QUANTITATIVE DISTRIBUTIONAL ASPECTS OF GENERIC DIAGONAL FORMS

- Quantitative Oppenheim theorem
- Eigenvalue spacing for rectangular billiards
- Pair correlation for higher degree diagonal forms

EFFECTIVE ESTIMATES ON INDEFINITE TERNARY FORMS

Theorem (**Dani–Margulis**) *Let Q be an indefinite, irrational, ternary quadratic form. Then the set*

$$\{Q(v) : v \in \mathbb{Z}^3 \text{ primitive}\}$$

is dense in the real line.

There is the following quantification obtained by an effective version of Ratner's theorem.

Theorem (**Lindenstrauss–Margulis**) *Let Q be a given indefinite, ternary quadratic form with $\det Q = 1$ and $\varepsilon > 0$. Then, for T large enough, at least one of the following holds*

- *There is an integral quadratic form Q_1 such that $|\det Q_1| < T^{\varepsilon}$ and*

$$\|Q - \lambda Q_1\| \ll T^{-1} \text{ where } \lambda = |\det Q_1|^{-1/3}$$

- *For any $\xi \in \big[-(\log T)^c, (\log T)^c \big]$, there is primitive $v \in \mathbb{Z}^3, 0 < \|v\| < T^C$ such that*

$$|Q(v) - \xi| \ll (\log T)^{-c}$$

Let Q be as above. For which functions $A(N) \to \infty$ and $\delta(N) \to 0$ (depending on Q) does the statement

$$\max_{|\xi|<A(N)} \min_{\substack{x\in\mathbb{Z}^3 \\ 0<|x|<N}} |Q(x)-\xi| < \delta(N) \qquad (*)$$

hold? In the [L-M] result, $A(N)$ and $\delta(N)$ depend logarithmically on N.

Consider diagonal forms of signature (2, 1) $Q(x) = x_1^2 + \alpha_2 x_2^2 - \alpha_3 x_3^2$ $(\alpha_2, \alpha_3 > 0)$

There is the stronger statement for one-parameter families

Theorem (**B**) *Let $\alpha_2 > 0$ be fixed. For most $\alpha_3 > 0$, the following holds*

(i) *Assuming the Lindelöf hypothesis*

$\min_{x\in\mathbb{Z}^3, 0<|x|<N} |Q(x)| \ll N^{-1+\varepsilon}$ *and* $(*)$ *holds provided* $A(N)\delta(N)^{-2} \ll N^{1-\varepsilon}$

(ii) *Unconditionally,*

$\min_{x\in\mathbb{Z}^3, 0<|x|<N} |Q(x)| \ll N^{-\frac{2}{5}+\varepsilon}$ *and* $(*)$ *holds assuming* $A(N)^3\delta(N)^{-\frac{11}{2}} \ll N^{1-\varepsilon}$

REMARKS

- The statement

$$\min_{x\in\mathbb{Z}^3,0<|x|<N} |Q(x)| \ll N^{-1+\varepsilon}$$

is essentially best possible

- For generic diagonal forms, i.e. considering both α_2 and α_3 as parameters, (i) holds without need of Lindelöf

- **Ghosh** and **Kelmer** obtained similar results for generic elements in the full space of indefinite ternary quadratic forms

EIGENVALUE SPACINGS OF FLAT TORI

Berry–Tabor Conjecture: local statistics of 'generic' integrable quantum Hamiltonian coincide with those of random numbers generated by Poisson process

PAIR CORRELATION FOR 2D FLAT TORI

$\Gamma \subset \mathbb{R}^2$ lattice and $M = \mathbb{R}^2/\Gamma$ associated flat torus.

