Supersymmetric approach to random band matrices

Tatyana Shcherbina

Institute for Advanced Study, Princeton, USA

Analysis seminar, IAS, March 2, 2016

Local statistics, localization and delocalization

One of the key physical parameter of models is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length ℓ is comparable with the matrix size, and it is called localized otherwise.

- Localized eigenvectors: lack of transport (insulators), and Poisson local spectral statistics
- Delocalization: diffusion (electric conductors), and GUE local statistics.

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory.

From the RMT point of view, the main objects of the local regime are k-point correlation functions R_k $(k = 1, 2, \ldots)$, which can be defined by the equalities:

$$\mathbb{E}\left\{\sum_{j_1\neq\ldots\neq j_k}\varphi_k(\lambda_{j_1}^{(N)},\ldots,\lambda_{j_k}^{(N)})\right\}$$
$$=\int_{\mathbb{R}^k}\varphi_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})R_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})d\lambda_1^{(N)}\ldots d\lambda_k^{(N)},$$

where $\varphi_k : \mathbb{R}^k \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments.

Universality conjecture in the bulk of the spectrum (hermitian case, deloc.eg.s.) (Wigner – Dyson):

$$(N\rho(\lambda_0))^{-k}R_k(\{\lambda_0 + \xi_j/N\rho(\lambda_0)\}) \xrightarrow{N\to\infty} \det\left\{\frac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^{k}.$$

- Wigner matrices, β-ensembles with $\beta = 1, 2$, sample covariance matrices, etc.: delocalization, GUE/GOE local spectral statistics
- Anderson model (Random Schrödinger operators):

$$\mathrm{H_{RS}} = -\triangle + \mathrm{V},$$

where $\triangle$ is the discrete Laplacian in lattice box $\Lambda = [1, \mathrm{n}]^{\mathrm{d}} \cap \mathbb{Z}^{\mathrm{d}}$, V is a random potential (i.e. a diagonal matrix with i.i.d. entries). In $\mathrm{d} = 1$: narrow band matrix with i.i.d. diagonal

$$\mathrm{H_{RS}} = \begin{pmatrix} \mathrm{V}_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \mathrm{V}_2 & 1 & 0 & \dots & 0 \\ 0 & 1 & \mathrm{V}_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \mathrm{V}_{\mathrm{n}-1} & 1 \\ 0 & \dots & 0 & 0 & 1 & \mathrm{V}_{\mathrm{n}} \end{pmatrix}.$$

Localization, Poisson local spectral statistics (Fröhlich, Spencer, Aizenman, Molchanov, ...)

Random band matrices

Intermediate model that interpolates between random Schrödinger operator and Wigner matrices.

$\Lambda = [1, n]^d \cap \mathbb{Z}^d$ is a lattice box, $N = n^d$.

$$H = \{H_{jk}\}_{j,k \in \Lambda}, \quad H = H^*, \quad \mathbb{E}\{H_{jk}\} = 0.$$

Entries are independent (up to the symmetry) but not identically distributed. Variance is given by some function J (even, compact support or rapid decay)

$$\mathbb{E}\{|H_{jk}|^2\} = \frac{1}{W^d} J\Big(\frac{|j-k|}{W}\Big)$$

Main parameter: band width $W \in [1; N]$.

It also has nontrivial spatial structure like RS, but technically more simple.

Anderson transition in random band matrices

$W = O(1)$ [$\sim$ random Schrödinger] $\longleftrightarrow$ $W = N$ [Wigner matrices]

Varying W, we can see the transition between localization and delocalization

Conjecture (in the bulk of the spectrum):

$d = 1:$	$\ell \sim W^2$	$W \gg \sqrt{N}$	Delocalization, GUE statistics
		$W \ll \sqrt{N}$	Localization, Poisson statistics
$d = 2:$	$\ell \sim e^{W^2}$	$W \gg \sqrt{\log N}$	Delocalization, GUE statistics
		$W \ll \sqrt{\log N}$	Localization, Poisson statistics
$d \geq 3:$	$\ell \sim N$	$W \geq W_0$	Delocalization, GUE statistics

At the present time only some upper and lower bounds on the order of localization length are proved rigorously ($d = 1$).

