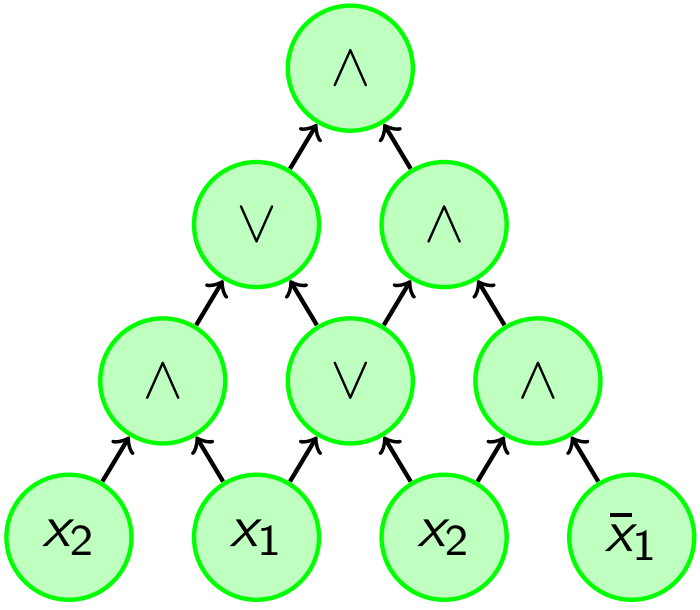
Pebble Games and Complexity

Siu Man Chan

Princeton CCI

21 Jan, 2014 @ IAS

Circuit Evaluation Problem



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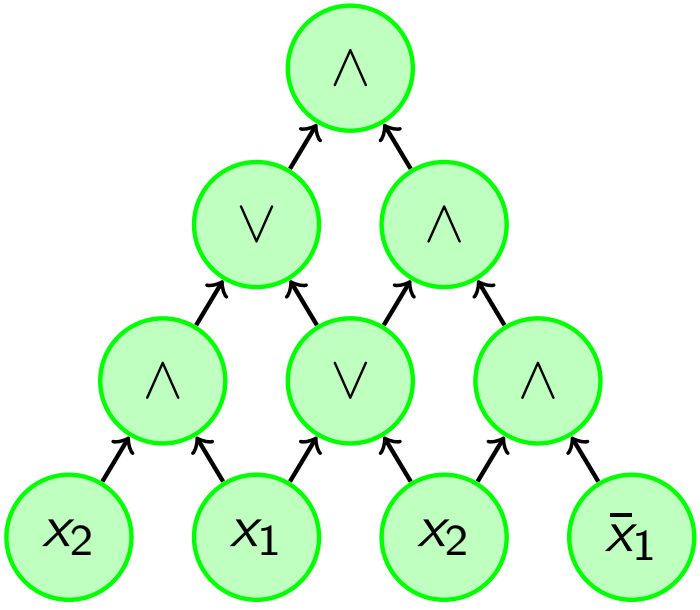
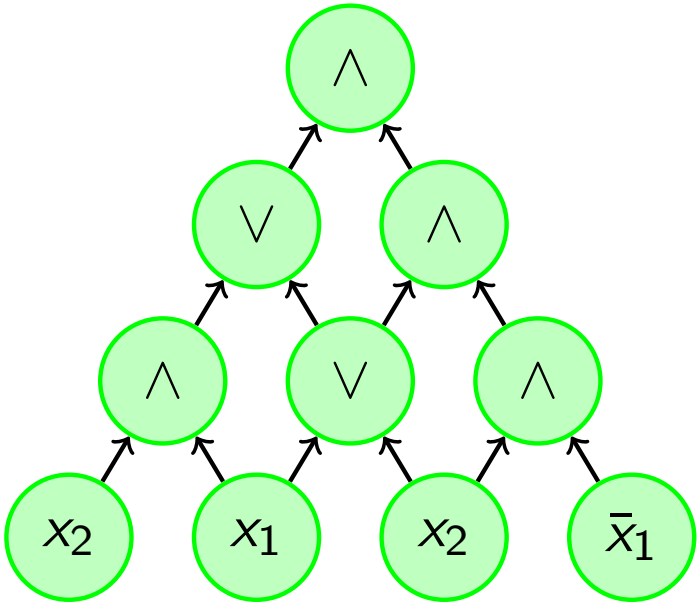


- Input:
  - circuit C
  - instance x

Circuit Evaluation Problem



- Input:
  circuit C
  instance x
- Output:
  Result of evaluating C on x

Circuit Evaluation Problem

- Complete for P

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- "Complete" for NC^i for restricted C of depth $\leq O(\log^i n)$