Eigenvalues of M given by $4\pi^2\|v\|^2, v \in \Gamma^*$ (= dual lattice). These are the values at integral points of binary quadratic form $B(m,n) = 4\pi^2\|n_1 v_1 + n_2 v_2\|^2$, with $\{v_1, v_2\}$ a $\mathbb{Z}$-basis for Γ^*. **Weyl** law holds

$$|\{j : \lambda_j(M) \leq T\}| \sim c_M T \text{ with } c_M = (\text{area } M)/4\pi$$

Distribution of local spacings $\lambda_j - \lambda_k, j \neq k$

$$R(a,b,T) = \frac{1}{T}\left|\left\{(j,k) : j \neq k, \lambda_j \leq T, \lambda_k \leq T, a \leq \lambda_j - \lambda_k \leq b\right\}\right|$$

where $-\infty < a < b < \infty$.

According to the Berry–Tabor conjecture, $\lambda_j - \lambda_k (j \neq k)$ should be uniformly distributed on $\mathbb{R}$. Thus for given $a < b$,

$$\lim_{T \to \infty} R(a, b, T) = c_M^2 (b - a) \qquad (*)$$

Theorem (**Sarnak**, 1996) $(*)$ *holds for a set of full measure in the space of all tori.*

Analytical approach which seems to require 2 parameters and not applicable to rectangular billiards with eigenvalues $\alpha m^2 + n^2$ (where $\sqrt{\alpha}$ is the aspect ratio between height π and width $\pi/\sqrt{\alpha}$).

Theorem (**Eskin-Margulis-Mozes**) $(*)$ *holds for (weakly) diophantine billiards, assuming* $0 \notin (a, b)$

In particular $(*)$ is valid for rectangular billiards provided α is diophantine, in the sense that $\|q\alpha\| > q^{-C}$ for some $C > 0$.

Ergodic theory approach with no (or very poor) quantitative versions if we let $b - a \to 0$ with $T \to \infty$.

MINIMAL GAP FUNCTION OF SPECTRUM OF RECTANGULAR BILLIARD

For α irrational, we get simple spectrum $0 < \lambda_1 < \lambda_2 < \cdots$, with growth

$$|\{j : \lambda_j \leq T\}| = |\{(m,n); m,n \geq 1, \alpha m^2 + n^2 \leq T\}| \sim \frac{\pi}{4\sqrt{\alpha}} T$$

Definition

$$\delta_{\text{min}}^{(\alpha)}(N) = \min(\lambda_{i+1} - \lambda_i : 1 \leq i \leq N)$$

For a Poisson sequence of N uncorrelated levels with unit mean spacing, the smallest gap is almost surely of size $\approx \frac{1}{N}$.

Remark For the Gaussian unitary ensemble, on the scale of the mean spacing, smallest eigenphase gap is $\approx N^{-1/3}$, while for the Gaussian orthogonal ensemble, it is expected that the minimal gap is of size $N^{-1/2}$.

Theorem (**Bloomer–B–Radziwill–Rudnick**) *For almost all* $\alpha > 0$

- $\delta_{\text{min}}^{(\alpha)}(N) \ll \frac{1}{N^{1-\varepsilon}}$

- $\delta_{\text{min}}^{(\alpha)}(N) \gg \frac{(\log N)^c}{N}$ with $c = 1 - \frac{\log(e \log 2)}{\log 2} = 0,086\ldots$ infinitely often

PAIR CORRELATION FOR GENERIC 3D RECTANGULAR BILLIARDS

Consider generic positive definite ternary quadratic form $Q(x) = x_1^2 + \alpha x_2^2 + \beta x_3^2$ with $\alpha, \beta > 0$. Restricting the variables to $\mathbb{Z} \cap [0, N]$, mean spacing is $O(\frac{1}{N})$.

