Universal current fluctuations in non equilibrium systems

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IAS Princeton

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OUTLINE

- Current fluctuations
- The Fano factor on general graphs
- The macroscopic fluctuation theory
- One dimension versus higher dimension

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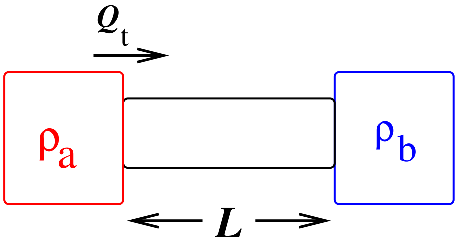
- The Fano factor on general graphs
- The macroscopic fluctuation theory
- One dimension versus higher dimension
- The parameter ω

  KPZ
- Quantum transport and full counting statistics

  Random Matrices

NON EQUILIBRIUM STEADY STATE



2 reservoirs of particles

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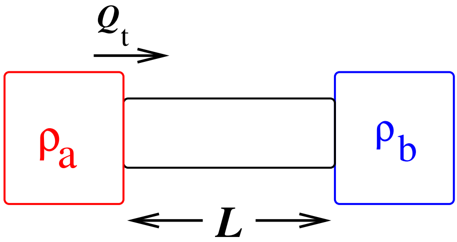

2 reservoirs of particles

$\langle Q_t \rangle$?

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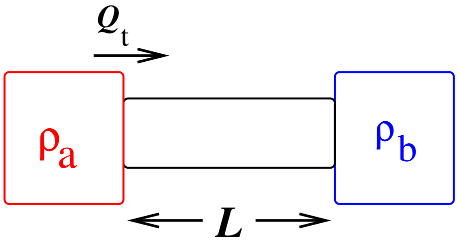

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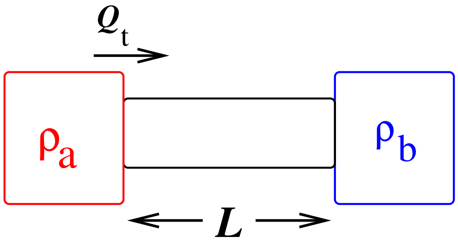

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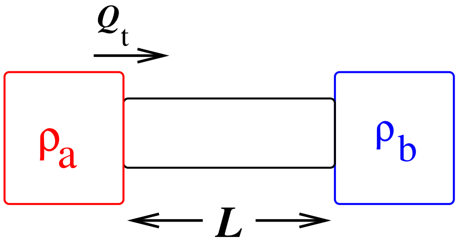
Distribution $P(Q_t)$ of Q_t

NON EQUILIBRIUM STEADY STATE



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Generating function (for large t)

$$\langle \exp[\lambda Q_t] \rangle \sim \exp[t\ \mu(\lambda)]$$

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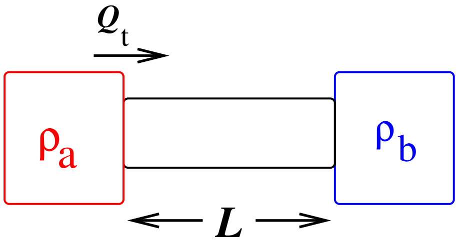

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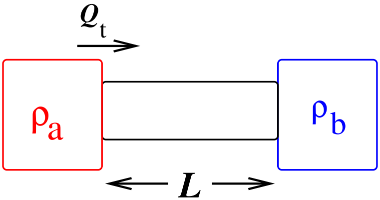
$$\langle \exp[\lambda Q_t] \rangle \sim \exp[t\ \mu(\lambda)]$$

($Q_t \simeq$ Sum of t/τ independent random variables)

INDEPENDENT PARTICLES

(Ballistic, random walkers, Levy flights,..)



with $\rho_a > 0$ and $\rho_b = 0$

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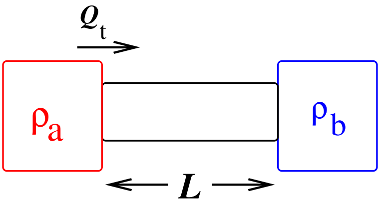


with $\rho_a > 0$ and $\rho_b = 0$

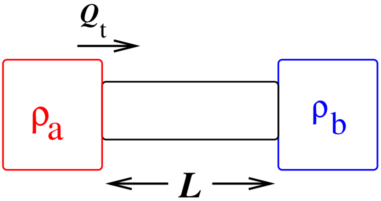
Poisson process:

$$\langle \exp[\lambda Q_t] \rangle \sim \exp[t \; \mu(\lambda)]$$

with $\mu(\lambda) = \kappa \times (e^{\lambda} - 1)$

INDEPENDENT PARTICLES

(Ballistic, random walkers, Levy flights,..)



