Interfaces in inhomogeneous media: pinning, hysteresis, and facets

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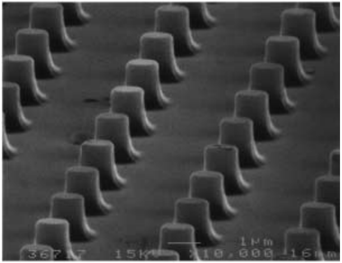
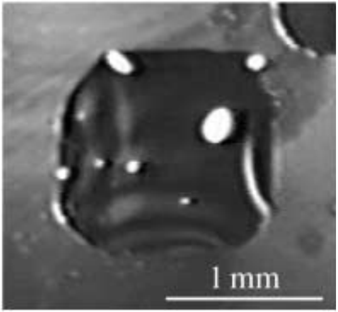
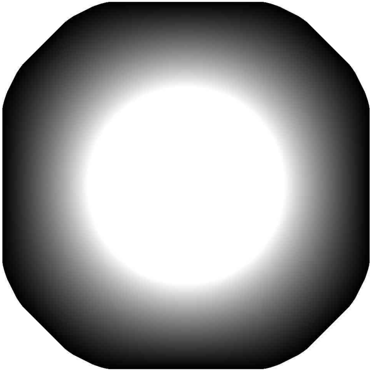
Liquid drops on rough surfaces

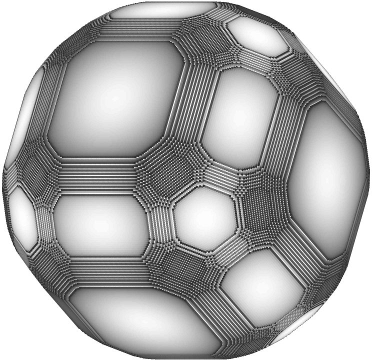




Marzolin, Smith, Prentiss and Whitesides *Adv. Mater.* (1998)
Bico, Tordeaux and Quéré *Euro. Phys. Lett.* (2001)

Scaling limit of boundary sandpile model





Simulations from Smart and F., *ARMA* '18
Model introduced by Aleksanyan and Shahgholian.

A model free boundary problem

Both pictures can be explained by a free boundary problem of the following form:

$$\begin{cases} \Delta \bar{u} = 0 \text{ in } \{\bar{u} > 0\} \text{ and} \\ |\nabla \bar{u}| \in [Q_*(n_x), Q^*(n_x)] \text{ on } \partial\{\bar{u} > 0\}. \end{cases}$$

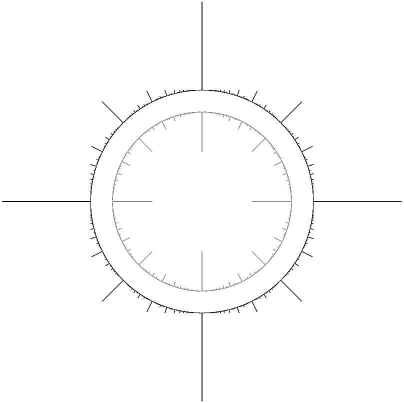
Facets caused by discontinuities of endpoints of the **pinning interval** $[Q_*(n_x), Q^*(n_x)]$, as a function of the normal direction (idea goes back to Caffarelli and Lee '07).

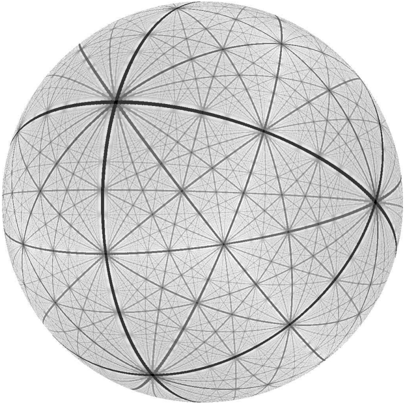
- Derived as scaling limit of boundary sandpile model (Smart and F., ARMA '18).
- Derived as homogenization limit of continuous free boundary problem (F., preprint '18):

  $$\Delta u^\varepsilon = 0 \text{ in } \{u^\varepsilon > 0\} \text{ and } |\nabla u^\varepsilon| = Q(\tfrac{x}{\varepsilon}) \text{ on } \partial\{u^\varepsilon > 0\}.$$

  Inhomogeneity modeled by $Q > 0$ and $\mathbb{Z}^d$-periodic.

Structure of the effective PDE





Smart and F., *ARMA* '18

Structure of the effective PDE

Theorem (Smart and F., ARMA '18)

Define $S : 2\pi\mathbb{T}^d \to \mathbb{R}$ *by* $S(\theta) = -\log(1 + \frac{1}{d}\sum_{j=1}^{d} \cos\theta_j)$*, and let* $\hat{S} : \mathbb{Z}^d \to \mathbb{C}$ *be the corresponding Fourier transform. Then* $\hat{S}$ *is real and positive on* $\mathbb{Z}^d$ *and for all* $e \in S^{d-1}$*,*

$$Q^*(e) = \frac{1}{\sqrt{2d}} \exp\left(\frac{1}{2} \sum_{k \in \mathbb{Z}^d :\ k \cdot e = 0} \hat{S}(k)\right).$$

Where $Q^*(e)$ *is the upper endpoint of pinning interval associated with boundary sandpile scaling limit.*

Theorem (F., preprint '18)

(Informally) Similar qualitative continuity properties for continuous case in $d = 2$*.*

Minimal and maximal solutions

Minimal (and maximal) solutions play a key role:

$$\begin{cases} \Delta \bar{u} = 0 \text{ in } \{\bar{u} > 0\} \text{ and} \\ |\nabla \bar{u}| = Q^*(n_x) \text{ on } \partial\{\bar{u} > 0\}. \end{cases}$$

New theory needs to be developed due to the discontinuous free boundary condition.

