Reinforced random walks and statistical physics

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- Edge-Reinforced random walk (ERRW): first results ('86-'09)

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- VRJP $\longleftrightarrow$ 2nd generalised Ray-Knight Theorem
- Application : recurrence/transience

Self-interacting RW: definition and introduction

- $G = (V, E)$ non-oriented locally finite graph
- $W_e : \mathbb{N} \longrightarrow \mathbb{R}_+$, $e \in E$, weight functions

• Vertex SIRW (VSIRW) (X_n) on V : $X_0 = i_0$ and, if $X_n = i$, then

$$\mathbb{P}(X_{n+1} = j \,|\, X_k,\ k \leqslant n) = \mathbb{1}_{\{j \sim i\}} \frac{W_{\{i,j\}}(Z_n(j))}{\sum_{k \sim X_n} W_{\{i,k\}} Z_n(k)}$$

where

$$Z_n(j) = \sum_{k=0}^{n} \mathbb{1}_{\{X_k = j\}}.$$

• Edge SIRW (ESIRW): replace $Z_n(j)$, $j \sim i$ by

$$Z_n(\{i,j\}) = \sum_{k=0}^{n} \mathbb{1}_{\{\{X_{k-1}, X_k\} = \{i,j\}\}}$$

Self-Interacting RW: definition and introduction

- Edge and Vertex reinforced random walk (ERRW and VRRW): (ESIRW and VSIRW) with $W_e(n) = a_e + n$, $a_e > 0$, $e \in E$
- ERRW introduced by Diaconis and Coppersmith in 1986 : behavior of person in a new city

Applications and related questions

- Reinforcement learning in game theory (see e.g. Erev-Roth'95): m people play repeatedly the same game, each having d possible strategies, and reinforce on strategies according to payoff : for instance creation of social networks, or of a new language (Kious and T.'16, Hu, Skyrms and T.'16)
- Biology:
  - Trail formation for ants, Schneirla'33
  - model for aggregation of myxobacteria, with short range interactions, Othmer and Stevens'97

Self-Interacting RW: definition and introduction

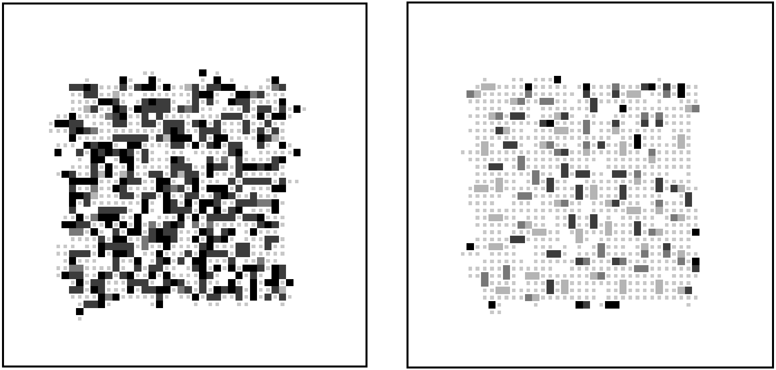
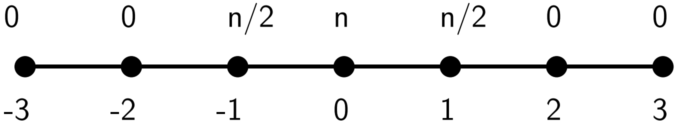


Figure : Model with 1000 particles after 1000 iterations, $W_e(n) = 2^n$: reinforcement by edges on the left, by sites on the right (Othmer and Stevens, 1997).
Black: 1 bacteria, dark grey 2 to 4, grey 5 or 6, light grey: 7 or more

Self-Interacting RW: preliminary remarks

- VRRW $\neq$ ERRW. $a_e := 1$.

On seven sites, at time k, $X_k = 1$ and



- VRRW: $\mathbb{P}(X_{k+2} = 3 \,|\, X_i,\ i \leqslant k) \sim 2/n^2$
- ERRW: $\mathbb{P}(X_{k+2} = 3 \,|\, X_i,\ i \leqslant k) \sim 1/(3n)$

Self-Interacting RWs: preliminary remarks

- Edge or Vertex SIRW on 3 vertices, $W_e(n) = W(n) = (n+1)^{\alpha}$, $\alpha \in \mathbb{R}$.
  - $\alpha < 1$. Then $Z_n(1)/Z_n(-1) \longrightarrow 1$ a.s.
  - $\alpha > 1$ (more generally $\sum W(n)^{-1} < \infty$).
    Then $Z_\infty(1) < \infty$ or $Z_\infty(-1) < \infty$ a.s.
  - $\alpha = 1$. Then $Z_n(1)/(Z_n(1) + Z_n(-1)) \longrightarrow \beta \in (0,1)$ r.v.

