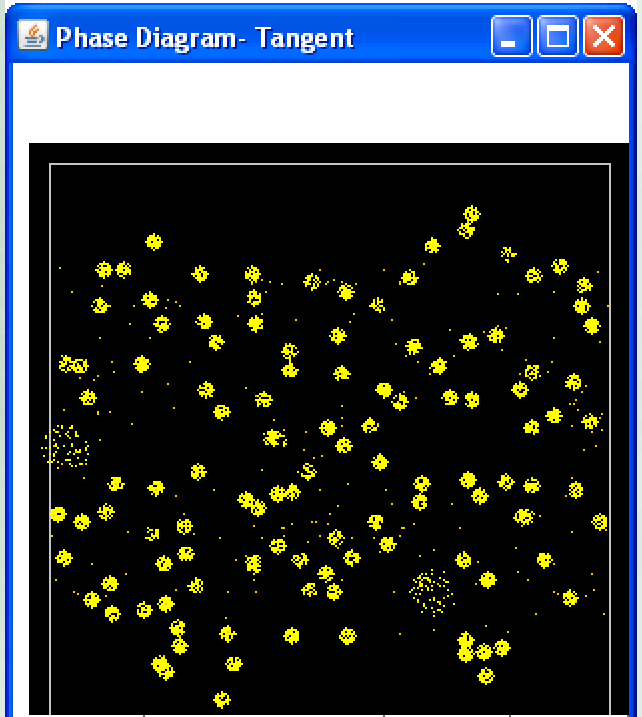




Bunimovich stadium

Bunimovich stadium

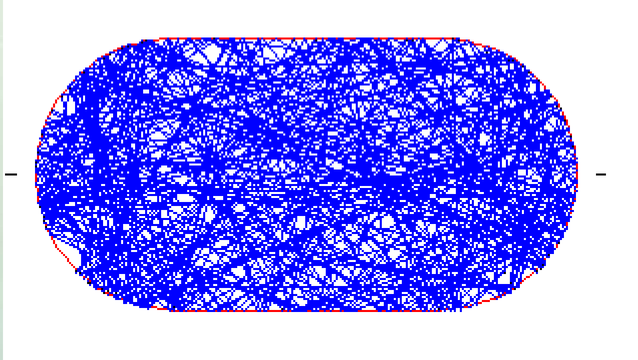




Classical ergodicity

The billiard flow Φ is **ergodic** if for generic initial conditions (position, direction)= (x, ξ)$\varepsilon S^*B = B \times S^1$, the orbit $\{\Phi^t(x,\xi)\}$ is **uniformly distributed** in phase space (in particular, is **dense**).

the **time average** along a **generic** trajectory of any observed quantity a(x,ξ) converges to its **space average**

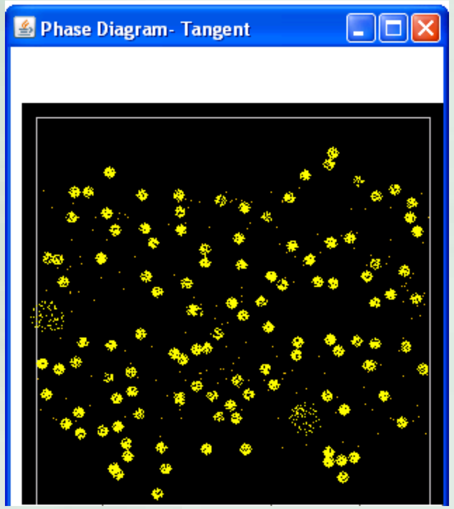


for almost all initial conditions $(x,\xi) \in S^*B$

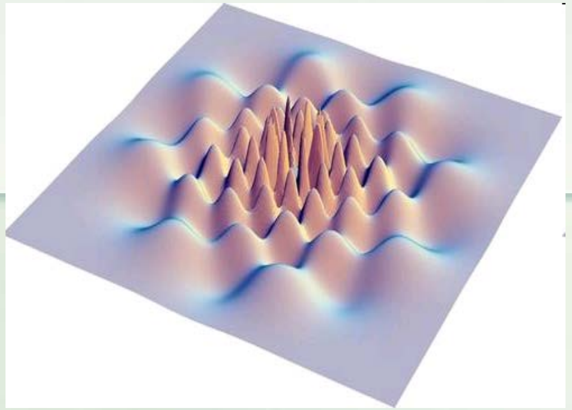
$$\langle a \rangle_T (x,\xi) := \frac{1}{2T} \int_{-T}^{T} a(\Phi^t(x,\xi))\, dt \xrightarrow[T\to\infty]{} \int_{S^*B} a(x,\xi)dxd\xi = \bar{a}$$