For T large and $a < b$, denote

$$R(a, b; T) = T^{-3/2} \Big| \Big\{ (m, n) \in \mathbb{Z}_+^3 \times \mathbb{Z}_+^3 : m \neq n, Q(m) \leq T, Q(n) \leq T \text{ and } Q(m) - Q(n) \in [a, b] \Big\} \Big|$$

and

$$c = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left| \left\{ (x, y) \in \mathbb{R}_+^3 \times \mathbb{R}_+^3 ; Q(x) \leq 1 \text{ and } |Q(x) - Q(y)| < \frac{\varepsilon}{2} \right\} \right|$$

Theorem (**B**) *Let* $Q = Q_{\alpha,\beta}$ *with* $\alpha, \beta > 0$ *parameters. Almost surely in* α, β*, the following holds. Fix* $0 < \rho < 1$*.*

Let $T \to \infty$ *and* $a < b$*,* $|a|, |b| < O(1)$ *such that* $T^{-\rho} < b - a < 1$*. Then*

$$R(a, b; T) \sim cT^{\frac{1}{2}}(b - a)$$

Remark **Vanderkam** established the pair correlation conjecture for generic four-dimensional flat tori using an unfolding procedure.

PAIR CORRELATION FOR HOMOGENOUS FORMS OF DEGREE k

Let $F(x_1,\ldots,x_k) = x_1^k + \alpha_2 x_2^k + \cdots + \alpha_k x_k^k$ with $\alpha_2,\ldots,\alpha_k > 0$. Thus average spacing between values of P is asymptotically constant. Define

$$R(a,b;T) = \frac{1}{T}\left|\left\{(m,n) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k : m \neq n, F(m) \leq T, F(n) \leq T \text{ and } F(m) - F(n) \in [a,b]\right\}\right|$$

Theorem **(B)** *Consider $\alpha_2,\ldots,\alpha_k$ as parameters. There is $\rho > 0$ such that almost surely in $\alpha_2,\ldots,\alpha_k$ the following statement holds. Let $T \to \infty$ and $|a|,|b| = O(1)$, $b - a > T^{-\rho}$. Then*

$$R(a,b;T) \sim c^2(b-a) \text{ where } c = |\{x \in \mathbb{R}_+^k : F(x) \leq 1\}|$$

Results of this type for generic elements in the full space of homogenous forms were obtained by **Vanderkam** (1999) using Sarnak's method.

QUANTITATIVE OPPENHEIM FOR 1-PARAMETER FAMILIES

Harmonic analysis approach but rather than expressing the problem using Gauss sums, reformulate it multiplicatively and rely on Dirichlet sum behavior. For instance, the sum

$$\sum_{n \sim N} n^{it}$$

is 'small' for say $|t| > N^{\varepsilon}$; under the Lindelöf hypothesis one gets

$$\Big| \sum_{n \sim N} n^{it} \Big| \ll N^{\frac{1}{2}} (1 + |t|)^{\varepsilon}$$

This permits to decompose in small and large t, where the small-t range provides main term and large-t contribution is an error term.

Consider one-parameter family

$$Q_\alpha(x) = x_1^2 + x_2^2 - \alpha x_3^2 \text{ with } \alpha > 0 \text{ a parameter}$$

Taking $x_1, x_2, x_3 \sim N$, condition $|x_1^2 + x_2^2 - \alpha x_3^2| < \delta$ is rewritten as

$$\left|\frac{x_1^2 + x_2^2}{\alpha x_3^2} - 1\right| \lesssim \frac{\delta}{N^2} \text{ or } \left|\log(x_1^2 + x_2^2) - 2\log x_3 - \log\alpha\right| \lesssim \frac{\delta}{N^2}$$

The number of $x_1, x_2, x_3 \sim N$ for which this inequality holds is then expressed using Fourier transform as

$$\frac{1}{T}\int_{|t|<T} F_1(t)\,\overline{F_2(2t)}\,e^{it\log\alpha}\,dt \text{ with } T = \frac{N^2}{\delta}$$

$$F_1(t) = \sum_{x_1, x_2 \sim N} (x_1^2 + x_2^2)^{it}$$ (partial sum of Epstein zeta function)

$$F_2(t) = \sum_{x \sim N} x^{it}$$ (partial sum of Riemann zeta function)

Split

$$\int_{|t|<T} = \int_{|t|<N^{\frac{1}{2}}} + \int_{N^{\frac{1}{2}}<|t|<T} = (1) + (2)$$