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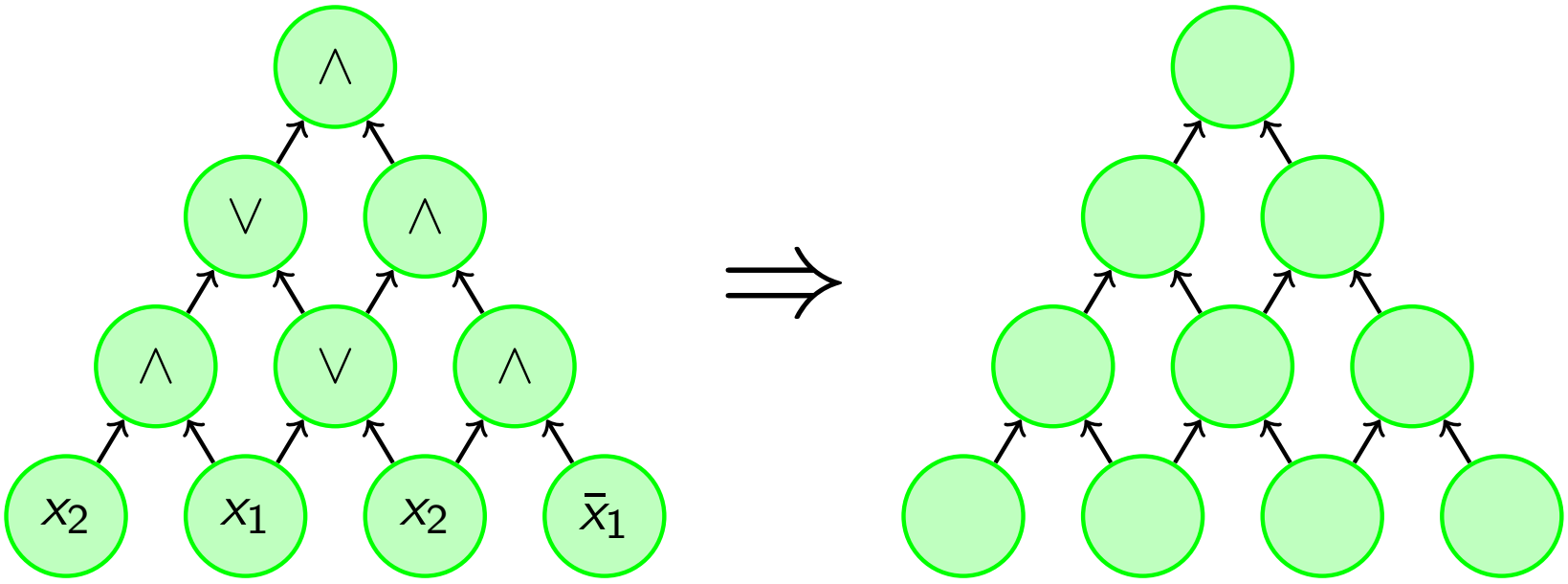
- Concerns space and parallel complexity
- Applications: Database query algorithms, Data flow models, Big Data computation, etc.

Black Pebble Game



Black Pebble Game

Algorithms for Circuit Evaluation

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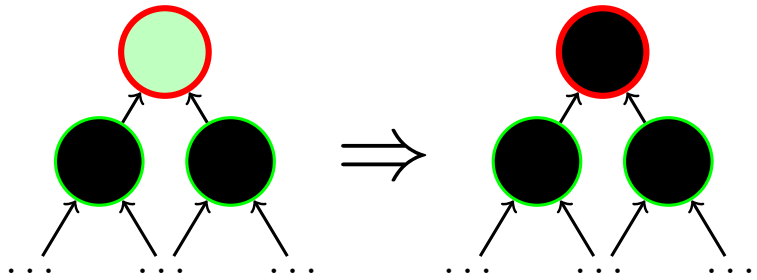
- Rule 1: add pebble to v if all immediate predecessors of v are pebbled



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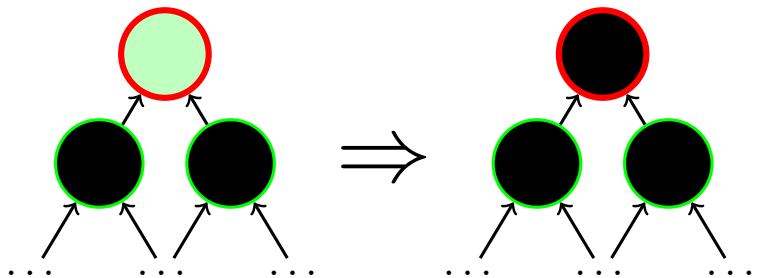

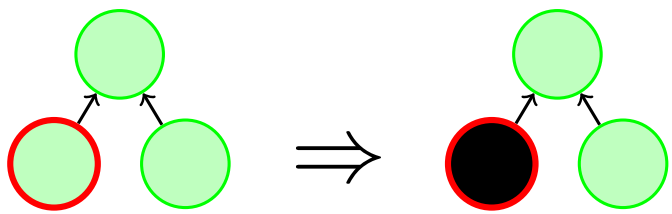

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Algorithms for Circuit Evaluation

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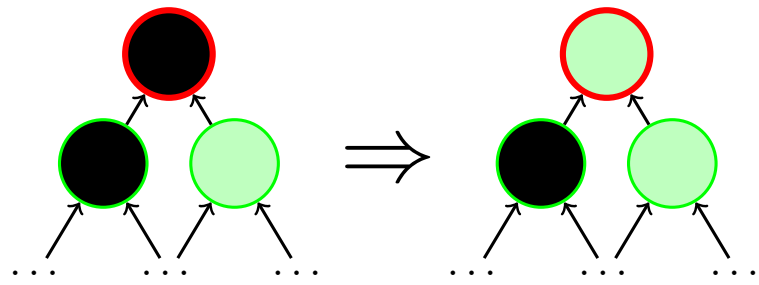
- Rule 2: remove pebble at any time



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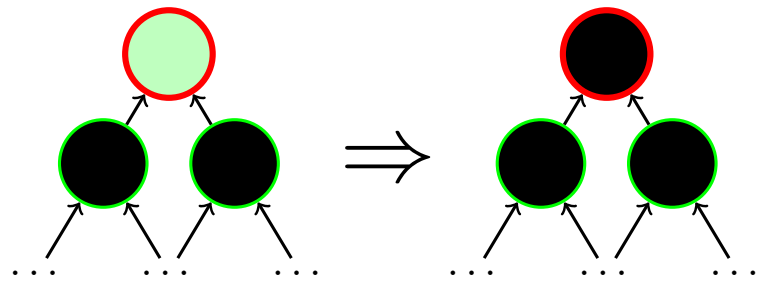

