Sidon sets in bounded orthonormal systems

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$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n\in\Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

$\Lambda \subset \mathbb{Z}$ is randomly Sidon if

$$\sum_{n\in\Lambda} \pm a_n e^{int} \in C(\mathbf{T})\ a.s. \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

$\Lambda \subset \mathbb{Z}$ is subGaussian if

$$\sum_{n\in\Lambda} |a_n|^2 < \infty \Rightarrow \int \exp|\sum_{n\in\Lambda} a_n e^{int}|^2 < \infty.$$

They are all equivalent !

Obviously Sidon $\Rightarrow$ randomly Sidon

Rudin (1961): Sidon $\Rightarrow$ subGaussian

Rider (1975) : Sidon $\Leftrightarrow$ randomly Sidon

P (1976) : Sidon $\Leftrightarrow$ subGaussian

Results hold more generally for any subset $\Lambda \subset \widehat{G}$
when G is any compact Abelian group

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Rudin (1961): Sidon $\Rightarrow$ subGaussian

Rider (1975) : Sidon $\Leftrightarrow$ randomly Sidon
(Note: This refines Drury's celebrated 1970 union Theorem)

P (1976) : Sidon $\Leftrightarrow$ subGaussian

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when G is any compact Abelian group

Examples

Hadamard lacunary sequences

$$n_1 < n_2 < \cdots < n_k, \cdots$$

such that

$$\inf_k \frac{n_{k+1}}{n_k} > 1$$

Explicit example

$$n_k = 2^k$$

Basic Example: Quasi-independent sets

Λ is quasi-independent if all the sums

$$\{\sum_{n \in A} n \mid A \subset \Lambda, |A| < \infty\}$$

are distinct numbers

$$\text{quasi-independent} \Rightarrow \text{Sidon}$$

Arithmetic characterization

P. (1983) A set Λ is **Sidon**
IFF
$\exists \delta > 0$ such that $\forall A \subset \Lambda$ $(|A| < \infty)$
$\exists B \subset A$ **quasi-independent** with $|B| \geq \delta |A|$.
Proof uses random Fourier series and the metric entropy condition

Most Delicate part is the IF part

Bourgain (1985) gave a different proof and proved the following strengthening:
Λ is **Sidon** IFF $\exists \delta > 0$
such that for any probability Q on Λ $\exists B \subset \Lambda$ **quasi-independent** such that

$$Q(B) \geq \delta$$

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Main Open Problem

Is every **Sidon** set a finite union of **quasi-independent** sets ?

This is reduced to **a purely combinatorial problem**
whether for a finite set Λ

$$\begin{cases} \forall A \subset \Lambda \ (|A| < \infty) \\ \exists B \subset A \ \mathbf{quasi-independent} \ \textit{with} \ |B| \geq \delta |A| \end{cases}$$

implies

$$\exists k = k(\delta) \quad \Lambda = \cup_1^k B_k \qquad \text{with } B_k \text{ quasi-independent}$$

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Paul Erdos

(met in 1983 in Warsaw)

got interested in the problem and worked on it as well as on variants of the problem in **graph theory**:

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Main known cases

$$G = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \quad \textit{Malliavin–Malliavin}, \text{ 1967 } [\textit{using Horn's theorem}]$$

for p prime

Extended by Bourgain (1983) to $p = p_1 \cdots p_n$
with p_k distinct primes

The recent rebirth

Bourgain and Lewko (arxiv 2015) wondered whether a group environment is needed for all the preceding

Question

What remains valid if $\Lambda \subset \widehat{G}$ is replaced by a *uniformly bounded* orthonormal system ?

Let $\Lambda = \{\varphi_n\} \subset L_\infty(T, m)$ orthonormal in $L_2(T, m)$ ((T, m) any probability space)

(i) We say that (φ_n) is Sidon with constant C if for any n and any complex sequence (a_k) we have

$$\sum\nolimits_1^n |a_k| \le C\|\sum\nolimits_1^n a_k\varphi_k\|_\infty.$$

(ii) We say that (φ_n) is randomly Sidon with constant C if for any n and any complex sequence (a_k) we have

$$\sum\nolimits_1^n |a_k| \le C\mathrm{Average}_{\pm 1}\|\sum\nolimits_1^n \pm a_k\varphi_k\|_\infty,$$

(iii) Let $k \ge 1$. We say that (φ_n) is $\otimes^k$-Sidon with constant C if the system $\{\varphi_n(t_1)\cdots\varphi_n(t_k)\}$ (or equivalently $\{\varphi_n^{\otimes k}\}$) is Sidon with constant C in $L_\infty(T^k, m^{\otimes k})$.