with $\rho_{\rm a} > 0$ and $\rho_{\rm b} = 0$

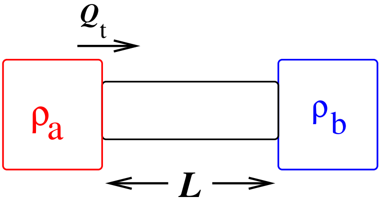
Poisson process:

$$\langle \exp[\lambda Q_t] \rangle \sim \exp[t\ \mu(\lambda)] \quad \text{with} \quad \mu(\lambda) = \kappa \times (e^{\lambda} - 1)$$

Fano factor $\quad F = \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{\langle Q_t \rangle} = 1$

INDEPENDENT PARTICLES

(Ballistic, random walkers, Levy flights,..)



with $\rho_a > 0$ and $\rho_b = 0$

Poisson process:

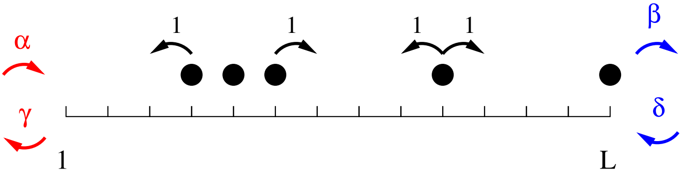
$\langle \exp[\lambda Q_t] \rangle \sim \exp[t \ \mu(\lambda)]$ with $\mu(\lambda) = \kappa \times (e^\lambda - 1)$

If during dt, there is a probability pdt that a particle is emitted by the left reservoir which will hit the right resevoir before the left reservoir

$$\langle e^{\lambda Q_t} \rangle = \prod_{dt} (1 - pdt + pdte^\lambda) = \exp\left[t \times p(e^\lambda - 1)\right]$$

EXCLUSION PROCESSES

SSEP (Symmetric simple exclusion process)

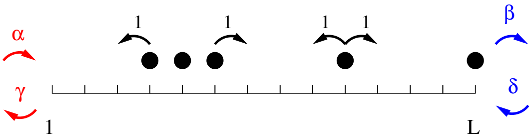


$$\rho_a = \frac{\alpha}{\alpha+\gamma} \,, \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

$$\boxed{\langle \exp\left[\lambda Q_t\right] \rangle \sim \exp\left[t \;\mu(\lambda)\right]} \quad ?$$

TWO APPROACHES

SSEP (Symmetric simple exclusion process)



Microscopic

Bethe ansatz, Perturbation theory, Computer,...

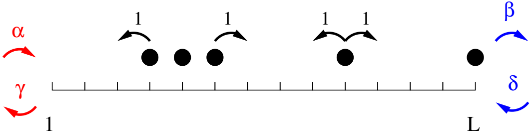
Macroscopic

$i = Lx \ , \ t = L^2 \tau$

$$\mathrm{Pro}(\{\rho(x,\tau), j(x,\tau)\}) \sim \exp\left[-L \int_0^{T/L^2} dt \int_0^1 dx \ \frac{[j+\rho']^2}{4\rho(1-\rho)}\right]$$

SSEP (Symmetric simple exclusion process)

D. Douçot Roche 2004



$$\lim_{t\to\infty} \frac{\langle Q(t)\rangle}{t} \simeq \frac{1}{L}[\rho_a - \rho_b] \qquad \text{Fick's law}$$

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle_c}{t} \simeq \frac{1}{L}\left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a\rho_b + \rho_b^2)}{3}\right]$$

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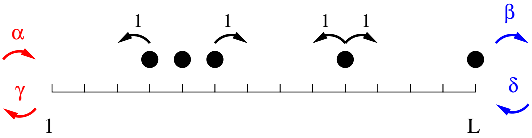