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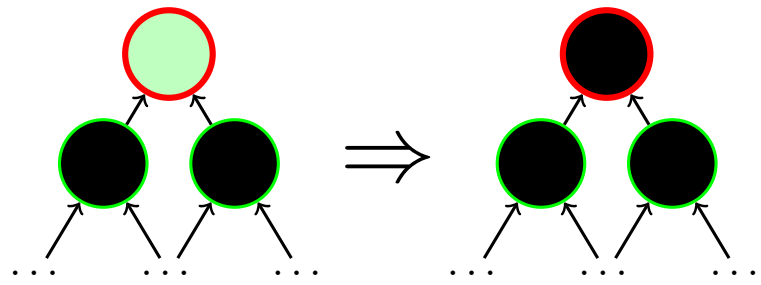

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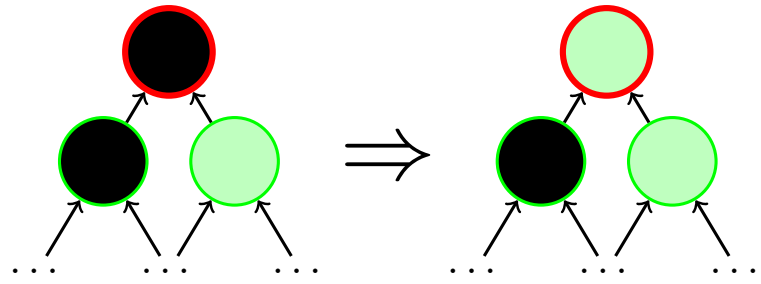

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$$\forall G$$
$$\#\text{Pebbles(G)} \leq O\big(|G|/\log|G|\big)$$

[Hopcroft–Paul–Valiant '77]

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$$\mathrm{Time}[t] \subseteq \mathrm{Space}[t/\log t]$$

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Lower Bounds by Pebble Games

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m-$\mathsf{NC}^1 \subsetneq$ m-NL m-circuits [Karchmer–Wigderson '90]

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Theorem (Karchmer–Wigderson)

Circuit Depth = Communication Complexity

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NC^1 vs NC^2 Universal composition relation [Edmonds–Impagliazzo–Rudich–Sgall '01]

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In the bounds for Iterated indexing:

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My audience

- Parallel Complexity
- Space Complexity
- Randomised Complexity
- Communication Complexity
- Decision Tree Complexity (Certificate Complexity)
- Proof Complexity
- Algebraic Complexity

Reversible game

$\|$

Dymond–Tompa game

$\|$

Raz–McKenzie game

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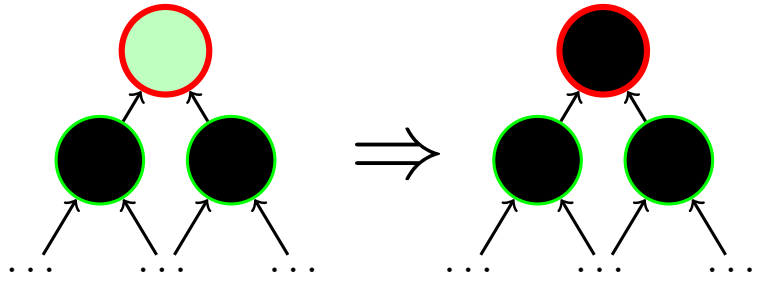

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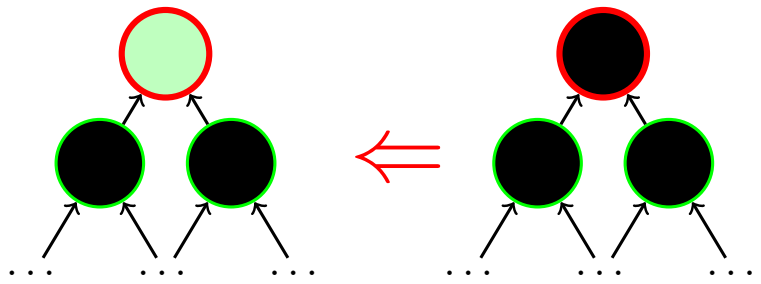

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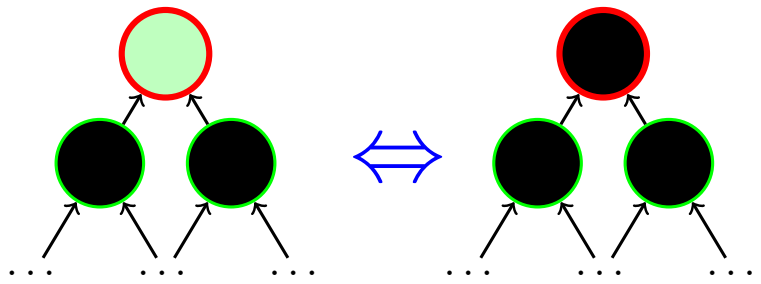

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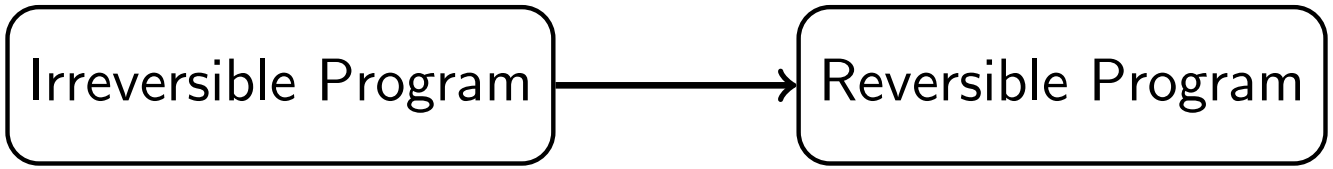
  - Reversible simulation of irreversible computation [Bennett '89]

    [Li–Vitanyi '96, '97] [Král'ovič '01]



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- Monotone space lower bounds [Potechin '10] [C.–Potechin '12]:

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  - Reversible simulation of irreversible computation [Bennett '89]

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- Monotone space lower bounds [Potechin '10] [C.–Potechin '12]:
  - Determinism equals reversibility/symmetry [Lange–McKenzie–Tapp '00]

    [Reingold '08]