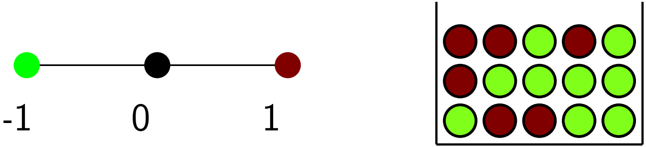


Figure : The VSIRW can be represented by successive draws into the urn: n-th draw, G_n and R_n numbers of balls of green and red color, then probability to pick a green ball is $W(G_n)/(W(G_n) + W(R_n))$.

Vertex Self-Interacting RWs: localisation results

Let R' be the asymptotic range of the process:

$$R' := \{v \in V \text{ s.t. } Z_\infty(v) = \infty\}.$$

VRRW with $W_e(n) = n + a$, $a > 0$, $e \in E$.

Theorem ($G = \mathbb{Z}$, T., '04 and '11, conjectured by Pemantle and Volkov '88, '01)

$$\mathbb{P}(|R'| = 5) = 1.$$

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Continuous-time version of RW in T.'11 used in other contexts:

- VSIRW with superlinear reinforcement (Basdevant, Schapira, Singh '12): localisation on 4 or 5 sites a.s., $W(n) = n \log\log n$
- Proof of conjecture of Erschler, Tóth and Werner (Kious '16): localisation of RWs with competing interactions: self-repelling at distance 1 and self-attracting at distance 2.

Vertex Self-Interacting RW: localisation results

VRRW with $W_e(n) = b_e(a_e + n)$, $a_e > 0$, $b_e > 0$, $e \in E$.

Theorem (G arbitrary, Volkov '03, Benaïm and T., '11)

If G is of bounded degree and under some assumptions on a, b, then

$$\mathbb{P}(|R'| < \infty) > 0.$$

• Generically, $R' = S \cup \partial S$, where S is a complete d-partite graph for some integer $d \geqslant 1$.

• Localisation sets are supports of equilibria of linear replicator dynamics in population biology (Fisher, Wright, Haldane'1920-30)

Vertex Self-Interacting RW: localisation results

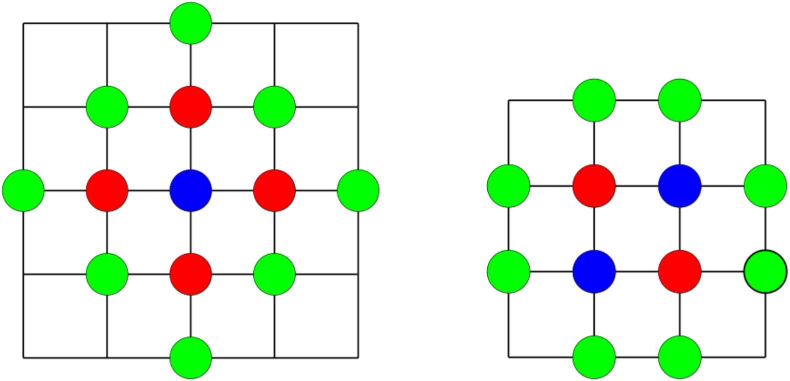

Figure : Two types of localisation on $\mathbb{Z}^2$ for the VRRW, respectively on 13 and 12 points. The seldom visited sites are green, and the other sites divide up into their blue and red parts of the complete bipartite subgraph.

Edge Self-Interacting RW: scaling and localisation results

Assume $W_e(n) = W(n)$.

Lemma

If $\sum W(n)^{-1} = \infty$ and G connected, then $R' = \emptyset$ or $R' = V$ a.s.

PROOF: Conditional Borel-Cantelli lemma. $\square$

Theorem (Tóth'94-98, $G = \mathbb{Z}$, $W(n) = (n+1)^\alpha$)

X_t/t^ν converges to non-trivial (non-gaussian) law, where

$$\begin{cases} \nu = 1/2 & \text{if } \alpha \leqslant 0 \\ \nu = (1-\alpha)/(2-\alpha) & \text{if } \alpha \in (0,1) \end{cases}$$

Theorem (Sellke'94, Limic'03, Limic-T.'07, Cotar-Thacker'15)

If $\sum W(n)^{-1} < \infty$ and G of bounded degree, then $|R'| = 2$ a.s.

