Quantitative decompositions of Lipschitz mappings

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supported by NSF DMS-1758709 and DMS-1763973

IAS Analysis Seminar

May 12, 2020

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such that

1. (i) each $f|_{E_i}$ is "simple",
2. (ii) the "size" of G is $< \alpha$,
3. (iii) the "simplicity" of f on each piece and the number M of pieces depends only on α (and maybe some ambient dimensions), but **not** on the particular map f.

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But this result is not quantitative: no control on number of pieces E_i, which is infinite, or the the bi-Lipschitz constant of f on each piece.

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The remainder of the talk will concern Euclidean domains and metric space targets, although the results are new even for Euclidean targets.

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Question

What can we say for Lipschitz mappings that lower dimension? e.g., from $[0,1]^3$ *to* $\mathbb{R}^2$*?*

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i.e. if you write points of $\mathbb{R}^{n+m}$ as $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, we would like our map to look like

$$(x,y) \mapsto x.$$

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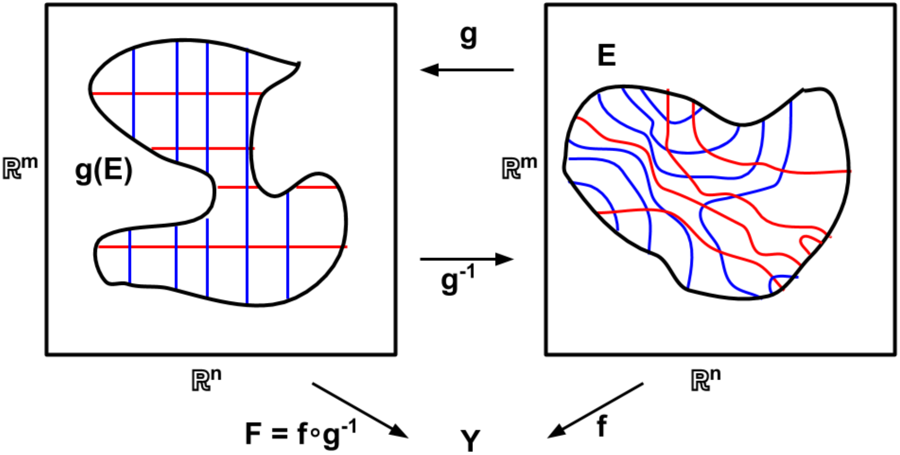
1. $F|_{g(E)}$ is **constant** on "vertical m-planes": $(\{x\} \times \mathbb{R}^m) \cap g(E)$, and
2. $F|_{g(E)}$ is **bi-Lipschitz** on "horizontal n-planes": $(\mathbb{R}^n \times \{y\}) \cap g(E)$.

Diagram of a "Hard Sard set"



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Answer: No!

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Theorem (Kaufman '79)

There is a C^1 *(hence Lipschitz)* surjection $f : [0,1]^3 \to [0,1]^2$ *such that* Df *has* rank ≤ 1 *everywhere.*

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Theorem (Azzam-Schul '12)

Suppose $\mathcal{H}^n(Y) \leq 1$. (Y is "n-dimensional".)
Then f has a Hard Sard set E with

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if and only if

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However, they did not completely decompose the domain of f (up to arbitrarily small error) into pieces on which it looks like a projection.

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Theorem (GCD-Schul '20)

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Proposition (GCD-Schul '20)

If $F \subseteq [0,1]^{n+m}$ is a set on which the map

$$(x,y) \mapsto (f(x,y),y)$$

is bi-Lipschitz, then F can be quantitatively decomposed into Hard Sard sets, plus a set of small measure.

Ideas behind the proof: Step 2

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Proposition (GCD-Schul '20; closely related to David-Semmes '00)

Given $\alpha > 0$, *we can write*

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Actually, this step works without any n-dimensionality assumption on Y.

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(For mappings to metric spaces, one has to interpret "close to linear" appropriately.)

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3. **Q such that $f|_Q$ is close to a linear map of rank $\geq n$.**

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If f was always close to the **same** linear map, say $(x, y) \mapsto y$, then we could pre-compose f by a rotation ϕ, and then supplement by $(x, y) \mapsto (f \circ \phi^{-1}(x, y), y)$.

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After this, one can form the change of coordinates ϕ by assembling different rotations at different scales, like a clockwork mechanism.

Thanks

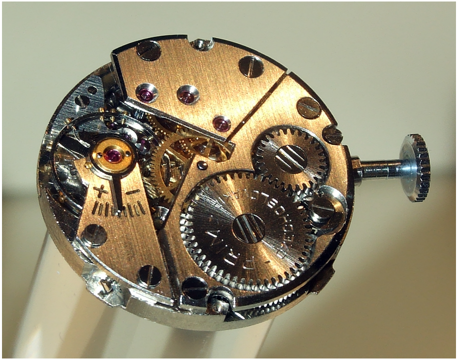

photo: `https://en.wikipedia.org/wiki/Clockwork##/media/File:Prim_clockwork.jpg`