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Parallel Evaluation, Recursively

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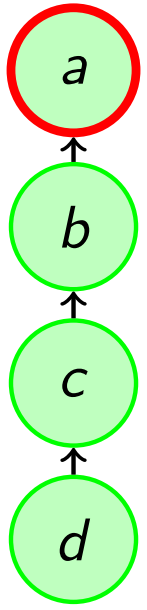

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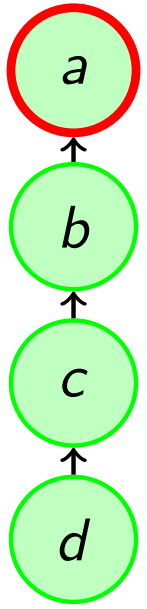

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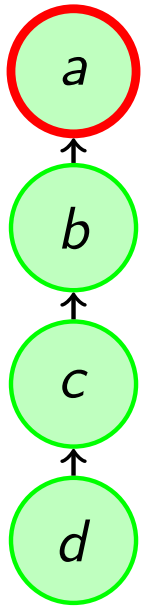

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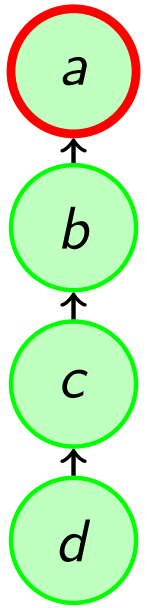

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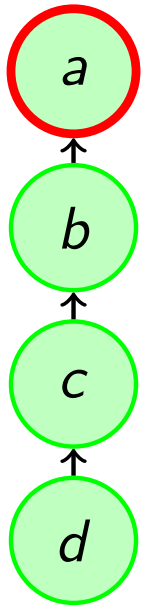

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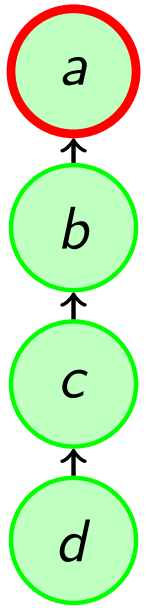

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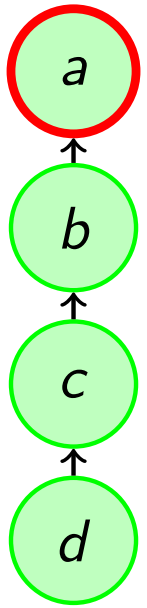

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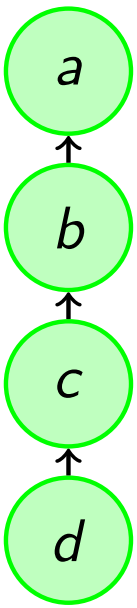

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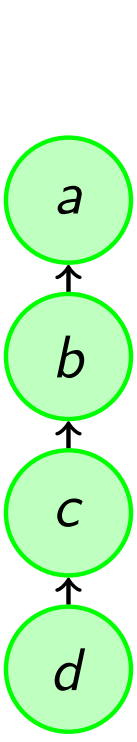

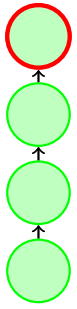

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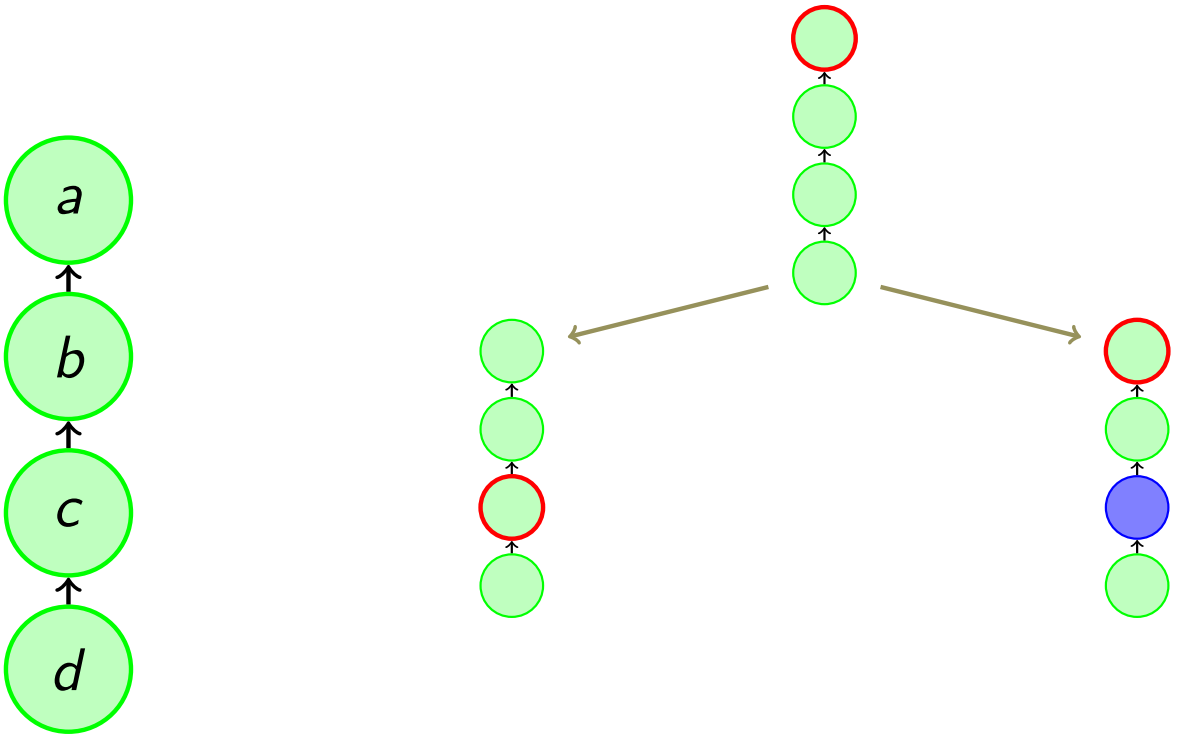