- Schenker (2009) $\ell \leq W^8$ – localization techniques;
- Erdős, Yau, Yin (2011) $\ell \geq W$ – RM methods;
- Bourgade, Erdős, Yau, Yin (Feb. 2016) gap universality for $W \sim N$.

At the present time only some upper and lower bounds on the order of localization length are proved rigorously ($d = 1$).

- Schenker (2009) $\ell \leq W^8$ – localization techniques;
- Erdős, Yau, Yin (2011) $\ell \geq W$ – RM methods;
- Bourgade, Erdős, Yau, Yin (Feb. 2016) gap universality for $W \sim N$.

By the developing the Erdős-Yau approach, other results were obtained. In these bounds the localization length is controlled in a rather weak sense, i.e. the estimates hold for "most" eigenfunctions only:

- Erdős, Knowles (2011): $\ell \gg W^{7/6}$;
- Erdős, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$ (not uniform in N).

Main problem: to control of the resolvent $G(z) = (H - z)^{-1}$ for $\varepsilon := \operatorname{Im} z \sim 1/N$ (more precisely, to obtain the bounds for $\mathbb{E}\{|G(E + i\varepsilon)|^2\}$). The techniques allows to obtain the control only for $\varepsilon \sim 1/W$.

Another method, which allows to work with random operators with non-trivial spatial structures, is supersymmetry techniques (SUSY), which based on the representation of the determinant as an integral over the Grassmann variables.

This method is widely (and successively) used in the physics literature and is potentially very powerful but the rigorous control of the integral representations, which can be obtained by this method, is quite difficult.

Part of the formalism is rigorous and can be used. However, good understanding of saddle point approximation in Grassmann variables is still a major challenge for mathematicians.

The method has some restrictions. First of all, up to this point it was mainly applied to the matrices with Gaussian element's distribution (except the case of characteristic polynomials that we will discuss later). Besides, it is much simpler to consider covariance of a special form.

We consider the following two models:

- Random band matrices: specific covariance $J_{ij} = \left(-W^2\Delta + 1\right)^{-1}_{ij}$ where Δ is the discrete Laplacian with Neumann boundary conditions on $[1, n]^d$.

  Note that for $d = 1$ we have $J_{ij} \approx C_1 W^{-1} \exp\{-C_2|i-j|/W\}$, and so the variance of the matrix elements is exponentially small when $|i-j| \gg W$.

- Block band matrices

Assign to every site $j \in \Lambda$ one copy $K_j \simeq \mathbb{C}^W$ of an W-dimensional complex vector space, and set $K = \oplus K_j \simeq \mathbb{C}^{|\Lambda|W}$. From the physical point of view, we are assigning W valence electron orbitals to every atom of a solid with hypercubic lattice structure.

We start from the matrices $M : K \to K$ belonging to GUE, and then multiply the variances of all matrix elements of H connecting K_j and K_k by the positive number J_{jk}, $j, k \in \Lambda$ (which means that H becomes the matrix constructed of $W \times W$ blocks, and the variance in each block is constant).

Such models were first introduced and studied by Wegner.

Note that $P_N(dH)$ is invariant under conjugation $H \to U^*HU$ by $U \in \mathcal{U}$, where $\mathcal{U}$ is the direct product of all the groups of unitary transformations in the subspaces K_j. This means that the probability distribution $P_N(dH)$ has a local gauge invariance.

We consider

$$J = 1/W + \alpha\Delta/W, \quad \alpha < 1/4d.$$

This model is one of the possible realizations of the Gaussian random band matrices, for example for $d = 1$ they correspond to the band matrices with the width of the band $2W + 1$.

Density of states for both models:

Denote by $\lambda_1, \ldots, \lambda_N$ the eigenvalues of the random matrix H.