time average

space average



Quantum mechanics



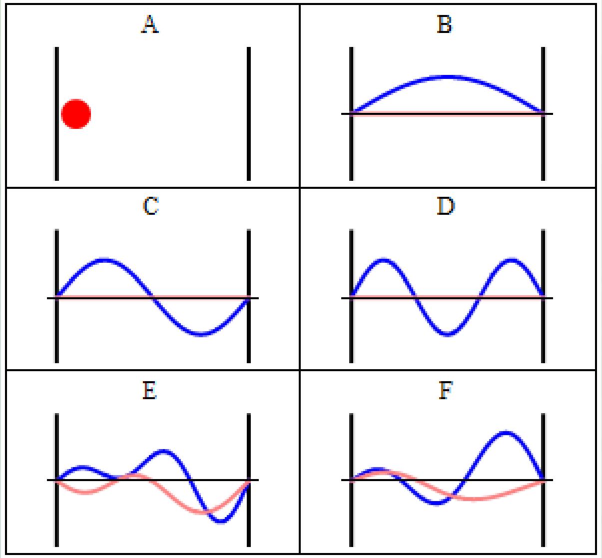
A particle at time t is described by its **wave function** $\Psi(q,t)$

$|\Psi(q,t)|^2$ = probability density of particle in state Ψ

Time evolution is described by **Schrödinger's equation** :

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\Psi$$

$\hbar = 1.054 \times 10^{-34}$ J-s



Stationary states: $\Psi(q,t) = \psi(q)e^{-itE/\hbar}$ with $\psi(q)$ an **eigenfunction** of Δ

$$-\frac{\hbar^2}{2}\Delta\psi = \mathbf{E}\psi$$

E=energy level

+ boundary condition

$$-\Delta u = Eu, \quad u|_{\partial D} = 0$$

The semi-classical eigenfunction hypothesis

The semiclassical limit $\hbar \rightarrow 0$ & the correspondence principle:

"classical mechanics is a special case of quantum mechanics".

If so, then:

How is the dichotomy "regular vs. chaotic" manifested in Quantum Mechanics ?

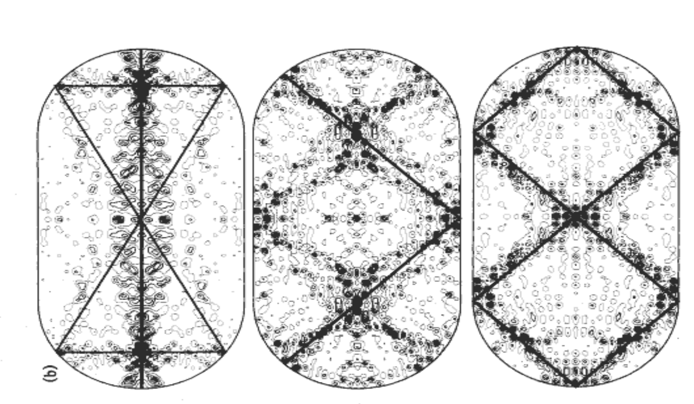
The semi-classical eigenfunction hypothesis of M.V. Berry and A.Voros (~ 1977):

"*Each semi-classical eigenstate has a Wigner function concentrated on the region explored by a typical orbit over infinite times*".

In particular, for chaotic systems, "The wave functions cover phase space uniformly"

However…..“scars” were found by Heller and by McDonald & Kauffman (1984-88)

Scars: Concentration of eigenfunctions on unstable periodic orbits (controversial)



E. Heller: Scarred stadium mode

A mathematical formulation

For a particle with wave function ψ, the expectation values of its position coordinates x_1 , x_2 are given by

$$\langle x_1 \rangle_\psi := \int_B x_1 |\psi(x_1, x_2)|^2 \, dx_1 dx_2$$

"Likewise", for any classical observable $a(x,\xi)$ of position $x=(x_1, x_2)$ and momentum $\xi=(\xi_1, \xi_2)$, one can define a pseudo-differential operator Op(a) so that the expected value of the observable a "at the state ψ" is the diagonal matrix element $\langle Op(a)\psi,\psi \rangle$