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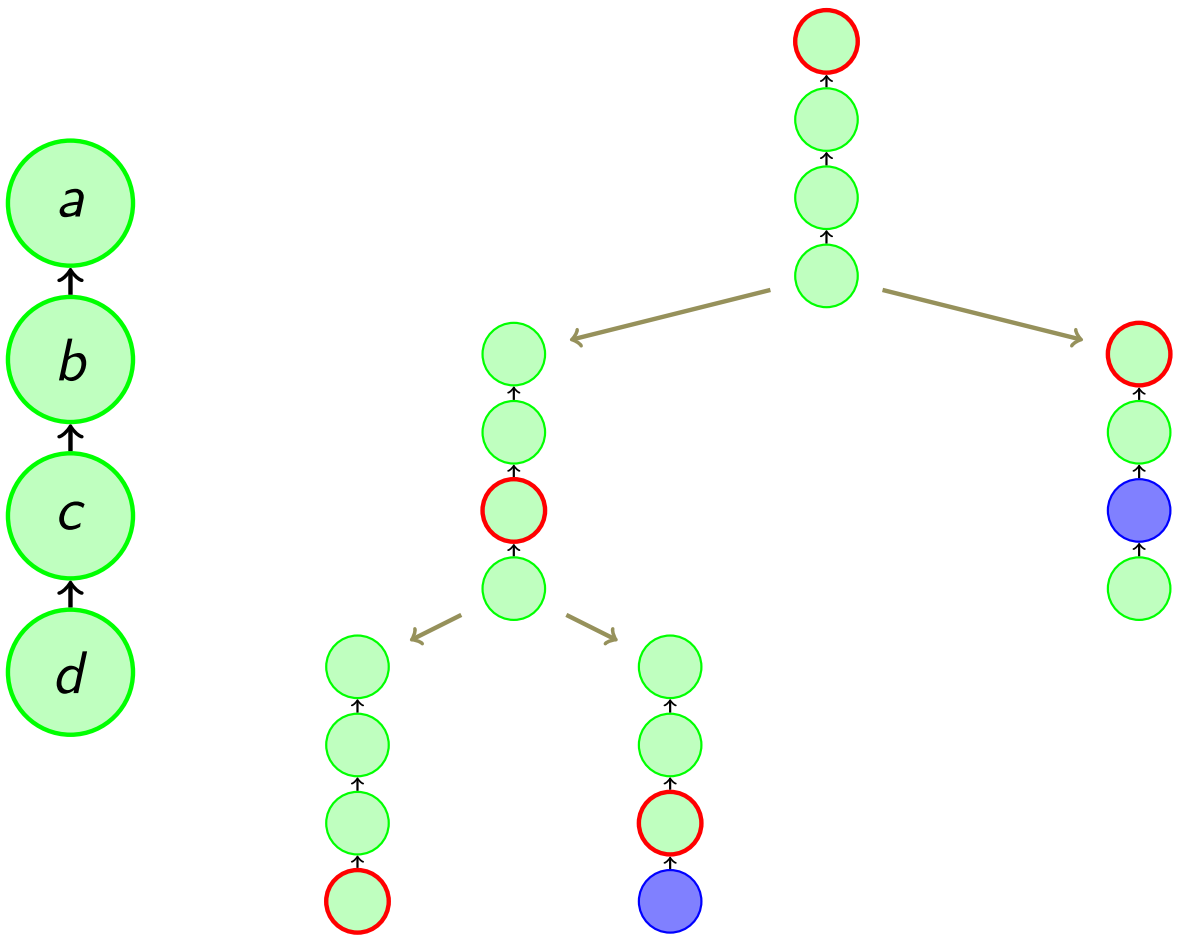

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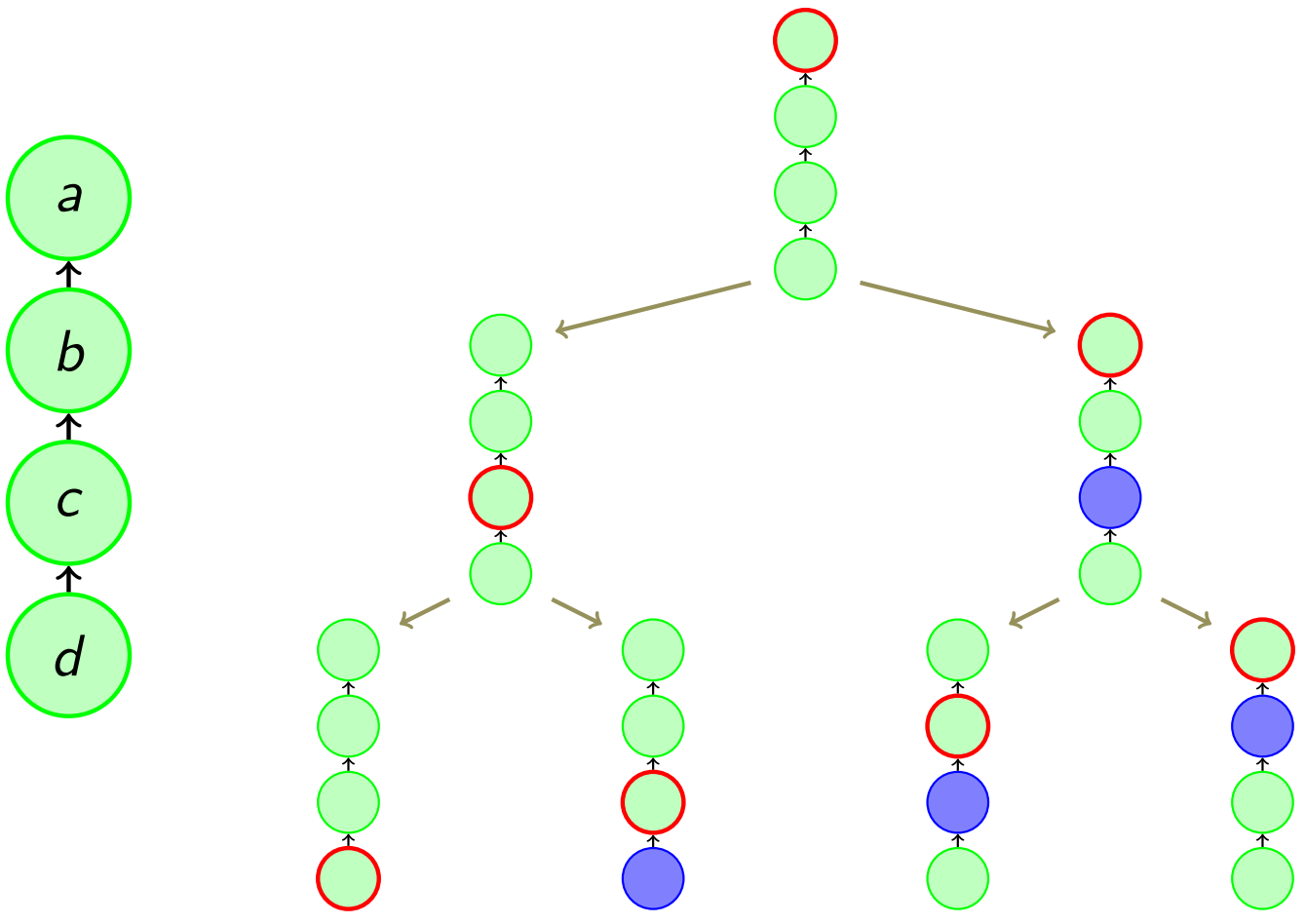