NCM and the density of states:

$$\mathcal{N}_N[\Delta] = N^{-1}\sharp\{\lambda_i \in \Delta\} \to \mathcal{N}(\Delta), \quad \rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \quad \lambda \in [-2, 2].$$

SUSY method is especially useful for characteristic polynomials.

Correlation functions of the characteristic polynomials:

$$F_{2k}(\Lambda) = \mathbb{E}\Big\{ \det(\lambda_1 - H) \dots \det(\lambda_{2k} - H) \Big\},$$

where $\Lambda = \mathrm{diag}\,\{\lambda_1, \dots, \lambda_{2k}\}$.
We are interested in the asymptotic behavior of this function for

$$\lambda_j = E + \frac{\xi_j}{N\rho(E)}, \quad j = 1, 2, \dots, \quad E \in (-2, 2).$$

Although $F_{2k}(\Lambda)$ is not a local object, it is also expected to be universal in some sense. Moreover, correlation functions of characteristic polynomials are expected to exhibit a crossover which is similar to that of local eigenvalue statistic (for 1d RBM: GUE/GOE for $W \gg \sqrt{N}$, and the different behavior for $W \ll \sqrt{N}$).

To prove universality, we want to obtain the control of $\mathbb{E}\,|G(E+i\varepsilon)|^2$ for $\varepsilon \sim N^{-1}$, where $G(z) = N^{-1}\mathrm{Tr}\,(H-z)^{-1}$.

This means that we have to control

$$G_2^{\pm}(z) := \mathbb{E}\left\{ \frac{\partial^2}{\partial x \partial y} \frac{\det\Big(H - E - i\varepsilon - \dfrac{x}{N}\Big) \cdot \det\Big(H - E + i\varepsilon - \dfrac{y}{N}\Big)}{\det(H - E - i\varepsilon)\cdot\det(H - E + i\varepsilon)}\Bigg|_{x,y=0}\right\}.$$

Grassmann (fermionic) variables can be used to represent the determinants in the numerator, and usual complex variables (bosonic) represents the determinants of the denominator. So from the SUSY point of view characteristic polynomials correspond to the so-called fermion-fermion sector of the supersymmetric full model (which describes the correlation functions R_k).

SUSY results for the characteristic polynomials:

Let $D_2 = F_2(\lambda_0, \lambda_0)$, $\bar{F}_2 = D_2^{-1} \cdot F_2$.

$$\lim_{n\to\infty} \bar{F}_2\Big(E+\frac{\xi}{2N\rho(E)}, E-\frac{\xi}{2N\rho(E)}\Big) = \begin{cases} \dfrac{\sin \pi\xi}{\pi\xi}, & W \geq N^{1/2+\theta}; \\ 1, & 1 \ll W \leq \sqrt{\dfrac{N}{C_* \log N}}. \end{cases}$$

First part: S., 2013 – saddle-point analysis; (the case of orthogonal symmetry is also done, S., 2015)

Second part: M. Shcherbina, S., 2016 – transfer matrix approach.

SUSY results for the density of states:

Let $g(z) = N^{-1}\mathbb{E}\{\mathrm{Tr}\,(H - z)^{-1}\}$, g_{sc} is a Stieltjes transform of semi-circle.

- Disertori, Pinson, Spencer, 2002: The smoothness and the local semicircle for averaged density for RBM in 3d, i.e.

$$|g(z) - g_{sc}(z)| \le C/W^2$$

  uniformly in $\mathrm{Im}\, z$, $W \ge W_0$.

- M. Shcherbina, S., in progress: local semicircle for averaged density for RBM in 1d (with an arrow W^{-1}).

First result uses the cluster expansion, the second one uses the supersymmetric transfer matrices.
All other result about the density for RBM deals with $\mathrm{Im}\, z \gg W^{-1}$ (but allows to control G_{ij}, which implies delocalization at this scale).