E.g. for "isotropic" observables $a(x,\xi) = a_0(x)$,

$$\langle \mathrm{Op}(a)\psi, \psi \rangle = \int_B a_0(x) \, |\psi(x)|^2 \, dx$$

A possible interpretation of the statement that "wave functions cover phase space uniformly" is that the matrix elements converge to the classical average of a:

$$\langle \mathrm{Op}(a)\psi, \psi \rangle \xrightarrow[E_\psi \to \infty]{} \iint_{S^*B} a(x,\xi) \, dx \, d\xi$$

Quantum Ergodicity

Schnirelman (1974): For a Riemannian manifold M (or billiard) with **ergodic** geodesic (billiard) flow, "**most**" eigenfunctions cover phase space uniformly: if $\{u_n\}$ is an ONB consisting of eigenfunctions of the Laplacian, then there is a subsequence of density one s.t. for all observables a(p,q)

$$\left\langle \mathrm{Op}(a)u_n, u_n \right\rangle \xrightarrow[n\to\infty]{} \int_{S^*B} a(x,\xi)\,dx\,d\xi$$

Zelditch (1987), Colin de Verdiere (1985), Gerard & Leichtnam (1993), Zelditch-Zworski (1996).

One interpretation of "scars" is as possible **exceptional** subsequences

ZR & Sarnak (1994): Conjecture that for **negatively curved** manifolds, **no** exceptional subsequence - **Quantum Unique Ergodicity** (QUE).

Progress: Ananthraman (2008), Lindenstrauss (2006)

New directions in quantum ergodicity

Non-ergodic systems

Small scale QE

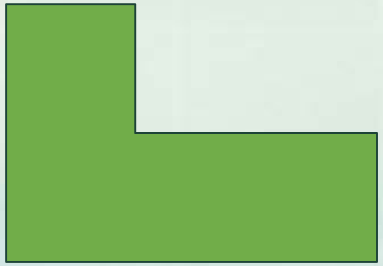
QE for billiards in rational polygons

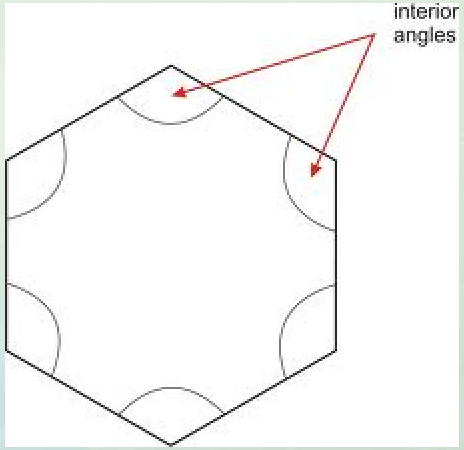
We have seen that for ergodic systems, almost all stationary states are uniformly distributed in phase space (quantum ergodicity).

For integrable systems this is not necessarily true. My goal is to explore a "pseudo-integrable" case – billiards in rational polygons.

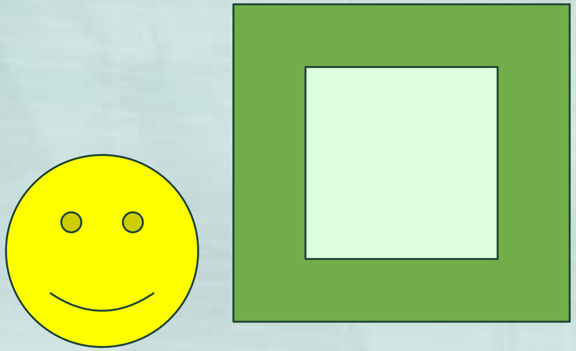
Rational polygons

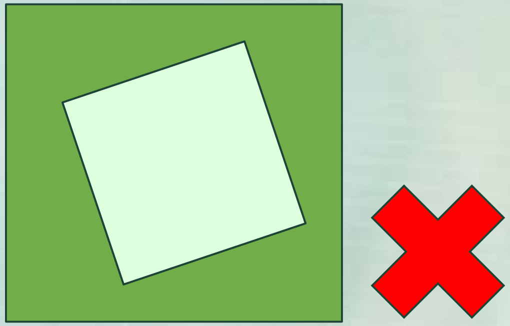
A simply connected polygon is rational if all interior angles are rational multiples of π