First results on Edge-Reinforced random walk ('86-'09)

- Partial exchangeability $\implies$ ERRW is a RWRE

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  Coppersmith-Diaconis '86, Keane-Rolles '00
- Pemantle '88: recurrence/transience phase transition on trees
- Merkl Rolles '09: recurrence on a $2d$ graph (but not $\mathbb{Z}^2$)

ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (I)

Let $(W_e)_{e\in E}$ be conductances on edges, $W_e > 0$.
VRJP $(Y_s)_{s\geqslant 0}$ is a continuous-time process defined by $Y_0 = i_0$ and, if $Y_s = i$, then, conditionally to the past,

$$Y \text{ jumps to } j \sim i \text{ at rate } W_{i,j} L_j(s),$$

with

$$L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u=j\}} du.$$

Proposed by Werner and first studied **on trees** by Davis, Volkov ('02,'04).

ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (II)

Theorem (T. '11, Sabot, T. '11)

$$\begin{array}{cl} & \textit{ERRW } (X_n)_{n\in\mathbb{N}} \textit{ with weights } (a_e)_{e\in E} \\ \overset{\text{"law"}}{=} & \textit{VRJP } (Y_t)_{t\geqslant 0} \textit{ with conductances } W_e \sim \Gamma(a_e) \textit{ indep.} \\ & \textit{(at jump times)} \end{array}$$

ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (II)

Theorem (T. '11, Sabot, T. '11)

ERRW $(X_n)_{n\in\mathbb{N}}$ *with weights* $(a_e)_{e\in E}$

$\overset{\text{"law"}}{=}$ *VRJP* $(Y_t)_{t\geqslant 0}$ *with conductances* $W_e \sim \Gamma(a_e)$ *indep.*

(at jump times)

- Similar equivalence applies to any linearly reinforced RW on its continuous time version (initially proved for VRRW, T'. 11)

ERRW $\longleftrightarrow$ VRJP : Rubin and Kendall

Theorem (T'.11, ST '11)

- $W_e \sim Gamma(a_e)$ *independent*, $e \in E$
- Y *VRJP with conductances* W_e.

Then $(Y_t)_{t\geqslant 0}$ *(at jump times)* $\overset{\text{"law"}}{=}$ (X_n) .

Three ingredients :

- Rubin construction (Davis '90, Sellke '94) : $(\tilde{X}_t)$ continuous-time version of (X_n).
- Kendall transform ('66): Representation of Yule process as Poisson Point Process with Gamma random parameter after change in time
- Another change of time

VRJP $\longleftrightarrow$ SuSy hyperbolic sigma model in QFT (I)

- $G = (V, E)$ finite, $N := |V|$
- Fixed conductances (W_e).
- $\mathbb{P}_{i_0}$ law of $(Y_s)_{s\geqslant 0}$ starting from $i_0 \in V$
- Change time at vertices $\ell_i = L_i^2 - 1$, $i \in V \longrightarrow (Z_t)_{t\geqslant 0}$

$$B(s) = \sum_{i\in V}(L_i(s)^2 - 1), \quad Z_t = Y_{B^{-1}(t)}.$$

Theorem (ST '11)

Under $\mathbb{P}_{i_0}$, $(Z_t)_{t\geqslant 0}$ is a mixture of Markov jump processes (MJPs) starting from i_0 with jump rate from i to j

$$\frac{1}{2} W_{i,j} e^{U_j - U_i}.$$

Let $\mathcal{Q}^{i_0,W}$ be the mixing measure on $U = (U_i)_{i\in V}$.

VRJP $\longleftrightarrow$ SuSy hyperbolic sigma model in QFT (II)

Theorem (ST '11 continued)

The measure $\mathcal{Q}^{i_0,W}(du)$ *has density on* $\mathcal{H}_0 = \{(u_i), \sum u_i = 0\}$

$$\frac{N}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)},$$

where

$$H(W,u) = 2 \sum_{\{i,j\}\in E} W_{i,j} \sinh^2\left((u_i - u_j)/2\right)$$

and

$$D(W,u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in T} W_{\{i,j\}} e^{u_i+u_j},$$

$\mathcal{T}$ *is the set of (non-oriented) spanning trees of* G.

