Independence of ℓ for Frobenius conjugacy classes attached to abelian varieties

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Structure of Talk

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1. Statement of Main Theorem

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3. Idea of Proof (Existence of CM-liftings, Lafforgue's Theorem on existence of compatible local systems)

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Fact: G_A is a reductive group.
Example: A an elliptic curve.

$$G_A \cong \begin{cases} \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m & \text{if } A \text{ has CM by } K/\mathbb{Q} \text{ quadratic imaginary} \\ GL_2 & \text{if } A \text{ does not have } CM \end{cases}$$

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It follows from the Weil conjectures that

$$\mathrm{Det}(I - \rho_\ell(\mathrm{Frob}_v)t|\mathrm{H}^1_{\mathrm{\acute{e}t}}(A_{\overline{E}}, \mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[t]$$

has coefficients in $\mathbb{Z}$ and is independent of ℓ.

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Let $\chi : G \to \mathrm{Conj}_G$ be the natural projection.
For v a prime of good reduction as above, obtain well-defined element

$$\gamma_\ell := \chi(\rho_\ell^{G_A}(\mathrm{Frob}_v)) \in \mathrm{Conj}_{G_A}(\mathbb{Q}_\ell).$$

the conjugacy class of ℓ-adic Frobenius at v.

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Theorem (1)(Kisin, Z.)

Let A be an abelian variety over a number field $E \subset \mathbb{C}$ and assume ρ_ℓ factors as $\rho_\ell^{G_A} : \mathrm{Gal}(\overline{E}/E) \to G_A(\mathbb{Q}_l)$. For $p > 2$ and $v|p$ a prime of good reduction for A, there exists $\gamma \in \mathrm{Conj}_{G_A}(\mathbb{Q})$ such that

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(iii) Obstruction in case $p = 2$ is related to construction of integral models for Shimura varieties.

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2. Noot proves result for Conj'_{G_A} under the additional assumption that γ_ℓ is weakly neat.
3. Kisin proves result when $G_{A,\mathbb{Q}_p} := G_A \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is unramified (i.e. is quasi-split and splits over an unramified extension). Follows from Langlands–Rapoport conjecture.

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Eg. If $G = GL_2$, $\mathrm{Sh}_{\mathrm{K}}(G, X)$ is the modular curve.
If A defined over E, A corresponds to a point $x_A \in \mathrm{Sh}_{\mathrm{K}}(G_A, X_A)(E)$.
Let $v'|p$ place of $\mathbf{E}$ and $S_K(G, X)$ “good” integral model for $\mathrm{Sh}_{\mathrm{K}}(G, X)$ over $\mathcal{O}_{\mathbf{E}_{(v')}}$.
Let $\mathcal{S}$ denote the special fiber of $S_K(G, X)$ over residue field $k_{v'} \cong \mathbb{F}_q$.
$\overline{A} := A \mod v$ corresponds to $\overline{x}_A \in \mathcal{S}(\mathbb{F}_{q^n})$ ($\mathbb{F}_{q^n}$ residue field at v),

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Applying the Theorem (2) to $(\mathrm{Res}_{F/\mathbb{Q}} G_{A,F}, X')$ for suitably chosen totally real F implies Theorem (1) using the Reduction Step.

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(3) Construct explicit smooth curves on $\mathcal{S}$ in order to apply (2) above.

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$$\tilde{\gamma} \in T(\mathbb{Q}) \subset G(\mathbb{Q})$$

such that $\gamma := \chi(\tilde{\gamma}) = \gamma_{x,\ell}$ in $\mathrm{Conj}_G(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
Upshot: If $x \in \mathcal{S}(\mathbb{F}_{q^n})$ admits a special lifting, then x satisfies (ℓ-ind).

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Upshot: All $x \in \mathcal{S}^{\mathrm{ord}}(\mathbb{F}_{q^n})$ satisfy (ℓ-ind).

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The composition $r \circ \rho_\ell^{\mathcal{S}} : \pi_1(\mathcal{S}, \overline{y}) \to G(\mathbb{Q}_\ell) \to GL_n(\mathbb{Q}_\ell)$ gives rise to lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\ell$ over $\mathcal{S}$.

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Problem: Deligne's Conjecture only known for smooth schemes (L. Lafforgue, Drinfeld).

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However for $\mathcal{S}$, we are able to prove a slightly weaker version of this statement which is enough to deduce Theorem (2).

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Let $x \in \mathcal{S}^{\lambda}(\mathbb{F}_{q^n})$ for some $\lambda \prec \mu$. There exists smooth curve $C/\mathbb{F}_{q^n}$ and a map $\phi : C \to \mathcal{S}_{\mathbb{F}_{q^n}}$ satisfying:

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Now use descending induction to prove (ℓ-ind) for all points in $\mathcal{S}$