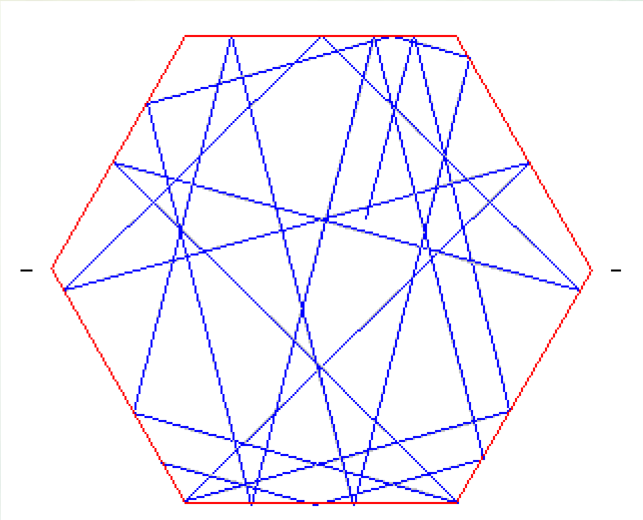


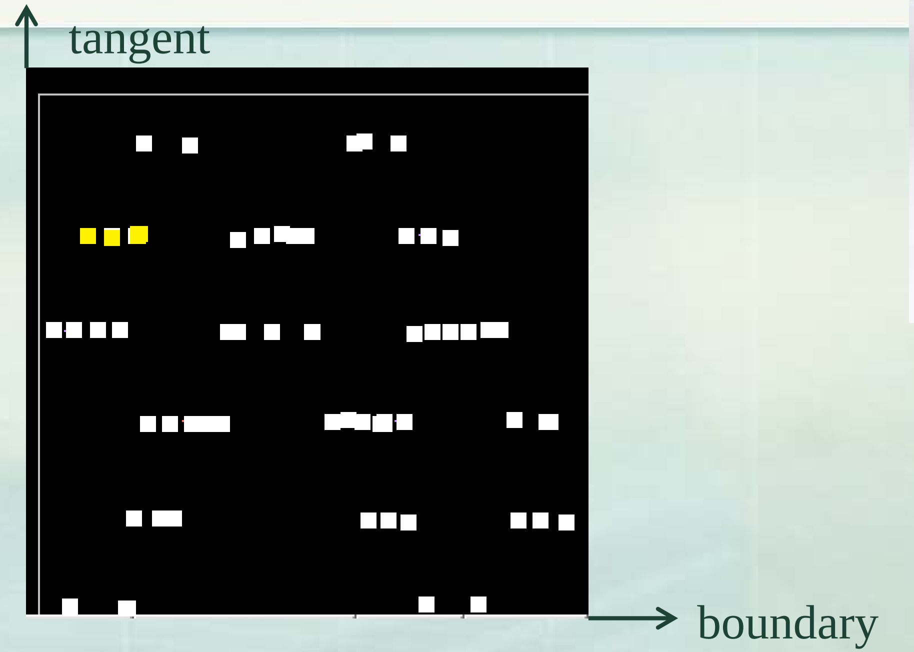
More generally: A connected polygon is **rational** if the group $\Gamma \subseteq O(2)$ generated by reflections in the sides is finite





Billiards in rational polygons





Conserved quantity: Γ-orbit of tangent angle θ

Extra constant of motion forces dynamics in phase space to be confined to invariant surfaces

$$B_\xi = \bigcup_{\gamma\in\Gamma} B\times\gamma\xi$$

Phase space $S^*B=B\times S^1$ is foliated by invariant surfaces - so **non-ergodic**