Other SUSY results for the full model:

- S., 2014: Gaussian case, three diagonal block band matrices with $J = \frac{\alpha}{W}\triangle + \frac{1}{W}$. If $W \sim N$, then

$$\frac{1}{(N\rho(\lambda_0))^2} R_2\big(\lambda_0{+}x/N\rho(\lambda_0), \lambda_0{+}y/N\rho(\lambda_0)\big) \stackrel{N\to\infty}{\longrightarrow} 1 - \frac{\sin^2(\pi(x-y))}{\pi^2(x-y)^2}$$

in any dimension.

Other SUSY results for the full model:

- S., 2014: Gaussian case, three diagonal block band matrices with $J = \frac{\alpha}{W}\triangle + \frac{1}{W}$. If $W \sim N$, then

$$\frac{1}{(N\rho(\lambda_0))^2}\, R_2\big(\lambda_0 + x/N\rho(\lambda_0), \lambda_0 + y/N\rho(\lambda_0)\big) \stackrel{N\to\infty}{\longrightarrow} 1 - \frac{\sin^2(\pi(x-y))}{\pi^2(x-y)^2}$$

  in any dimension.

- Erdős, Bao, 2015: Combining this techniques with Green's function comparison strategy (Erdős-Yau), they proved

$$\ell \geq W^{7/6}$$

  in a strong sense for the block band matrices with more or less general element's distribution (subexponential tails, four Gaussian moments).

Grassmann integration:

Anticommutation relations:

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0, \quad j,k = 1,\dots,n.$$

These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra $\mathfrak{A}$. We can also define functions of the Grassmann variables. Let χ be an element of $\mathfrak{A}$, i.e.

$$\chi = a + \sum_{j=1}^{n}(a_j\psi_j + b_j\overline{\psi}_j) + \sum_{j\neq k}(a_{j,k}\psi_j\psi_k + b_{j,k}\psi_j\overline{\psi}_k + c_{j,k}\overline{\psi}_j\overline{\psi}_k) + \dots .$$

For any smooth function f we mean by $f(\chi)$ the element of $\mathfrak{A}$ obtained by substituting $\chi - a$ in the Taylor series of f at the point a.

Examples:

$$\exp\{\mathrm{b}\,\overline{\psi}\psi\} = 1 + \mathrm{b}\,\overline{\psi}\psi + (\mathrm{b}\,\overline{\psi}\psi)^2/2 + \ldots = 1 + \mathrm{b}\,\overline{\psi}\psi,$$

$$\begin{aligned}
&\exp\{\mathrm{a}_{11}\overline{\psi}_1\psi_1 + \mathrm{a}_{12}\overline{\psi}_1\psi_2 + \mathrm{a}_{21}\overline{\psi}_2\psi_1 + \mathrm{a}_{22}\overline{\psi}_2\psi_2\} \\
&= 1 + \mathrm{a}_{11}\overline{\psi}_1\psi_1 + \mathrm{a}_{12}\overline{\psi}_1\psi_2 + \mathrm{a}_{21}\overline{\psi}_2\psi_1 \\
&+ \mathrm{a}_{22}\overline{\psi}_2\psi_2 + (\mathrm{a}_{11}\mathrm{a}_{22} - \mathrm{a}_{12}\mathrm{a}_{21})\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2.
\end{aligned}$$

Integral (Berezin):

$$\int d\,\psi_j = \int d\,\overline{\psi}_j = 0, \quad \int \psi_j d\,\psi_j = \int \overline{\psi}_j d\,\overline{\psi}_j = 1.$$

Thus, if

$$f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^{k} p_{j_1}\psi_{j_1} + \sum_{j_1<j_2} p_{j_1 j_2}\psi_{j_1}\psi_{j_2} + \ldots + p_{1,2,\ldots,k}\psi_1 \ldots \psi_k,$$

then

$$\int f(\psi_1, \ldots, \psi_k) d\,\psi_k \ldots d\,\psi_1 = p_{1,2,\ldots,k}.$$

Gaussian integration in Grassmann variables:

$$\int \exp\Big\{ -\sum_{j,k=1}^{n} A_{jk}\overline{\psi}_j\psi_k \Big\} \prod_{j=1}^{n} d\,\overline{\psi}_j d\,\psi_j = \det A. \tag{1}$$