Now assume merely that $\{\varphi_n\} \subset L_2(T, m)$.

(iv) We say that (φ_n) is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (a_k) we have

$$\|\sum\nolimits_1^n a_k\varphi_k\|_{\psi_2} \le C(\sum |a_k|^2)^{1/2}.$$

Here

$$\psi_2(x) = \exp x^2 - 1$$

and $\|f\|_{\psi_2}$ is the norm in associated Orlicz space
Again: We say that $\{\varphi_n\} \subset L_2(T, m)$ is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (a_k) we have

$$\|\sum_1^n a_k\varphi_k\|_{\psi_2} \leq C(\sum |a_k|^2)^{1/2}.$$

Equivalently, assuming w.l.o.g. $\int \varphi_k = 0, \forall k$
$\exists C$ such that $\forall (a_k)$

$$\int \exp Re(\sum_1^n a_k\varphi_k) \leq \exp C^2 \sum |a_k|^2$$

Important remark: Standard i.i.d. (real or complex) Gaussian random variables are subGaussian (Fundamental example !)

Easy Observation : *Sidon* $\not\Rightarrow$ *subGaussian*

By a much more delicate example Bourgain and Lewko proved:

$$subGaussian \not\Rightarrow Sidon$$

However, they proved

Theorem

$$subGaussian \Rightarrow \otimes^5 - Sidon$$

Recall $\otimes^5 - Sidon$ *means*

$$\sum_1^n |a_k| \leq C \| \sum_1^n a_k \varphi_k(t_1) \cdots \varphi_k(t_5) \|_{L_\infty(T^5)}.$$

This generalizes my 1976 result that subGaussian implies Sidon for characters $(\varphi_k(t_1) \cdots \varphi_k(t_5) = \varphi_k(t_1 \cdots t_5)$!)
They asked whether 5 can be replaced by 2 which would be optimal

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Indeed, it is so.

Theorem

For bounded orthonormal systems

$$subGaussian \Rightarrow \otimes^2 - Sidon$$

Recall $\otimes^2 - Sidon$ *means*

$$\sum_1^n |a_k| \leq C \| \sum_1^n a_k \varphi_k(t_1)\varphi_k(t_2) \|_{L_\infty(T^2)}.$$

Actually, we have more generally:

Theorem (1)

Let $(\psi_n^1), (\psi_n^2)$ be systems biorthogonal respectively to $(\varphi_n^1), (\varphi_n^2)$ on probability spaces $(T_1, m_1), (T_2, m_2)$ resp. and uniformly bounded respectively by C_1', C_2',
If $(\varphi_n^1), (\varphi_n^2)$ are subGaussian with constants C_1, C_2 then

$$\sum |a_n| \leq \alpha \ \mathrm{ess\,sup}_{(t_1,t_2)\in T_1\times T_2} \left|\sum a_n \psi_n^1(t_1)\psi_n^2(t_2)\right|,$$

where α is a constant depending only on C_1, C_2, C_1', C_2'.

To illustrate by a concrete (but trivial) example: take $\varphi_n^1 = \varphi_n^2 = g_n$ and $\psi_n^1 = \psi_n^2 = \mathrm{sign}(g_n)$

The key new ingredient is a corollary of a powerful result due to **Talagrand Acta (1985)** (combined with a soft Hahn-Banach argument)
Let (g_n) be an i.i.d. sequence of standard (real or complex) Gaussian random variables

Theorem

Let (φ_n) be C-subGaussian in $L_1(T, m)$. Then $\exists T : L_1(\Omega, \mathbb{P}) \to L_1(T, m)$ with $\|T\| \leq KC$ (K a numerical constant) such that

$$\forall n \quad T(g_n) = \varphi_n$$

Let $T \in L_1(m_1) \otimes L_1(m_2)$ (algebraic $\otimes$) say $T = \sum x_j \otimes y_j$ then the

projective and injective tensor product norm

denoted respectively by $\|\cdot\|_\wedge$ and $\|\cdot\|_\vee$ are very explicitly described by

$$\|T\|_\wedge = \int |\sum x_j(t_1)y_j(t_2)|dm_1(t_1)dm_2(t_2)$$

$$\|T\|_\vee = \sup\{|\sum \langle x_j, \psi_1\rangle\langle y_j, \psi_2\rangle| \mid \|\psi_1\|_\infty \leq 1, \|\psi_2\|_\infty\}.$$