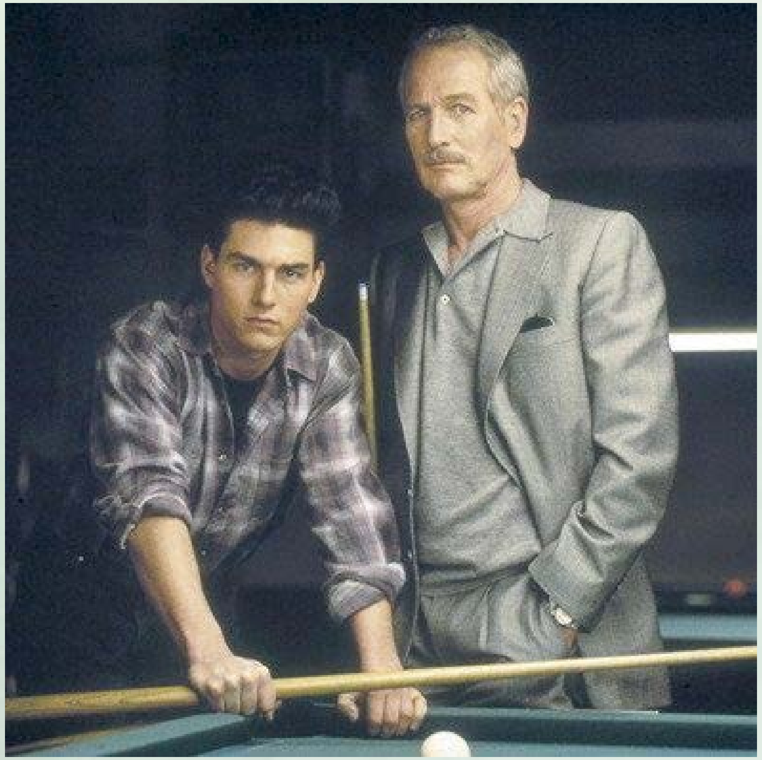
The restriction of the flow to the invariant surface is called the “directional flow”.

Quantum ergodicity in configuration space

Thm (J. Marklof & ZR, 2012): For billiards in rational polygons, **almost all** eigenfunctions are uniformly distributed in **configuration space**.

i.e. given any ONB of eigenfunctions u_n, there is a density one subsequence so that for any subset $\Omega \subseteq B$ of the billiard table,

$$\lim_{n\to\infty} \int_{\Omega} |u_n(x)|^2 dx = \frac{\text{area}(\Omega)}{\text{area}(B)}$$



Quantum vs. classical variance

Quantum variance

$$V(a,E) := \frac{1}{N(E)} \sum_{E_n \le E} \left| \langle \mathrm{Op}(a) u_n, u_n \rangle - \overline{a} \right|^2$$

Want to show: $V(a,E) \to 0$

Classical variance of time-averaged observable

$$\overline{a} := \int_{S^*B} a(x,\xi)\,dx\,d\xi$$

space average

$$C(a,T) := \int_{S^*B} \left| \langle a \rangle_T (x,\xi) - \overline{a} \right|^2 dx\,d\xi$$

$$\langle a \rangle_T := \frac{1}{2T} \int_{-T}^{T} a \circ \Phi^t \, dt$$

time average

Penultimate step in proof of Quantum Ergodicity **(**for **<u>any</u>** billiard): For all T>0,

$$\limsup_{E \to \infty} V(a,E) \le C(a,T)$$

- For **<u>ergodic</u>** case, the classical variance C(a,T) vanishes as T→∞ for **<u>all</u>** observables.
- For **<u>rational polygons</u>**, the classical variance vanishes for **<u>isotropic</u>** observables: $a(x,\xi)=a_0(x)$.

Directional flows

Kerckhoff, Masur & Smilie 1986: almost all directional flows are **uniquely ergodic** (analogue of Kronecker's theorem on irrational rotations)

For almost all directions ξ_0

$$\frac{1}{2T}\int_{-T}^{T} a(\Phi^t(x_0,\xi_0))dt \xrightarrow[T\to\infty]{} \int_{B_{\xi_0}=\cup_{\gamma\in\Gamma} B\times\gamma\xi_0} a(x,\xi_0)dx$$

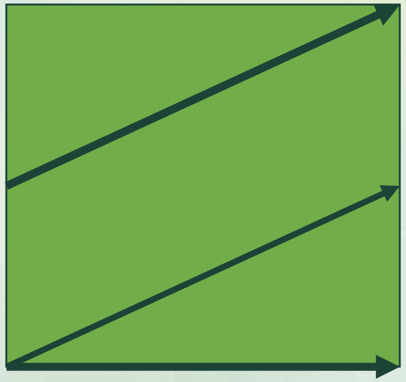
$$B_\xi = \bigcup_{\gamma\in\Gamma} B\times\gamma\xi$$

In particular, for isotropic observables, we recover the average on configuration space for almost all directions

$$\frac{1}{2T}\int_{-T}^{T} a(\Phi^t(x_0,\xi_0))dt \xrightarrow[T\to\infty]{} \int_B a_0(x)dx, \qquad a(x,\xi)=a_0(x)$$

