Affine Springer fibers and the small quantum group

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Introduction

Algebra/Representation Theory

Geometry

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- Frobenius kernel/ small quantum group

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- Affine Springer fiber $\mathcal{F}l_{ts}$

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- Frobenius kernel/ small quantum group
- Center of the category

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- Affine Springer fiber $\mathcal{F}l_{ts}$
- Cohomology of affine Springer fiber

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Algebra/Representation Theory

- Frobenius kernel/ small quantum group
- Center of the category
- Category of Representations

Geometry

- Affine Springer fiber $\mathcal{F}l_{ts}$
- Cohomology of affine Springer fiber
- Category of microlocal sheaves

Representation theory

Definition

A representation of a group G, is a vector space V, with a linear action of G.

This can be rephrased as a group homomorphism.

$$G \to GL(V)$$

Example

If S is a set with a G action, we can construct a representation $k[S]$, with a basis given by S and the linear extension of the action of G on S.

Affine Algebraic groups

Definition

An affine algebraic group G, is a subgroup $G \subset GL_n$ defined by the vanishing of a set of polynomials in the matrix entries.

We consider G a reductive group. Examples are GL_n, SL_n, SO_n,...

Definition

A representation of an algebraic group G on a vector space V, is a group homomorphism

$$G \to GL(V)$$

defined by polynomials

Algebraic groups Representation theory in char 0

In characteristic 0 an algebraic group G is always smooth.
$\Rightarrow$ There does not exist $H \subsetneq G$ containing an infinitesimal neighbourhood of the identity.

Theorem (Weyl's complete reducibility Theorem)

A representation V of a reductive algebraic group G in characteristic 0 breaks up as

$$V = \bigoplus V_i$$

for irreducible V_i

Algebraic groups Representation theory in char p

In characteristic p a group G comes with a group homomorphism

$$Fr : G \to G^{(1)}$$

given by the Frobenius morphism.

Question

What is the kernel of Fr? This group is known as the Frobenius kernel and denoted by $G_{(1)}$.

Answer

Fr is an isomorphism on closed points, so $G_{(1)}$ has a unique closed point, but it is non-trivial. So it is a non-trivial infinitesimal algebraic group.

Warning for experts: Our results are about the small quantum group u_q, an algebra over $\mathbb{C}$, with analogous representation theory to $G_{(1)}$, as shown by Andersen, Jantzen and Soergel

Affine flag variety and affine Springer fibers

Consider the group $G(\mathcal{K})$, the algebraic group G with entries in $\mathcal{K} = \mathbb{C}((t))$. Define the Iwahori subgroup

$$\begin{array}{ccc} I & \hookrightarrow & G(\mathbb{C}[[t]]) \\ \downarrow & & \downarrow{\scriptstyle t=0} \\ B & \hookrightarrow & G \end{array}$$

Define the affine flag variety $\mathcal{F}l$ as the ind-variety defined as the quotient $G(\mathcal{K})/I$.
For $e = ts$ for $s \in \mathfrak{g}$ a regular semisimple element, define the affine Springer fiber

$$\mathcal{F}l_e = \{gI \in \mathcal{F}l | e \in {}^g(Lie(I))\}$$

Geometry of affine Springer fiber

- $\mathcal{F}l_e$ has countably many components
- All components are of the same finite dimension
- There is an action of a lattice Λ such that the action on the set of components is free with finitely many orbits

Example SL_2

- $\mathcal{F}l_e$ is an infinite chain of $\mathbb{P}^1$'s
- $\Lambda = \mathbb{Z}$ and it acts by translating the components by 2

Results

Theorem (Bezrukavnikov, BA, Shan, Vasserot)

Denote by Z the center (of a regular block). We have a commutative diagram

$$\begin{array}{ccc} H^*(\mathcal{F}l) & \longrightarrow & Z^G \\ \downarrow & & \downarrow \\ H^*(\mathcal{F}l_e)^\Lambda & \hookrightarrow & Z^T \end{array}$$

Conjecture

$$\begin{array}{ccc} H^*(\mathcal{F}l_e)^{\widetilde{W}} & \xrightarrow{\simeq} & Z^G \\ \downarrow & & \downarrow \\ H^*(\mathcal{F}l_e)^\Lambda & \xrightarrow{\simeq} & Z^T \end{array}$$

Further results

For SL_n:(joint with Losev)

The inclusion

$$H^*(\mathcal{F}l_e)^{\widetilde{W}} \hookrightarrow H^*(\mathcal{F}l_e)^{\Lambda}$$

is an equality.

Assuming the conjecture it would follow that $Z^G = Z^T$, which implies $Z = Z^G$.
This equality is related to the n! conjecture proven by Haiman, related to the Hilbert scheme of point in $\mathbb{C}^2$.

Theorem(Bezrukavnikov, McBreen, Yun)

There is a relation (Koszul duality) between a category of sheaves on $\mathcal{F}l_e$ and the category of (graded) representations of the small quantum group.