Gibbs "measure" on supermanifold extension $H^{2|2}$ of hyperbolic plane with action $A_W(v,v) = \sum_{i,j} W_{ij}(v_i - v_j, v_i - v_j)$, taken in horospherical coordinates after integration over fermionic variables.

VRJP $\longleftrightarrow$ Random Schrödinger operator ST-Zeng '15 (I)

Let, for all $i \in V$,

$$\beta_i = \frac{1}{2}\sum_{j\sim i} W_{ij} e^{u_j - u_i} + \mathbf{1}_{i_0}\gamma,$$

$\gamma \sim \Gamma(1/2)$ indep. of u.

- $\forall i \neq i_0$, $\beta_i =$ jump rate from i
- β field 1-dependent: $\beta_{|V_1}$ and $\beta_{|V_2}$ are independent if $dist_{\mathcal{G}}(V_1, V_2) \geqslant 2$.
- On $\mathbb{Z}^d$ with $W_{ij} = W$ constant, $(\beta_i)_{i\in V}$ translation-invariant
- The marginals β_i are such that $(2\beta_i)^{-1}$ have inverse Gaussian law.

VRJP $\longleftrightarrow$ Random Schrödinger: Range and law of β (II)

- V finite
- $\Delta = (\Delta_{i,j})_{i,j \in V}$ discrete Laplacian, letting $W_i := \sum_{j \sim i} W_{i,j}$,

$$\Delta_{i,j} := \begin{cases} W_{i,j}, & \text{if } i \sim j,\ i \neq j \\ -W_i, & \text{if i=j,} \end{cases}$$

- $H_\beta := -\Delta + 2(\beta - W)$.
- $H_\beta > 0$ (positive definite): $\implies$ $(H_\beta)^{-1}$ has positive entries.
- $\beta = (\beta_i)_{i \in V}$ has distribution

$$\nu^W(d\beta) = \sqrt{\frac{2}{\pi}}^{|V|} \mathbb{1}_{\{H_\beta > 0\}} \frac{e^{\sum_{i \in V}(W_i - \beta_i)}}{\sqrt{|H_\beta|}} \prod_{i \in V} d\beta_i.$$

VRJP $\longleftrightarrow$ Random Schrödinger: Retrieve u from β (III)

- Set $G = (H_\beta)^{-1}$.
- Then

$$e^{u_i} = \frac{G(i_0, i)}{G(i_0, i_0)},$$

where $(u_i)_{i \in V}$ defined above and follows the law $\mathcal{Q}_{i_0}^{W}(du)$.

- Hence, time-changed VRJP starting from i_0 mixture of Markov jump processes with jump rate

$$\frac{1}{2} W_{i,j} e^{u_j - u_i} = \frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$$

VRJP $\longleftrightarrow$ Dynkin's isomorphism (ST'15)

Theorem (Generalized second Ray-Knight theorem)

For any $u > 0$, *letting* $\sigma_u = \inf\{t \geqslant 0;\ \ell_{i_0}(t) > u\}$,

$$\left(\ell_i(\sigma_u) + \frac{1}{2}\varphi_i^2\right)_{i\in V} \quad \textit{under } \mathbb{P}_{i_0} \otimes P^{G,U}, \ \textit{has the same law as}$$

$$\left(\frac{1}{2}(\varphi_i + \sqrt{2u})^2\right)_{i\in V} \quad \textit{under } P^{G,U}.$$

Let

$$\Phi_i = \sqrt{\varphi_i^2 + 2\ell_i(\sigma_u)}.$$

Theorem (Sabot, T. '14)

$$\mathcal{L}\left(\varphi | \Phi\right)$$

can be retrieved from a magnetized version of the reverse VRJP.

ERRW and statistical physics: implications

Using link of QFT and localisation/delocalisation results of Disertori, Spencer, Zirnbauer '10 :

Theorem (ST'12, Angel-Crawford-Kozma'12, G bded degree)

ERRW is positive recurrent at strong reinforcement, i.e. for a_e is uniformly small in $e \in E$.

Theorem (Disertori-ST'14, $G = \mathbb{Z}^d$, $d \geqslant 3$)

ERRW is transient at weak reinforcement, i.e. for a_e uniformly large in $e \in E$.

Using link with Random Schrödinger operator:

Theorem (Sabot-Zeng '15,Merkl-Rolles '09)

ERRW with constant weights $a_e = a$ is recurrent in dimension 2.