Theorem (Smart and F., ARMA '18)

Strict comparison principle holds for

$$\Delta u = 0 \;\; \textit{in} \;\; \{u > 0\} \;\; \textit{with} \;\; |\nabla \bar{u}| = Q^*(n_x) \;\; \textit{on} \;\; \partial\{u > 0\}$$

when $d = 2$ or in arbitrary dimension and convex setting.

Future Directions / Open Questions

- Mathematical follow-up questions:
  - General comparison principle and explaining facet shapes in $d \geq 3$?
  - Optimal regularity of the free boundary for the discontinuous free boundary condition?
  - Presence of facets with co-dimension ≥ 2?
- Energy based approach, perhaps via dissipative evolutions, volume constrained solutions.
- Generic discontinuities of pinning interval in continuous model?
- What phenomena need to be explained with rough surface (as opposed to chemically patterned)?
- Random media...

Thank you for your attention!

Moving interfaces in random media

Forced mean curvature flow

Interface Γ_t evolving by normal velocity with planar initial data $e \cdot x = 0$ (outward normal e)

$$V_n = -\kappa + c(x) + F.$$

Here κ is mean curvature, $c(x)$ is inhomogeneous environment, constant F is large scale external driving force (e.g. pressure, contact angle, or magnetic field).

Model for

- Flow in porous media
- Contact line motion
- Domain boundaries in magnetic materials

Pinning and depinning

Expectation: there is a pinning interval $[F_*(e), F^*(e)]$

$$F^*(e) = \inf\{F : \liminf_{t\to\infty} \frac{u(0,t)}{t} > 0\}.$$

Front has positive speed outside of the pinning interval. Can also define the depinning transition value

$$F^d(e) = \inf\{F : \lim_{t\to\infty} u(0,t) = +\infty \quad \text{a.s}\}$$

Open questions:

- Are the depinning and positive speed transitions the same?
- Is the propagating interface flat for $F > F^*$?
- What is the behavior of $\bar{c}(F)$ near the depinning threshold? Conjectured universality $\bar{c}(F) \sim (F - F^*)^\theta$.

Propagation as a flat front

Take now $F = 0$. Periodic media:

- (Lions and Souganidis, '05) Lipschitz estimates and existence of correctors under the coercivity condition

$$\inf_{\mathbb{R}^d}[c(x)^2 - (d-1)|Dc|] > 0.$$

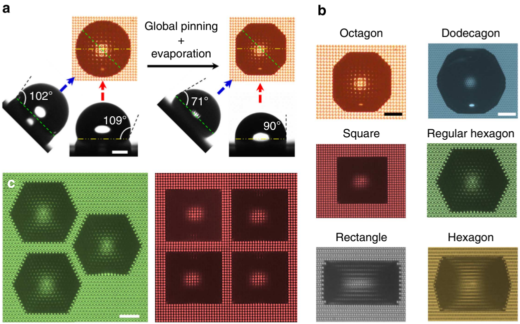
- (Caffarelli and Monneau, '14) Counter-example in $d \geq 3$, homogenization in $d = 2$ without a Lipschitz estimate with weaker coercivity

$$\inf_{\mathbb{R}^d} c > 0.$$

Random media:

- (Armstrong and Cardaliaguet, '15) Homogenization in $d \geq 2$ with the L-S coercivity condition (finite range).
- (F., in preparation '19) Homogenization in $d = 2$ with C-M condition, counter-example in $d \geq 3$.

Liquid drops on rough surfaces



Raj, Adera, Enright and Wang *Nat. Commun.* (2014)

Flow in lattice porous medium

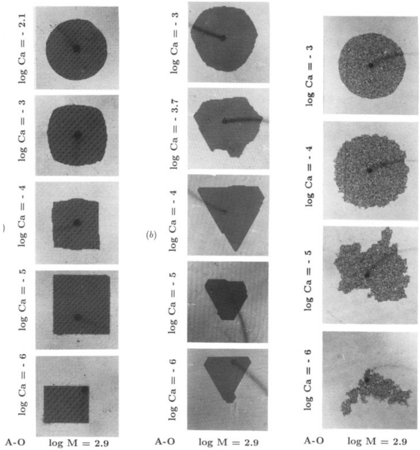


Figure 5. Imbibition. (*a*) Injection of oil (black) displacing air in a square network with a narrow pore size distribution. (*b*) Injection of oil (black) displacing air in a triangular network with a narrow pore size distribution. (*c*) Injection of oil (black) displacing air in a square network with a wide pore size distribution.

R Lenormand *J. Phys.: Condens. Matter* (1990)