Contribution of (1) amounts to

$$\frac{\delta}{N^2} N^{\frac{1}{2}} \sum_{x_1, x_2, x_3 \sim N} 1_{[|\log(x_1^2+x_2^2)-2\log x_3 - \log \alpha|<N^{-\frac{1}{2}}]} \sim \delta N$$

without further assumptions on α

To bound contribution of (2), average in α and use Parseval $\Rightarrow$

$$\frac{1}{T}\Big[\int_{N^{\frac{1}{2}}<|t|<T} |F_1(t)|^2 |F_2(t)|^2 dt\Big]^{\frac{1}{2}} \leq T^{\frac{1}{2}}\Big(\max_{N^{\frac{1}{2}}<|t|<T} |F_2(t)|\Big).\Big(\frac{1}{T}\int_{|t|<T} |F_1(t)|^2 dt\Big)^{\frac{1}{2}}$$

Assuming Lindelöf

$$\max_{N^{\varepsilon}<|t|<T} |F_2(t)| \ll N^{\frac{1}{2}} T^{\varepsilon}$$

For second factor

$$\frac{1}{T}\int_{|t|<T} |F_1(t)|^2 dt$$

$$\sim \#\Big\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4; x_i\sim N \text{ and } |\log(x_1^2+x_2^2)-\log(x_3^2+x_4^2)|<\frac{1}{T}\Big\}$$

$$\sim \#\Big\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4; x_i\sim N \text{ and } |x_1^2+x_2^2-x_3^2-x_4^2|<\frac{N^2}{T}=\delta\Big\}$$

$$\ll N^{2+\varepsilon}$$

This gives

$$c\delta N + O\Big(\frac{\delta^{\frac{1}{2}}}{N}.N^{\frac{1}{2}+\varepsilon}N^{1+\varepsilon}\Big) \sim \delta N \text{ provided } N^{-1+\varepsilon}<\delta$$

Unconditionally, one may show that a.s. in α

$$\min_{\substack{x \in \mathbb{Z}^3 \setminus \{0\} \\ |x| < N}} |Q_\alpha(x)| \ll N^{-\frac{2}{5}+\varepsilon}$$

by bounding error term using distributional estimates. Main ingredients

- Large value estimate for Dirichlet polynomials

Lemma *Define* $S(t) = \sum_{n \sim M} a_n \, n^{it}$ *where* $|a_n| \leq 1$*. Then for* $T > M$

$$\text{mes}\,[|t| < T : |S(t)| > V] \ll M^\varepsilon (M^2 V^{-2} + M^4 V^{-6} T)$$

(due to **Jutila**, 1977)

Applied here with $M = N^2$ and $S(t) = F_2(t)^2$.

- Bounds on partial sums of the Epstein zeta function

Lemma *For $|t| > N^2$, one has the estimate*

$$\Big| \sum_{m,n \sim N} (m^2 + n^2)^{it} \Big| \ll N |t|^{\frac{1}{3}+\varepsilon}$$

Follows the steps of Van der Corput's third derivative estimate, i.e.

$$\left(\frac{1}{6}, \frac{2}{3}\right) \text{ is a (2-dim) exponent pair}$$

Appears in a paper of V. Blomer on Epstein zeta functions where it is used to establish a bound

$$\left| E\left(\frac{1}{2} + it\right) \right| \ll |t|^{\frac{1}{3}+\varepsilon} \text{ where } E(s) = \sum_{\substack{x \in \mathbb{Z}^2 \\ x \neq 0}} Q(x)^{-s}$$

SMALL GAPS IN SPECTRUM OF RECTANGULAR BILLIARD

Recall that $\{\lambda_1 < \lambda_2 < \cdots\} = \{\alpha m^2 + n^2 : m, n \in \mathbb{Z}\}$

$$\delta^{(\alpha)}_{\text{min}}(N) = \min_{i \le N}(\lambda_{i+1} - \lambda_i)$$

In order to bound $\delta^{(\alpha)}_{\text{min}}(N)$, consider equivalently $(M = \sqrt{N})$

$$\min\left\{n_1^2 - n_2^2 + \alpha(n_3^2 - n_4^2)| : n_i < M \text{ and } (n_1, n_3) \neq (n_2, n_4)\right\}$$

or

$$\min\left\{|n_1 n_2 - \alpha n_3 n_4| : 0 < n_i < M\right\} \qquad (*)$$

Problem Is it true that $(*) \ll \frac{1}{M^{2-\varepsilon}}$ for all α?