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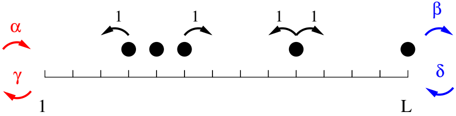
SSEP with $\rho_a = 1$ and $\rho_b = 0$ $\Rightarrow$

$$F = \frac{\langle Q_t^2\rangle_c}{\langle Q_t\rangle} = \frac{1}{3} \quad ; \quad \frac{\langle Q_t^3\rangle_c}{\langle Q_t\rangle} = \frac{1}{15} \quad \cdots$$

CURRENT FLUCTUATIONS IN THE SSEP



$$\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\ \mu(\lambda)}$$

$\mu(\lambda)$ gives all the cumulants of Q_t

For large L

$$\mu(\lambda, \rho_a, \rho_b, \text{contacts}) = \frac{1}{L} R(\omega)$$

where

$$\omega = \rho_a\,(e^{\lambda} - 1) + \rho_b\,(e^{-\lambda} - 1) + \rho_a\,\rho_b\,(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

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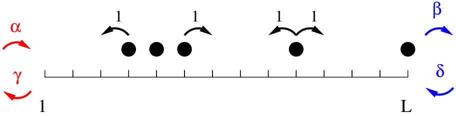

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Result:

$$R(\omega) = [\log(\sqrt{1+\omega} + \sqrt{\omega})\ ]^2$$

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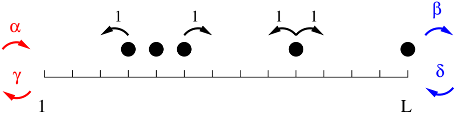

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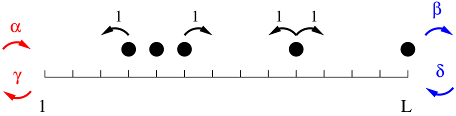
Same as the universal statistics of transport of fermions in disordered conductors (suppression of shot noise)

Beenakker, Buttiker 1992, Lee, Levitov, Yakovets 1995......

CURRENT FLUCTUATIONS IN THE SSEP



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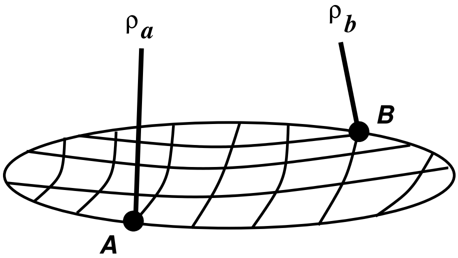
Result: $$R(\omega) = [\log(\sqrt{1+\omega} + \sqrt{\omega})\ ]^2$$

For $\rho_a = \rho_b = \frac{1}{2}$ $$R(\omega) = \frac{\lambda^2}{4} \quad \Leftrightarrow \quad Q_t \text{ is Gaussian}$$

GENERAL GRAPH

D. Gerschenfeld 2009



$$\langle e^{\lambda Q_t} \rangle \sim e^{t\ \mu(\lambda)}$$

$$\mu(\lambda, \rho_a, \rho_b, \text{contacts}) = R(\omega, \text{graph}, \text{contacts})$$

where

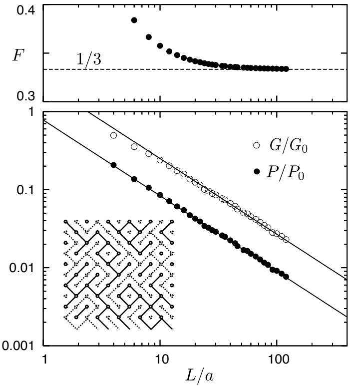
$$\omega = \rho_a\,(e^{\lambda} - 1) + \rho_b\,(e^{-\lambda} - 1) + \rho_a\,\rho_b\,(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

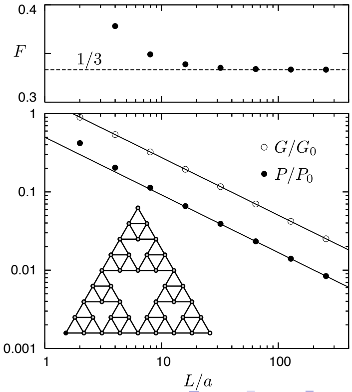
FANO FACTOR:

$$F = \lim_{t \to \infty} \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{\langle Q_t \rangle}$$

Groth, Tworzydło, Beenakker (2008)