Let

$$F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix},$$

where a and $b > 0$ are Hermitian complex $k \times k$ matrices and σ, ρ are $k \times k$ matrix of the anticommuting Grassmann variables, and let

$$\Phi = (\psi_1, \dots, \psi_k, z_1, \dots, z_k)^t,$$

where $\{\psi_j\}_{j=1}^k$ are independent Grassmann variables and $\{z_j\}_{j=1}^k$ are complex variables. Combining (1) with usual Gaussian integration we obtain

$$\int \exp\{-\Phi^+ F \Phi\} \prod_{j=1}^k d\overline{\psi}_j \, d\psi_j \prod_{j=1}^k \frac{\operatorname{Re} z_j \operatorname{Im} z_j}{\pi} = \operatorname{sdet} F,$$

where

$$\operatorname{sdet} F = \frac{\det(a - \sigma\, b^{-1}\, \rho)}{\det b}.$$

Integral representation for characteristic polynomials

Using representation of the determinant in Grassmann variables we obtain

$$F_2(\Lambda) = \mathbb{E}\Big\{ \int e^{-\sum\limits_{\alpha=1}^{2}\sum\limits_{j,k=1}^{N}(\lambda_\alpha - H)_{jk}\overline{\psi}_{j\alpha}\psi_{k\alpha}} \prod_{\alpha=1}^{2}\prod_{q=1}^{N} d\,\overline{\psi}_{q\alpha} d\,\psi_{q\alpha} \Big\}$$

$$= \mathbb{E}\Big\{ \int e^{-\sum\limits_{\alpha=1}^{2}\lambda_\alpha \sum\limits_{p=1}^{N}\overline{\psi}_{p\alpha}\psi_{p\alpha}} \exp\Big\{ \sum_{j<k}\sum_{\alpha=1}^{2}\Big(\mathrm{Re}\, H_{jk}\cdot(\overline{\psi}_{j\alpha}\psi_{k\alpha} + \overline{\psi}_{k\alpha}\psi_{j\alpha})$$

$$+ i\,\mathrm{Im}\, H_{jk}\cdot(\overline{\psi}_{j\alpha}\psi_{k\alpha} - \overline{\psi}_{k\alpha}\psi_{j\alpha})\Big) + \sum_{j=1}^{N} H_{jj}\cdot\sum_{\alpha=1}^{2}\overline{\psi}_{j\alpha}\psi_{j\alpha}\Big\} d\Psi \Big\},$$

where $\{\psi_{j\alpha}\}$, $j = 1, \dots, N$, $\alpha = 1, 2$ are the Grassmann variables (N variables for each determinant).

$$F_2(\Lambda) = \int \exp\Big\{ -\sum_{s=1}^{2} \lambda_s \sum_{p=1}^{N} \overline{\psi}_{ps}\psi_{ps} + \sum_{j=1}^{N} \frac{J_{j,j}}{2}(\overline{\psi}_{j1}\psi_{j1} + \overline{\psi}_{j2}\psi_{j2})^2 \Big\}$$
$$\times \exp\Big\{ \sum_{j<k} J_{j,k}(\overline{\psi}_{j1}\psi_{k1} + \overline{\psi}_{j2}\psi_{k2})(\overline{\psi}_{k1}\psi_{j1} + \overline{\psi}_{k2}\psi_{j2}) \Big\} d\Psi.$$

Hubbard-Stratonovich transform: $e^{a^2/2} = (2\pi)^{-1/2}\int e^{-x^2/2+ax}dx$.
Applying many times we get

$$(2\pi^2)^N \det{}^2 J \exp\Big\{ \frac{1}{2}\sum_{j,k} J_{j,k}(\overline{\psi}_{j1}\psi_{k1} + \overline{\psi}_{j2}\psi_{k2})(\overline{\psi}_{k1}\psi_{j1} + \overline{\psi}_{k2}\psi_{j2}) \Big\}$$
$$= \int \exp\Big\{ -\frac{1}{2}\sum_{j,k} J^{-1}_{j,k} \operatorname{Tr} X_j X_k - i\sum_j (\overline{\psi}_{j1}, \overline{\psi}_{j2}) X_j \begin{pmatrix} \psi_{j1} \\ \psi_{j2} \end{pmatrix} \Big\} \prod_{j=1}^{N} dX_j,$$