In [B-B-R-R], this was shown for certain quadratic irrationals and also generically

Proposition $(*) \ll \frac{1}{M^{2-\varepsilon}}$ *for almost all* α.

SKETCH OF THE PROOF

Taking $n_i \sim M$, condition

$$|n_1 n_2 - \alpha n_3 n_4| < \delta$$

is replaced by

$$|\log n_1 + \log n_2 - \log n_3 - \log n_4 - \log \alpha| < \frac{\delta}{M^2} = \frac{1}{T}$$

This leads to evaluation of

$$\frac{1}{T} \int_{|t|<T} |F(t)|^4 \, e^{it(\log \alpha)} dt \quad \text{with} \quad F(t) = \sum_{n \sim M} n^{it}$$

Split again

$$\int_{|t|<T} = \int_{|t|<M^{\frac{1}{2}}} + \int_{M^{\frac{1}{2}}<|t|<T} = (1) + (2)$$

Contribution of (1) evaluated independently of α and gives $c\delta M^2$

Contribution of (2) estimated in the α-mean and by Parseval bounded by

$$\frac{1}{T}\Big[\int_{M^{\frac{1}{2}}<|t|<T} |F(t)|^8 dt\Big]^{\frac{1}{2}}$$

Assuming Lindelöf, one obtains

$$\int_{M^{\frac{1}{2}}<|t|<T} |F(t)|^8 dt \ll T^{1+\varepsilon} M^4$$

leading to the condition $\delta M^2 > T^{-\frac{1}{2}+\varepsilon} M^2$, hence $\delta > M^{-2+\varepsilon}$.

The Lindelöf hypothesis may be avoided in the following way.

Replace $n_i \sim M$ by $n_i = n_i' n_i''$ with $n_i' \sim M^{1-\sigma}$ and $n_i'' \sim M^\sigma$ (σ arbitrary small and fixed).

We obtain

$$\frac{1}{T}\int_{|t|<T} |F(t)|^4 |G(t)|^4 \, e^{it(\log \alpha)} dt$$

where

$$F(t) = \sum_{n \sim M^{1-\sigma}} n^{it} \text{ and } G(t) = \sum_{n \sim M^\sigma} n^{it}$$

Error term becomes

$$\frac{1}{T}\left[\int_{M^{\frac{1}{2}}<|t|<T} |F(t)|^8 \, |G(t)|^8 dt\right]^{\frac{1}{2}}$$

Estimate

$$\int_{M^{\frac{1}{2}}<|t|<T} |F(t)|^8 \, |G(t)|^8 \, dt \leq \Big(\max_{M^{\frac{1}{2}}<|t|<T} |G(t)|^8\Big). \int_{|t|<T} |F(t)^4|^2 dt$$

$$< M^{8\sigma(1-\tau)} \, (T + M^{4(1-\sigma)}) \, M^{4(1-\sigma)} \text{ for some } \tau = \tau(\sigma) > 0$$

This gives

$$\delta M^{2-\varepsilon} + O\Big\{\Big(T^{-\frac{1}{2}} + \frac{1}{T} M^{2(1-\sigma)}\Big) M^{2+2\sigma-4\sigma\tau}\Big\}$$

$$= \delta M^{2-\varepsilon} + O\Big\{(\delta^{\frac{1}{2}} M^{-1+2\sigma} + \delta) M^{2-4\sigma\tau}\Big\}$$

and condition

$$\delta > M^{-2+4\sigma}$$

The lower bound

Proposition *For almost all α, letting $c = 1 - \frac{\log(e \log 2)}{\log 2} = 0.086\ldots$*

$$\delta_{\min}^{(\alpha)}(N) \gg \frac{(\log N)^c}{N} \text{ infinitively often}$$

uses K. Ford's result

$$\#\left\{u.v : u, v \in \mathbb{Z}_+, u, v < N^{\frac{1}{2}}\right\} \asymp N(\log N)^{-c}(\log\log N)^{-2/3}$$

and shows small deviation from expected Poisson statistics for the eigenvalues.