SSEP with $\rho_a = 1$ and $\rho_b = 0$





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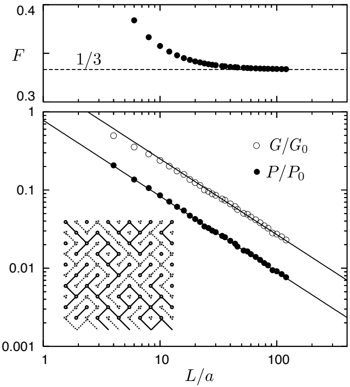

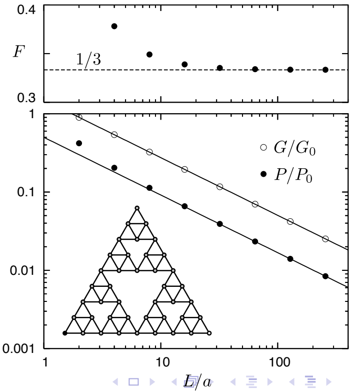

FANO FACTOR:

Akkermans, Bodineau, D., Shpielberg (2013)

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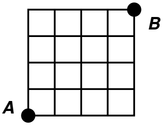
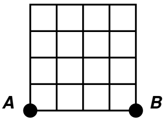
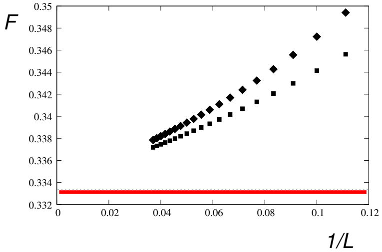
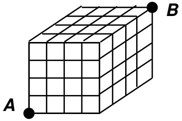
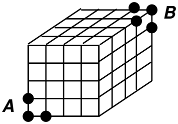








Cube 1



Cube 2

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Akkermans, Bodineau, D., Shpielberg (2013)

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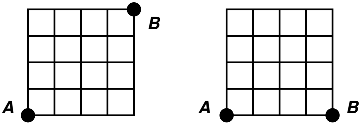
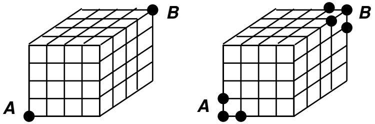
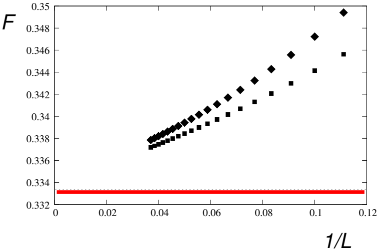
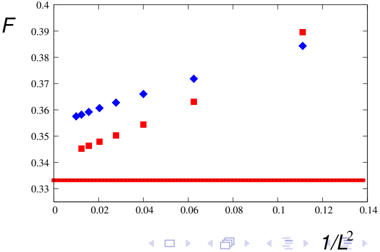




Cube 1 *Cube 2*





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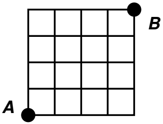
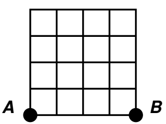
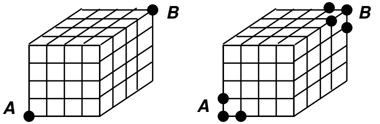
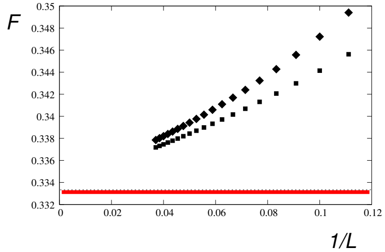




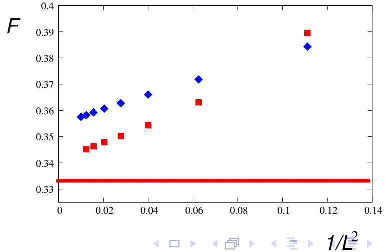


Cube 1

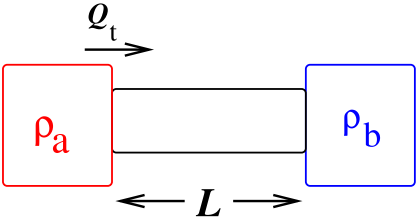
Cube 2



$F = \frac{1}{3}$



$F \neq \frac{1}{3}$



Diffusive system

- For $\rho_a - \rho_b$ small : $$\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)(\rho_a - \rho_b)}{L}$$

- $\rho_a = \rho_b = \rho$: $$\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$$