where X_j is hermitian 2×2 matrix and

$$dX_j = d\,\mathrm{Re}\,X_{12}\, d\,\mathrm{Im}\,X_{12}\, dX_{11}\, dX_{22}.$$

Integral representation for characteristic polynomials for RBM

$$F_2(\hat{\xi}) = C_N \int\limits_{\mathcal{H}_2^N} \exp\Big\{ -\frac{W^2}{2}\sum_{j=2}^{N} \mathrm{Tr}\,(X_j - X_{j-1})^2\Big\}\times$$
$$\exp\Big\{ -\frac{1}{2}\sum_{j=1}^{N} \mathrm{Tr}\,\Big(X_j + \frac{iE\cdot I}{2} + \frac{i\hat{\xi}}{2N\rho(\lambda_0)}\Big)^2\Big\}\prod_{j=1}^{N}\det\,(X_j - iE\cdot I/2)d\overline{X},$$

where $\{X_j\}$ are hermitian 2×2 matrices and $\hat{\xi} = \xi\sigma_3$.

Transfer matrix for the determinants

Set $\mathcal{H} = L_2[\mathrm{Herm}(2)]$, and let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be the operator of multiplication by

$$\mathcal{F}(X) = \exp\Big\{ -\frac{1}{4}\,\mathrm{Tr}\,\Big(X + \frac{iE\cdot I}{2} + \frac{i\hat{\xi}}{N\rho(E)}\Big)^2 + \frac{1}{2}\,\mathrm{Tr}\,\log\big(X - iE\cdot I/2\big) - C \Big\}.$$

Define $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ the operator with the kernel

$$\mathcal{K}(X, Y) = \frac{W^4}{2\pi^2}\,\mathcal{F}(X)\,\exp\Big\{ -\frac{W^2}{2}\mathrm{Tr}\,(X - Y)^2 \Big\}\,\mathcal{F}(Y).$$

Then

$$F_2\Big(E\cdot I + \frac{\widehat{\xi}}{N\rho(E)}\Big) = -W^{-4N}\det{}^{-2}J\cdot\mathrm{Tr}\,\mathcal{K}^N,$$

and thus we are interested in the asymptotic behaviour of $\mathrm{Tr}\,\mathcal{K}^N$.

For arbitrary compact operator M we denote $\lambda_j(M)$ the jth (by its modulo) eigenvalue of M, so that $|\lambda_0(M)| \ge |\lambda_1(M)| \ge \dots$
Part of $\mathcal{K}$ with $\hat{\xi}$ can be considered as a perturbation operator $\tilde{K}$ (of order N^{-1}).
Localization side: We are going to prove

$$\lambda_0(\mathcal{K}) = \Big(1 + \frac{C}{W}\Big)^{-1} + O(W^{-2}), \quad |\lambda_1(\mathcal{K})| \le |\lambda_0(K)| - C_0/W^2.$$

Then $\mathrm{Tr}\,\mathcal{K}^N = \lambda_0^N(\mathcal{K})\,(1+r)$, where

$$|r| = \Big|\sum_{j=1}^{\infty}(\lambda_j(\mathcal{K})/\lambda_0(\mathcal{K}))^N\Big| \le \Big|\frac{\lambda_1(\mathcal{K})}{\lambda_0(\mathcal{K})}\Big|^{N-2}\sum_{j=0}^{\infty}|\lambda_j(\mathcal{K})|^2$$
$$\le C e^{-CN/W^2}\int |\mathcal{K}(X,Y)|^2 dX dY \le C W^4 e^{-CC_*\log N} = o(1).$$

The main difficulty is that $\mathcal{K} = K + \widetilde{K}$ is not self-adjoint; thus perturbation theory is not easily applied in a rigorous way. For instance, it is not enough to estimate the gap for K only.