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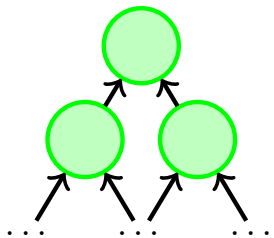

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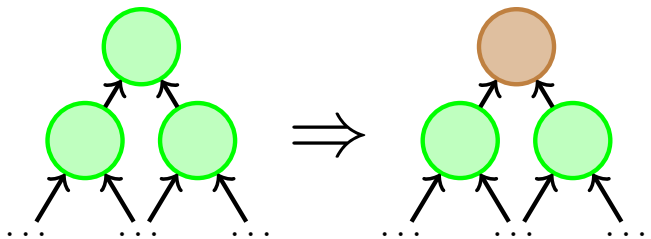

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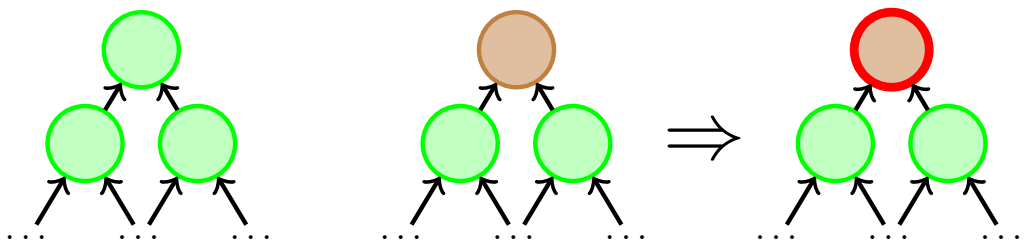

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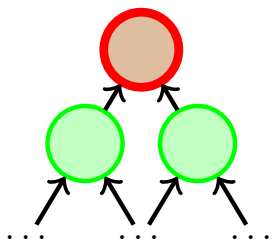

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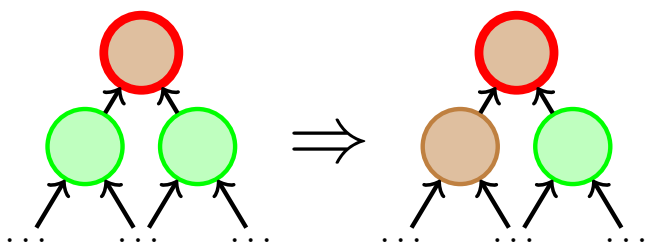

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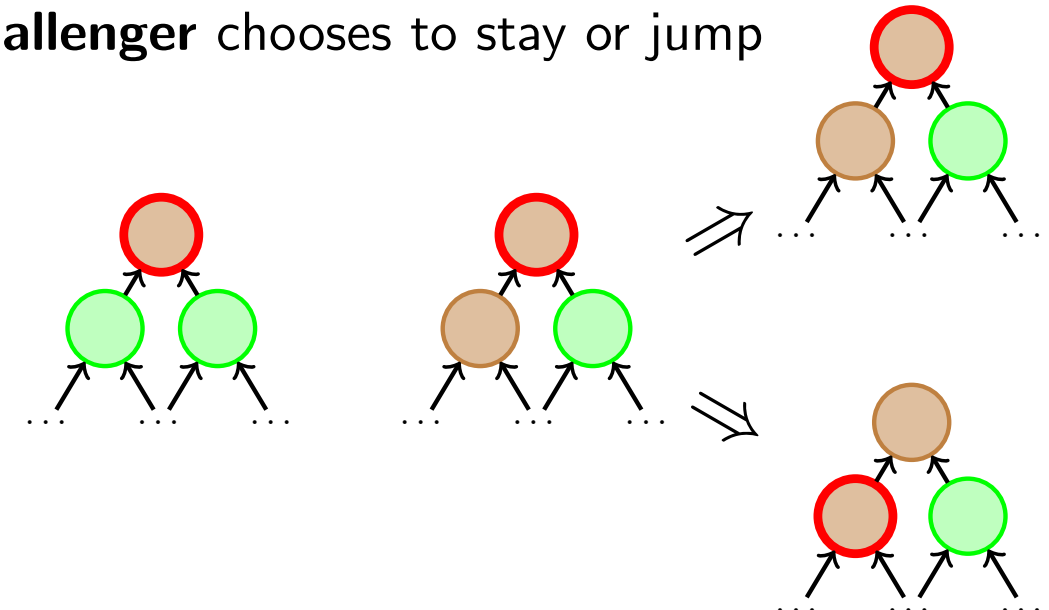