SSEP: $D = 1$ and $\sigma = 2\rho(1-\rho)$

MACROSCOPIC FLUCTUATION THEORY FOR DIFFUSIVE SYSTEMS

Kipnis Olla Varadhan 89
Spohn 91
⋮
Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001 →

Onsager Machlup theory for non equilibrium

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad \text{(conservation law)}$$

Evolution of a profile $\rho(x,t)$ for $0 \leq t \leq T$

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\})$$
$$\exp\left[-L\int_0^{T/L^2} dt \int_0^1 dx \; \frac{[j(x,t) + \rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

$$\Updownarrow$$

$$j(x,t) = -\rho'(x,t)D(\rho(x,t)) + \frac{1}{\sqrt{L}}\eta(x,t)$$

MACROSCOPIC FLUCTUATION THEORY FOR DIFFUSIVE SYSTEMS

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$$\Updownarrow$$

$$j(x,t) = -\rho'(x,t)D(\rho(x,t)) + \frac{1}{\sqrt{L}}\eta(x,t)$$

with the white noise $\langle \eta(x,t)\eta(x',t')\rangle = 2\sigma(\rho(x,t))\delta(x-x')\delta(t-t')$

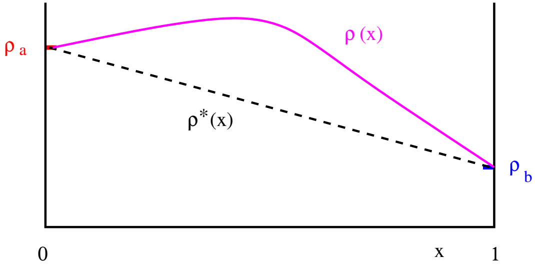
VARIATIONAL PRINCIPLE IN ONE DIMENSION

Bodineau D. 2004

Assuming that $j(x,t) = j$

Generating function $\langle e^{\lambda Q_t} \rangle \sim \exp[t\mu_L(\lambda)]$

$$\mu_L(\lambda, \rho_a, \rho_b) = \frac{1}{L} \max_{j,\rho(x)} \left[\lambda j - \int_0^1 dx \, \frac{[j\ L\ + \rho'(x)\ D(\rho(x))]^2}{2\sigma(\rho(x))} \right]$$



CONSEQUENCES

$$\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t \;\mu(\lambda)}$$

For large L

$$\mu(\lambda, \rho_a, \rho_b, \text{contacts}) = \frac{1}{L} R(\omega)$$

where

$$\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a \; \rho_b \; (e^{\lambda} - 1) \; (e^{-\lambda} - 1)$$

Result:

$$R(\omega) = [\log(\sqrt{1+\omega} + \sqrt{\omega}) \;]^2$$

TRUE VARIATIONAL PRINCIPLE

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2005

$$\mu(\lambda) = \frac{1}{L} \lim_{T\to\infty} \max_{\rho,j} \frac{1}{T} \int_0^T dt \int_0^1 dx \ \lambda j(x,t) - \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation), $\rho_t(0) = \rho_a$, $\rho_t(1) = \rho_b$

- Sufficient condition for the optimal profile to be time independent
- Dynamical phase transition

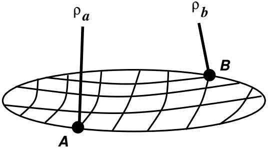
the optimal $\rho_t(x)$ starts to become time dependent

VARIATIONAL PRINCIPLE IN HIGHER DIMENSION

$$\mu(\lambda) = L^{d-2} \min_{\{\vec{j}(\vec{r}),\rho(\vec{r})\}} \int d\vec{r} \left(-\lambda \vec{\nabla} v(\vec{r}) \,.\, \vec{j}(\vec{r}) \;-\; \frac{[\vec{j}(\vec{r}) + D(\rho(\vec{r}))\; \vec{\nabla}\rho(\vec{r})\,]^2}{2\sigma(\rho(\vec{r}))} \right)$$

where

$$v(\partial A) = 1 \quad \text{and} \quad v(\partial B) = 0$$



Optimization equations:

$$\vec{j}(\vec{r}) = -D(\rho(\vec{r}))\, \vec{\nabla}\rho(\vec{r}) + \sigma(\rho(\vec{r}))\vec{\nabla} H(\vec{r})$$

Equations to solve:

$$\vec{\nabla}.\Big(D(\rho(\vec{r}))\, \vec{\nabla}\rho(\vec{r})\Big) = \vec{\nabla}.\,\Big(\sigma(\rho(\vec{r}))\, \vec{\nabla} H(\vec{r})\Big)$$

$$D(\rho(\vec{r}))\; \Delta H(\vec{r}) = -\frac{\sigma'(\rho(\vec{r}))}{2} \left(\vec{\nabla} H(\vec{r})\right)^2$$

with boundary conditions on ρ and H.