Theorem

Let (φ_n^1) and (φ_n^2) $(1 \leq n \leq N)$ are subGaussian with constants C_1, C_2. Then for any $0 < \delta < 1$ there is a decomposition in $L_1(m_1) \otimes L_1(m_2)$ of the form

$$\sum\nolimits_1^N \varphi_n^1 \otimes \varphi_n^2 = t + r$$

satisfying

$$\|t\|_\wedge \leq w(\delta)$$

$$\|r\|_\vee \leq \delta,$$

where $w(\delta)$ depends only on δ and C_1, C_2.
Moreover

$$w(\delta) = O(\log((C_1 C_2)/\delta)$$

Proof reduces to the case $\varphi_n^1 = \varphi_n^2 = g_n$

Let $f = \sum a_n \psi_n^1(t_1)\psi_n^2(t_2)$
$|a_n| = s_n a_n$

Note: (φ_n) *subGaussian* $\Rightarrow$ $(s_n\varphi_n)$ *subGaussian* (same constant)

$S = \sum_1^N s_n \varphi_n^1 \otimes \varphi_n^2 = t + r$

$$\langle f, S\rangle = \sum |a_n|$$

Therefore

$$\sum |a_n| = \langle f, t + r\rangle \leq |\langle f, t\rangle| + |\langle f, r\rangle|$$

$$\leq w(\delta)\|f\|_\infty + \sum |a_n||\langle \psi_n^1 \otimes \psi_n^2, r\rangle| \leq w(\delta)\|f\|_\infty + (\delta C_1' C_2') \sum |a_n|$$

and hence

$$\sum |a_n| \leq (1 - \delta C_1' C_2')^{-1} w(\delta)\|f\|_\infty$$

Bourgain and Lewko noticed that Slepian's classical comparison Lemma for Gaussian processes implies that randomly $\otimes^k$-Sidon and randomly Sidon are the same property, not implying Sidon. However, we could prove that this implies $\otimes^4$-Sidon:

Theorem (2)

Let (φ_n, ψ_n) be biorthogonal systems both bounded in L_∞. The following are equivalent:

(i) *The system (ψ_n) is randomly Sidon.*
(ii) *The system (ψ_n) is $\otimes^4$-Sidon.*
(iii) *The system (ψ_n) is $\otimes^k$-Sidon for all $k \geq 4$.*
(iv) *The system (ψ_n) is $\otimes^k$-Sidon for some $k \geq 4$.*

This generalizes Rider's result that randomly Sidon implies Sidon for characters

Open question: What about $k = 2$ or $k = 3$?

Non-commutative case

G compact non-commutative group
$\widehat{G}$ the set of distinct irreps, $d_\pi = \dim(H_\pi)$
$\Lambda \subset \widehat{G}$ is called Sidon if $\exists C$ such that for any finitely supported family (a_π) with $a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum\nolimits_{\pi\in\Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C\|\sum\nolimits_{\pi\in\Lambda} d_\pi \mathrm{tr}(\pi a_\pi)\|_\infty.$$

$\Lambda \subset \widehat{G}$ is called randomly Sidon if $\exists C$ such that for any finitely supported family (a_π) with $a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum\nolimits_{\pi\in\Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C\mathbb{E}\|\sum\nolimits_{\pi\in\Lambda} d_\pi \mathrm{tr}(\varepsilon_\pi \pi a_\pi)\|_\infty$$

where (ε_π) are an independent family such that each ε_π is uniformly distributed over $O(d_\pi)$.

Important Remark (easy proof) Equivalent definitions:

• unitary matrices (u_π) uniformly distributed over $U(d_\pi)$

• Gaussian random matrices (g_π) normalized so that $\mathbb{E}\|g_\pi\| \approx 2$ $(\{d_\pi^{1/2} g_\pi \mid \pi \in \Lambda, 1 \leq i,j \leq d_\pi\}$ forms a standard Gaussian (real or complex) i.i.d. family

Rider (1975) extended all results previously mentioned to arbitrary compact groups

Note however that the details of his proof that randomly Sidon implies Sidon (solving the non-commutative union problem) never appeared

I plan to remedy this on arxiv soon

General matricial systems

Assume given a sequence of finite dimensions d_n.
For each n let (φ_n) be a random matrix of size $d_n \times d_n$ on (T, m).
We call this a "matricial system".
Let g_n be an independent sequence of random $d_n \times d_n$-matrices, such that $\{{d_n}^{1/2} g_n(i,j) \mid 1 \le i,j \le d_n\}$ are i.i.d. normalized $\mathbb{C}$-valued Gaussian random variables. Note $\|g_n(i,j)\|_2 = {d_n}^{-1/2}$.

The **subGaussian condition** becomes: for any N and $y_n \in M_{d_n}$ ($n \leq N$) we have

$$\|\sum d_n \mathrm{tr}(y_n \varphi_n)\|_{\psi_2} \leq C(\sum d_n \mathrm{tr}|y_n|^2)^{1/2} = \|\sum d_n \mathrm{tr}(y_n g_n)\|_2. \tag{1}$$

In other words, $\{d_n^{1/2}\varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$ is a subGaussian system of functions.