Kronecker's theorem

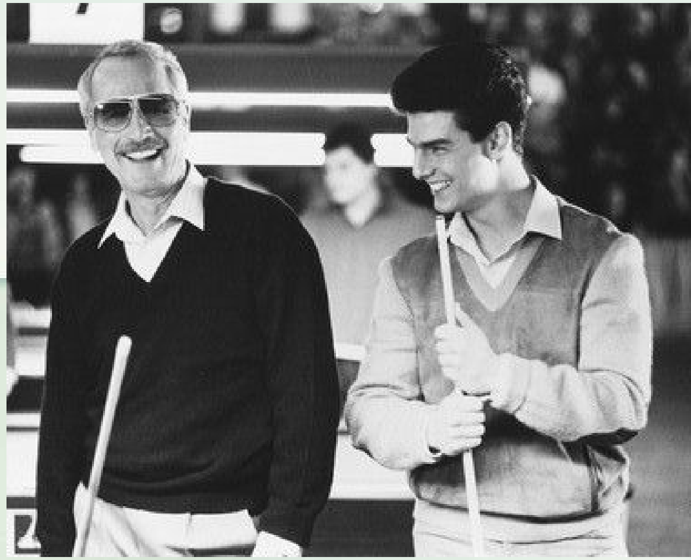
Kronecker's theorem: An **irrational** line is uniformly distributed in the torus, i.e. time average=space average:



For any nice function $a_0(x)$ on the torus,

$$\underbrace{\lim_{T\to\infty}\frac{1}{T}\int_0^T a_0\left(t(1,\sqrt{2})\right)dt}_{\text{time average}} = \underbrace{\iint_{\mathbb{R}^2/\mathbb{Z}^2} a_0(x_1,x_2)dx_1dx_2}_{\text{space average}}$$

conclusion



The **classical** variance vanishes for isotropic observables

$$\lim_{T\to\infty} C(a,T) := \lim_{T\to\infty} \int_{S^*M} \left|\langle a\rangle_T (x,\xi) - \bar{a}\right|^2 dx d\xi = 0, \qquad a(x,\xi) = a_0(x)$$

the **quantum** variance vanishes for **isotropic** observables:

$$V(a,E) := \frac{1}{N(E)} \sum_{E_n \le E} \left| \int_B a_0(x) \,|u_n(x)|^2 \, dx - \bar{a} \right|^2 \le \lim_{T\to\infty} C(a,T) = 0$$

almost all eigenfunctions are uniformly distributed in configuration space

$$\frac{1}{\text{area}(B)} \int_\Omega |u_n(x)|^2 \, dx \to \frac{\text{area}(\Omega)}{\text{area}(B)}, \qquad \text{for almost all } n$$

QED

Small scale Quantum Ergodicity

In the physics literature, the semiclassical eigenfunction hypothesis is viewed as a distributional statement, in particular that it holds on any scale where the eigenfunctions still oscillates, i.e. any scale larger than the wavelength (Planck scale).

In configuration space that means that we want to to study averages of the probability amplitudes $|\psi_n(x)|^2$ on balls $B(y, r_n)$ of size bigger than several wavelengths: $r_n \gg \lambda_n^{-1/2}$

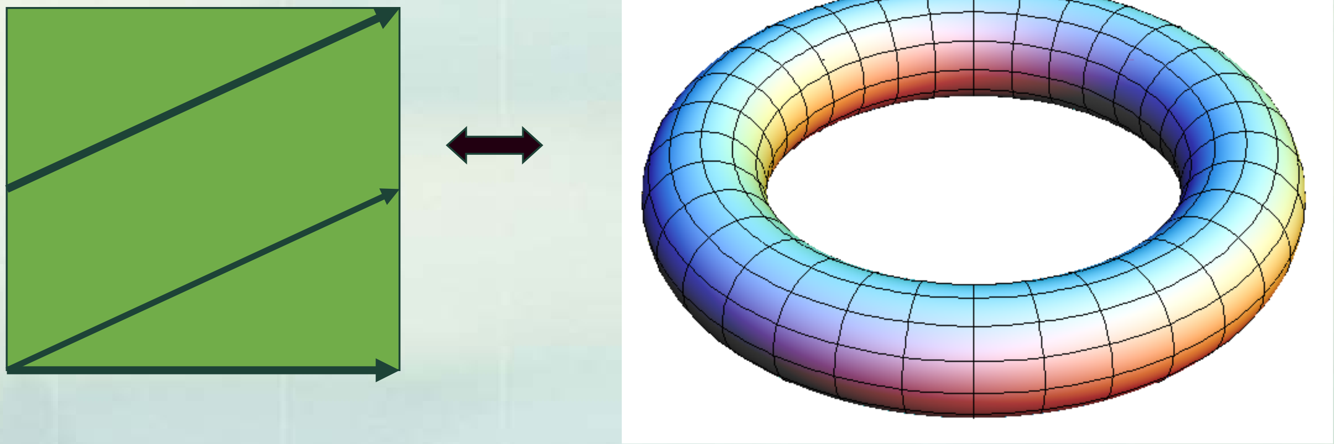
$$\left\langle |\psi_n(x)|^2 \right\rangle_{B(y,r_n)} := \frac{1}{B(y,r_n)} \int_{B(y,r_n)} |\psi_n(x)|^2 \, dx \xrightarrow{?} ??$$