Main operator K

Changing the variables

$$X = U\Lambda U^*, \quad \Lambda = \mathrm{diag}\{a, b\}, \quad a > b, \quad U \in U(2),$$

we obtain that K can be represented as an integral operator in $L_2[\mathbb{R}^2, \frac{\pi}{2}(a-b)^2\, da\, db] \times L_2[U(2), dU]$ defined by the kernel

$$K(\bar{a}, \bar{b}, \bar{U}) = t^{-1} A(a_1, a_2)\, A(b_1, b_2)\, K_*(t, U_1, U_2)$$

with

$$A(x, y) = (2\pi)^{-1/2} W \cdot e^{-W^2(x-y)^2/2 - f(x) - f(y)};$$
$$f(x) = \big((x + i\lambda_0/2)^2/2 - \log(x - i\lambda_0/2) - C\big)/2;$$
$$K_*(t, U_1, U_2) := W^2 t \cdot e^{tW^2 \mathrm{Tr}\, U_1 U_2^* \sigma_3 (U_1 U_2^*)^* \sigma_3/4 - tW^2/2},$$

where $\bar{a} = (a_1, a_2)$, $\bar{b} = (b_1, b_2)$, $\bar{U} = (U_1, U_2)$, and

$$t = (a_1 - b_1)(a_2 - b_2).$$

Density:

Integral representation for the density of states for RBM

$$g(E) = \int e^{-\frac{W^2}{2}\sum_{j=2}^{N} \mathrm{str}\,(F_j - F_{j-1})^2 - \frac{1}{2}\sum_{j=1}^{N} \mathrm{str}\,\left(F_j + \frac{iE}{2}\right)^2} \prod_{j=1}^{N} \mathrm{sdet}\,\left(F_j - \frac{iE}{2}\right) d\overline{F},$$

where

$$F_j = \begin{pmatrix} ia_j & \bar\rho_j \\ \rho_j & b_j \end{pmatrix}, \quad \mathrm{str}\, F_j = ia_j - b_j, \quad \mathrm{sdet}\, F_j = \frac{ia_j - \bar\rho_j b_j^{-1} \rho_j}{b_j}.$$

Transfer matrix: acts on $(f_1, f_2)^t := f_1(a,b) + f_2(a,b)\bar\rho\rho$

$$\begin{pmatrix} A(a_1,a_2)A_1(b_1,b_2) & 0 \\ 0 & A(a_1,a_2)A_1(b_1,b_2) \end{pmatrix} \cdot \begin{pmatrix} 1 + \frac{L(a,b)}{W^2} & -L(a,b) \\ -\frac{1}{W^2} & 1 \end{pmatrix}$$

Integral representation for the second correlation function for block RBM

$$C_N \int \exp\Big\{ -\frac{\alpha W}{2} \sum_{j\sim j'} \mathrm{str}\,(F_j - F_{j'})^2 \Big\} \times$$

$$\exp\Big\{ \frac{W}{2} \sum_{j\in\Lambda} \mathrm{str}\,\Big(F_j + \frac{i\Lambda_0}{2} + \frac{i\widehat{\xi}}{N\rho(\lambda_0)}\Big)^2 \Big\} \prod_{j\in\Lambda} \mathrm{sdet}^{-W}\big(F_j - i\Lambda_0/2\big) d\overline{F},$$

where

$$F_j = \begin{pmatrix} U_j & \rho_j \\ \tau_j & B_j \end{pmatrix},$$

U_j is 2×2 unitary matrix, B_j is 2×2 positive Hermitian matrix times σ_3, and ρ, τ are 2×2 matrices whose entries are independent Grassmann variables.

Here

$$\mathrm{sdet}\, F_j = \frac{\det\,(U_j - \tau_j B_j^{-1} \rho_j)}{\det\ B_j}, \quad \mathrm{str}\, F_j = \mathrm{Tr}\, U_j - \mathrm{Tr}\, B_j.$$