SOLUTION IN HIGHER DIMENSION DIMENSION

$$\Delta v(\vec{r}) = 0 \quad ; \quad v(\partial A) = 1 \quad \text{and} \quad v(\partial B) = 0$$

Solution:

$$H(\vec{r}) = H_1(v(\vec{r}))$$

$$\rho(\vec{r}) = \rho_1(v(\vec{r}))$$

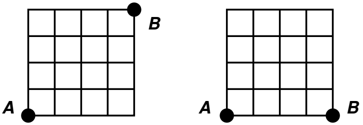
and

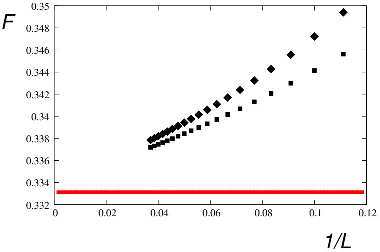
$$\boxed{\mu(\lambda) = \kappa(L) \times \mu_1(\lambda, \rho_a, \rho_b)}$$

FANO FACTOR:

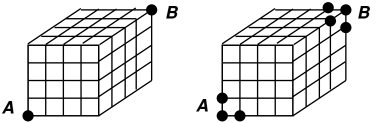
Akkermans, Bodineau, D., Shpielberg (2013)

SSEP with $\rho_a = 1$ and $\rho_b = 0$

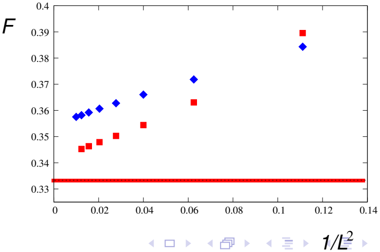




$F = \frac{1}{3}$



Cube 1 ***Cube 2***



$F \neq \frac{1}{3}$

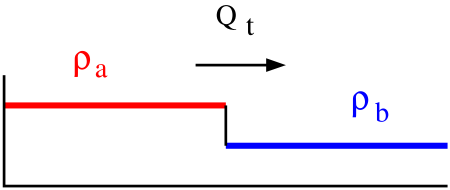
THE ω DEPENDENCE of $\left\langle e^{\lambda Q_t} \right\rangle$

- A finite graph $A \cup B$ with N_A sites in A occupied with density ρ_a and N_B sites in B occupied with density ρ_b
- Q_t is the flux from A to B during time t

1. $\langle Q_t^n \rangle$ is a polynomial of degree n in ρ_a and ρ_b
2. $\left\langle e^{\lambda Q_t} \right\rangle = e^{-N_B \lambda} \times \mathrm{Polynomial}(e^{\lambda})$ of degree $N_A + N_B$
3. $\Rightarrow \quad \left\langle e^{\lambda Q_t} \right\rangle = \mathrm{Polynomial}(\rho_a(e^{\lambda} - 1), \rho_n(e^{-\lambda} - 1))$ of degrees N_A, N_B
4. Hole particle symmetry: $\left\langle e^{\lambda Q_t} \right\rangle$ remains unchanged by
$$\{\lambda, \rho_a, \rho_b\} \quad \leftrightarrow \{-\lambda, 1 - \rho_a, 1 - \rho_b\}$$
5. $\Rightarrow \quad \left\langle e^{\lambda Q_t} \right\rangle$ is a function of the single variable

$$\omega = \rho_a (e^{\lambda} - 1) + \rho_b (e^{-\lambda} - 1) + \rho_a \rho_b (e^{\lambda} - 1)(e^{-\lambda} - 1)$$

STEP INITIAL CONDITION



SSEP

D Gerschenfeld 2009

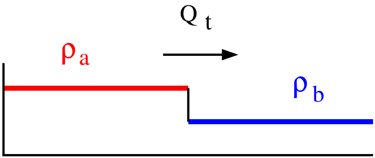
$$\langle e^{\lambda Q_t} \rangle \simeq \exp[\sqrt{t}\ F(\omega)]$$

with

$$F(\omega) = \frac{1}{\pi} \int_{\infty}^{\infty} dk \log \left[1 + \omega e^{-k^2} \right]$$

For large Q_t

$$\mathrm{Pro}(Q_t) \sim \exp\left[-\frac{\pi^2}{12}\ Q_t^3 / t \right]$$



Sethuraman, Varadhan 2013

exponent 1/3 related to KPZ?

