Exponential decay of correlations for Anosov flows

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Definition

A flow $g^t : M \to M$ on a compact Riemannian manifold M is a **Anosov flow** *if there is a continuous splitting*

$$TM = E^s \oplus N \oplus E^u$$

where N is the flow direction, E^s is uniformly contracted by Dg^1 and E^u is uniformly expanded by Dg^1.

It is known that for any Anosov flow, E^s and E^u integrate to stable foliation W^s and unstable foliaton W^u respectively (but they don't have to be jointly integrable).

The topological and statistical properties of Anosov flow were studied by many authors: Anosov, Bowen, Margulis, Plante, Ratner, Ruelle, Sinai, Smale, etc.
For transitive Anosov flows, the following theorem is well-known.

Theorem (Anosov alternative)

A transitve Anosov flow is topologically mixing if and only if the stable and unstable foliations are not jointly integrable.

We say that g^t is topologically mixing if for any non-empty open sets $A, B \subset M$, for all sufficiently large $t > 0$, we have $g^t(A) \cap B \neq \emptyset$.

It is also important to study measure-theoretical mixing. For a Anosov flow g^t, for any Hölder function F, there is a unique measure μ which maximizes

$$\int F d\mu + h_\mu(g^1).$$

We call μ an **equilibrium measure** for g with potential F.

1. when $F = 0$, μ is the entropy maximizing measure;
2. when $F = -\log|\det(Dg^1|_{E^u})|$, μ is called the Sinai-Ruelle-Bowen measure

$$\mu = \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} (g^i)_* Leb.$$

In particular, when g^t is volume preserving, then μ is the volume.

It is known that when g^t is topologically mixing, then g^t is also mixing with respect to any equilibrium measure with Hölder potential. Namely, for any $A, B \in L^2(M)$, we have

$$\int A \circ g^t B d\mu \to \int A d\mu \int B d\mu, \quad t \to \infty.$$

People are interested in the speed of the convergence when A, B are Hölder functions.

Conjecture (Bowen-Ruelle)

If g^t is topologically mixing, then g^t is exponentially mixing with respect to Hölder functions and equilibrium measures with Hölder potential.

Progress on Bowen-Ruelle conjecture:

1. (Dolgopyat) it is true when E^s and E^u are of class C^1 for SRB measure (and all equilibrium measure when $\dim E^u = 1$);
2. (Liverani) when g^t preserves a contact form, exp. mixing w.r.t. volume;
3. (Tsujii) in $3D$, when g^t preserves a volume form, open and dense condition;
4. (Butterley-War) when E^s is $C^{1+\epsilon}$ for SRB measure.

Usually, E^u and E^s do not need to be smooth. They are always Hölder. But in many cases they are strictly Hölder and exponent can be arbitrarily small (Plante, Hasselblatt, Wilkinson).

Theorem (Tsujii-Z)

There is an open and dense set of C^∞ 3D Anosov flows for which Bowen-Ruelle's conjecture holds.

Besides topological genericity, another natural notion for genericity is **prevalence** (Hunt-Sauer-Yorke).
Let B be a Banach space. We say that a subset $A \subset B$ is **shy** if there is a compactly supported measure ν on B so that

$$\nu(x + A) = 0, \quad \forall x \in B.$$

A property is prevalent if the set of elements without this property is shy.

Theorem (Tsujii-Z)

For any C^r Anosov flows, exponentially mixing is prevalent among the its C^r time-reparametrisations.