Understanding these averages is important for studying finer properties of the eigenfunctions, e.g. the structure of the nodal sets, L^p norms,….

<u>Zelditch (1994), Han (2014), Hezari & Riviere (2014):</u> For a **<u>negatively curved</u>** manifold, there are 0<c<C so that for any ONB $\{\psi_n\}$ there is a density one subsequence so that for all logarithmically small balls,

$$c \leq \left\langle |\psi_n(x)|^2 \right\rangle_{B(y,r_n)} \leq C, \qquad r_n > 1/(\log \lambda_n)^{1/3d}$$

Example: the flat torus $M = \mathbb{R}^2 / 2\pi\mathbb{Z}^2$



- Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.
- Basis for eigenfunctions: $e^{\mathbf{i}(mx+ny)}$, m,n ε**Z**
- Eigenvalues E = m^2+n^2.
- General eigenfunction $\psi(x,y) = \sum_{m^2+n^2=E} c_{m,n} e^{\mathbf{i}(mx+ny)}$

Small scale quantum ergodicity on flat tori

S. Lester & ZR (August 2015): On the flat 2-dim torus, for any ONB of eigenfunctions, **almost all** eigenfunctions are uniformly distributed in configuration space in any ball B(y,r) of size up to the Planck scale:

$$\lim_{n\to\infty} \frac{1}{\text{area}\, B(y,r_n)} \int_{B(y,r_n)} |\psi_n(x)|^2\, dx = 1, \quad r_n > \lambda_n^{-1/2+o(1)}$$

Same statement in any dimension, except that we do not reach the Planck scale, instead require $r_n > \lambda_n^{\frac{1}{2(d-1)}}$.

Hezari & Riviere (2014): fixed ball, $r_n > \lambda_n^{\frac{1}{4(d+1)}}$

J. Bourgain (October 2015): On smaller scales there are ONB's where a positive proportion of eigenfunctions are **not** uniformly distributed in some balls B(0,r), $r_n < \lambda_n^{\frac{1}{2(d-1)}}$

QE in eigenspaces for high-dim tori

On $\mathbb{T}^d$, the dimension of the λ-eigenspace is $\#\{x_1^2 + \cdots + x_d^2 = \lambda\} \approx \lambda^{\frac{d-2}{2}}$

THM (Lester & ZR): For d=3,4, for any ONB $\{\psi_{n,\lambda}\}$ of a λ-eigenspace, almost all elements are uniformly distributed in any ball of size $r_n > \lambda_n^{1/2(d-1)}$

- same result as without restricting to eigenspaces!

Arithmetic input: Statistics of lattice points in small caps on the sphere of radius λ in dimension d

For a lattice point $|\mu|^2 = \lambda$, let

$$n(\mu;Y) \coloneqq \#\{\nu \in \mathbb{Z}^d : |\nu|^2 = \lambda, \nu \neq \mu\}$$= # of **other** lattice points in a cap of size Y around μ.

Then for small caps of size $Y < \lambda^{1/2(d-1)}$,

$$\frac{1}{\#\{|\mu|^2 = \lambda\}} \sum_{|\mu|^2=\lambda} n(\mu;Y) \xrightarrow[\lambda\to\infty]{} 0, \qquad Y \ll \lambda^{1/2(d-1)}$$

- Expect this also holds for all $d \geq 5$

Summary

Formulation of Quantum Ergodicity: connects classical and quantum mechanics

Recent work on Quantum Ergodicity in configuration space:

- nonergodic - (pseudo-) integrable systems
- Small scale Quantum Ergodicity
- Question: small scale QE for rational polygons?

Thank you for your attention!

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