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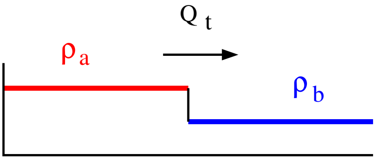

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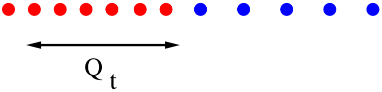
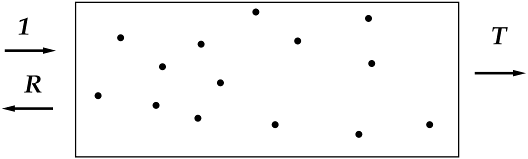


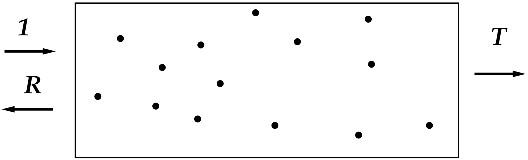
$$\mathrm{Pro}(Q_t) \sim \prod_{x=1}^{Q_t} \exp\left[-\frac{x^2}{t}\right]$$

QUANTUM TRANSPORT AND FULL COUNTING STATISTICS



Many channels

$$\frac{\langle Q_t \rangle}{t} = \sum_n T_n \quad ; \quad \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{t} = \sum_n T_n (1 - T_n)$$



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t electrons scatterred in each channel

$$\left\langle e^{\lambda Q_t} \right\rangle = \prod_n \left(1 - T_n + T_n e^{\lambda}\right)^t$$

STATISTICS OF THE T_n's

$$\langle e^{\lambda Q_t} \rangle = \prod_n \left(1 - T_n + T_n e^{\lambda}\right)^t$$

Many channels

$$\rho(T) = \sum_n \delta(T - T_n) = \frac{A}{T\sqrt{1 - T}}$$

leads to the same result as for the SSEP

Beenakker, Buttiker 1992, Lee, Levitov, Yakovets 1995

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Fokker Planck equation for $P_L(T_1, \cdots T_n)$ when $L \to L + DL$ is similar to the Fokker Planck equation of Dyson's Brownian motion for random matrices.

Dorokhov 82
Mello Pereyra and Kumar 88
Mello Pichard 89, 91
Beenakker 92
Nazarov 94
Jordan Sukhorukov, Pilgram 2004

CONCLUSION

Universal Statistics (for large systems and large enough contacts)

SSEP and other diffusive systems

OPEN QUESTIONS

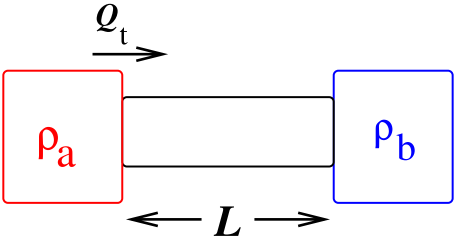
Time dependent profiles

Three or more reservoirs

Infinite domain

Random graphs

MICROSCOPIC APPROACH



$$\langle e^{\lambda Q_t} \rangle \sim e^{t\,\mu(\lambda)} \quad ?$$

The evolution (Markov process)

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} W(\mathcal{C},\mathcal{C}')P(\mathcal{C}') - W(\mathcal{C}',\mathcal{C})\mathcal{P}(\mathcal{C})$$

One can decompose

$$W(\mathcal{C},\mathcal{C}') = W_1(\mathcal{C},\mathcal{C}') + W_0(\mathcal{C},\mathcal{C}') + W_{-1}(\mathcal{C},\mathcal{C}')$$

$W_q(\mathcal{C},\mathcal{C}')$ represents a jump $\mathcal{C}' \to \mathcal{C}$ with $Q_t \to Q_t + q$

$$\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} W_1 + W_0 + e^{-\lambda} W_{-1}$$