The **uniform boundedness condition** becomes

$$\exists C' \; \forall n \quad \|\varphi_n\|_{L_\infty(M_{d_n})} \leq C'. \tag{2}$$

As for the **orthonormality condition** it becomes

$$\int \varphi_n(i,j)\overline{\varphi_{n'}(k,\ell)} = d_n^{-1}\delta_{n,n'}\delta_{i,k}\delta_{j,\ell}. \tag{3}$$

In other words, $\{d_n^{1/2}\varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$ is an orthonormal system.

The definition of $\otimes^k$-**Sidon** it now means that the family of *matrix products* $(\varphi_n(t_1)\cdots\varphi_n(t_k))$ is Sidon on $(T,m)^{\otimes^k}$

Theorem (3)

Theorems (1) and (2) are still valid with the corresponding assumptions:

- *subGaussian implies $\otimes^2$-Sidon*
- *randomly subGaussian is equivalent to $\otimes^k$-Sidon*

Example of application

Let $\chi \geq 1$ be a constant. Let T_n be the set of $n \times n$-matrices $a = [a_{ij}]$ with $a_{ij} = \pm 1/\sqrt{n}$. Let

$$A_n^\chi = \{a \in T_n \mid \|a\| \leq \chi\}.$$

This set **includes** the famous Hadamard matrices. We have then

Corollary

There is a numerical $\chi \geq 1$ such that for some C we have

$$\forall n \geq 1\ \forall x \in M_n \quad \mathrm{tr}|x| \leq C \sup_{a', a'' \in A_n^\chi} |\mathrm{tr}(xa'a'')|.$$

Equivalently, denoting $A_n^\chi A_n^\chi = \{a'a'' \mid a', a'' \in A_n^\chi\}$ its absolutely convex hull satisfies

$$(\chi)^2 absconv[A_n^\chi A_n^\chi] \subset B_{M_n} \subset C absconv[A_n^\chi A_n^\chi]$$

Curiously, even the case when $|\Lambda| = 1$ (only a single irrep) is interesting
The simplest (and prototypical) example of this situation with $|\Lambda_n| = 1$ is the case when $G_n = U(n)$ the group of unitary $n \times n$-matrices, and Λ_n is the singleton formed of the irreducible representation (in short irrep) defining $U(n)$ as acting on $\mathbb{C}^n$. Sets of this kind and various generalizations were tackled early on by Rider under the name "local lacunary sets"

The next Theorem of course is significant only if $\dim(\pi_n) \to \infty$

Theorem (Characterizing SubGaussian characters)

Let G_n be compact groups, let $\pi_n \in \widehat{G_n}$ be nontrivial irreps, let $\chi_n = \chi_{\pi_n}$ as well as $d_n = d_{\pi_n}$. The following are equivalent.

(i) *$\exists C$ such that the singletons $\{\pi_n\} \subset \widehat{G}_n$ are Sidon with constant C, i.e. we have*

$$\forall n\ \forall a \in M_{d_n} \quad \mathrm{tr}|a| \leq C \sup_{g \in G} |\mathrm{tr}(a\pi_n(g))|.$$

(ii) *$\exists C$ such that $\forall n \quad \|\chi_n\|_{\psi_2} \leq C$.*

(iii) *For each (or some) $0 < \delta < 1$ there is $0 < \theta < 1$ such that*

$$\forall n \quad m_{G_n}\{Re(\chi_n) > \delta d_n\} \leq e\theta^{d_n^2}.$$

(iv) *$\exists C$ such that*

$$\forall n \quad d_n \leq C \int_{U(d_n)} \sup_{g \in G_n} |\mathrm{tr}(u\pi_n(g))| m_{U(d_n)}(du).$$

Although I never had a concrete example, I believed naively for many years that this Theorem could be applied to finite groups. To my surprise, Emmanuel Breuillard showed me that it is not so. By a Theorem of Jordan, any finite group $\Gamma \subset U(d)$ (Breuillard extended this to amenable subgroups of $U(d)$) has an Abelian subgroup of index at most $(d+1)!$
This implies for any representation $\pi : \; G \to U(d)$ with **finite range**

$$\int_{U(d)} \sup_{g \in G} |\mathrm{tr}(u\pi(g))| m_{U(d)}(du) \leq c(d\log(d))^{1/2} << d$$

and also

$$\|\chi_\pi\|_{\psi_2} \geq c\sqrt{d/\log(d)} >> 1.$$