PAIR CORRELATION FOR HOMOGENOUS DIAGONAL FORMS

Recall that

$$F(x_1, \dots, x_k) = x_1^k + \alpha_2 x_2^k + \dots + \alpha_k x_k^k \quad (\alpha_i > 0)$$

and

$$R(a, b; T) = \frac{1}{T}\big|\{(m, n) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k; m \neq n, F(m) \leq T, F(n) \leq T \text{ and } F(m) - F(n) \in [a, b]\}\big|$$

A first step consists in a localization of the variables $m_i \in I_i, n_i \in J_i$ where I_i, J_i are intervals of size $T^{\frac{1}{k}-2\varepsilon}$ satisfying a separation condition dist $(I_i, J_i) > T^{\frac{1}{k}-\varepsilon}$.

Let $\xi = \frac{a+b}{2}$ and $\delta = \frac{b-a}{2}$. Evaluate

$$\frac{1}{T}\Big|\Big\{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k; m_i \in I_i, n_i \in J_i \text{ and } |F(m) - F(n) - \xi| < \delta\Big\}\Big| \qquad (*)$$

Condition $|F(m) - F(n) - \xi| < \delta$ is replaced by

$$\Big|\log\big(m_1^k - n_1^k + \alpha_2(m_2^k - n_2^k) + \dots + \alpha_{k-1}(m_{k-1}^k - n_{k-1}^k) - \xi\big) - \log(m_k^k - n_k^k) - \log \alpha_k\Big| < \frac{\delta}{\alpha_k \Delta} = \frac{1}{B}$$

where $\Delta \sim m_k^k - n_k^k$ for $m_k \in I_k, n_k \in J_k$

Evaluate $(*)$ by

$$\int S_1(t)\overline{S_2(t)}\, e^{-it\log\alpha_k}\ \widehat{1}_{[-\frac{1}{B},\frac{1}{B}]}\,(t)dt \qquad (**)$$

with

$$S_1(t) = \sum_{m_i\in I_i, n_i\in J_i} \left(m_1^k - n_1^k + \alpha_2(m_2^k - n_2^k) + \cdots + \alpha_{k-1}(m_{k-1}^k - n_{k-1}^k) - \xi\right)^{it}$$

$$S_2(t) = \sum_{m\in I_k, n\in J_k} (m^k - n^k)^{it}$$

Set $B_0 = \frac{\alpha_k\Delta}{T^{1-\kappa}}$ and decompose $1_{[-\frac{1}{B},\frac{1}{B}]} = \frac{B_0}{B}\,1_{[-\frac{1}{B_0},\frac{1}{B_0}]} + \left(1_{[-\frac{1}{B},\frac{1}{B}]} - \frac{B_0}{B}1_{[-\frac{1}{B_0},\frac{1}{B_0}]}\right)$ producing in $(**)$ the main contribution and an error term.

Evaluation of the error term uses pointwise bound on $S_2(t)$ and an estimate

$$\frac{1}{T}\int_{|t|<T} |S_1(t)|^2 dt \ll T^{3-\frac{4}{k}+\varepsilon} \text{ in average over } \alpha_2,\dots,\alpha_{k-1}$$

The following question seems open.

Problem It is true that for all $\alpha \in \mathbb{R}$

$$\inf_{m,n \in \mathbb{Z}_+} (m+n)\|mn\alpha\| = 0 \qquad ?$$