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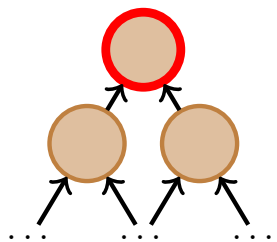

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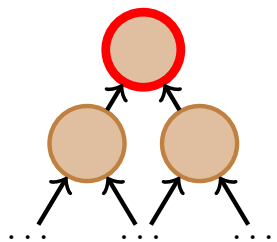

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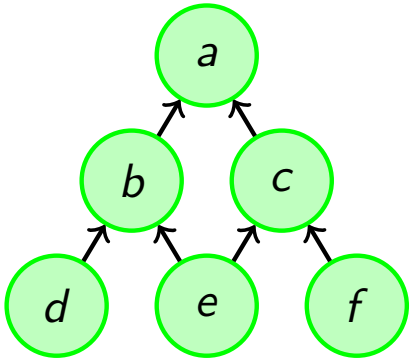

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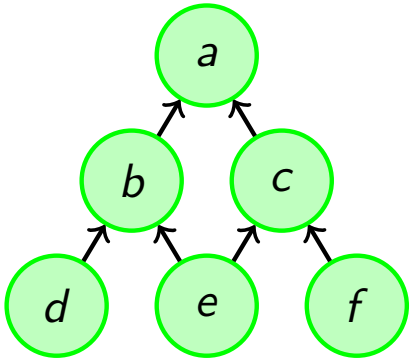

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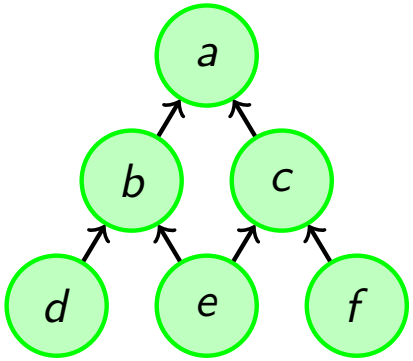

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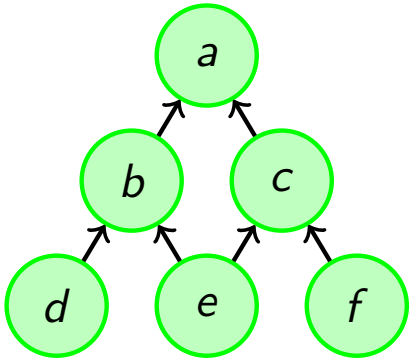

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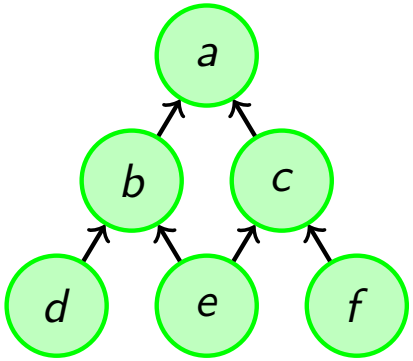

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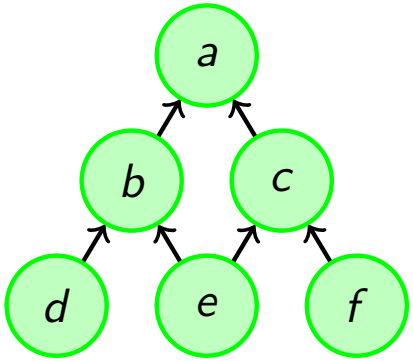

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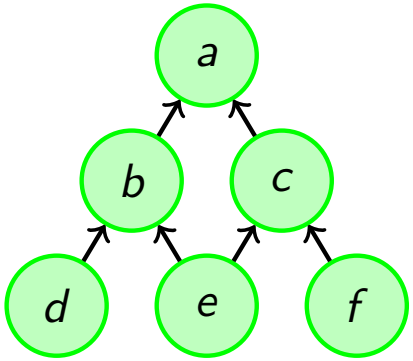

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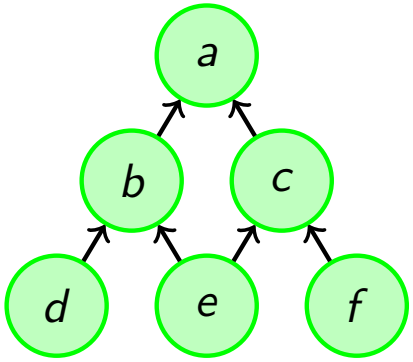

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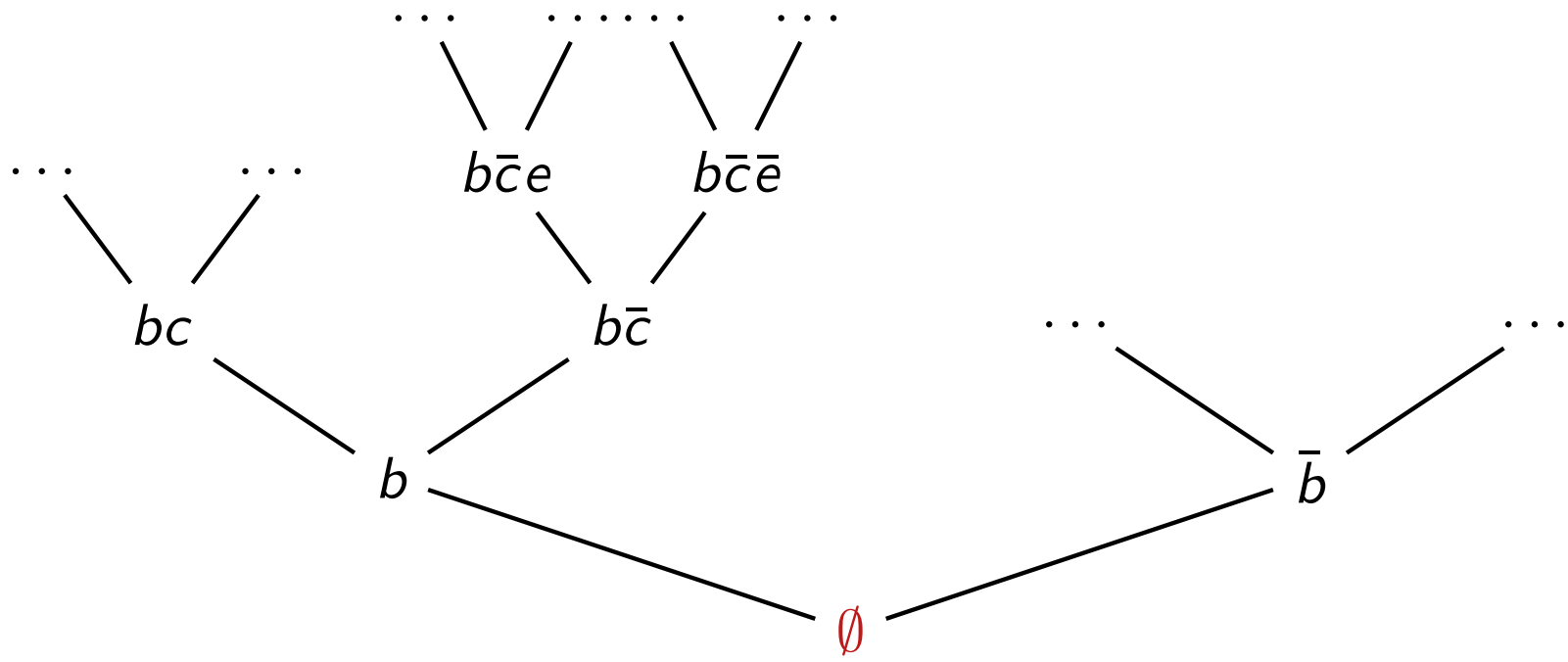

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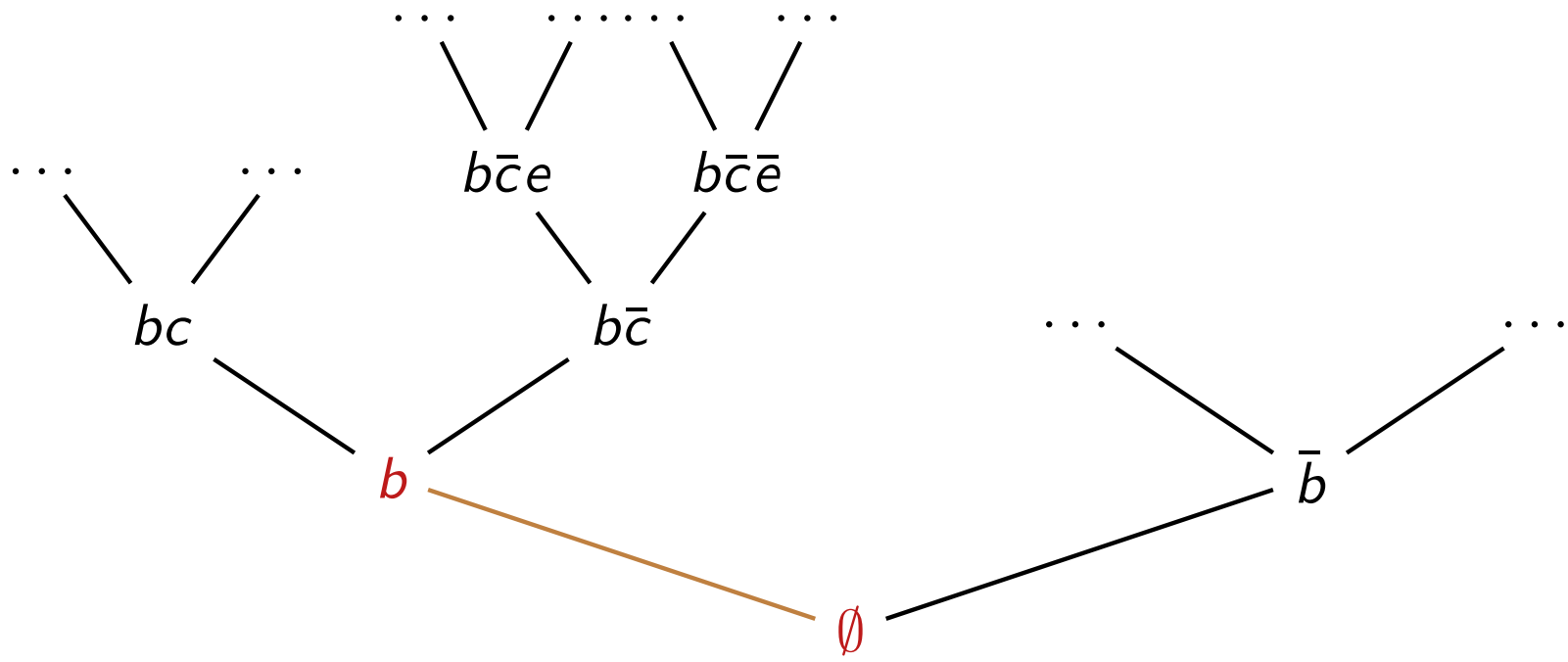

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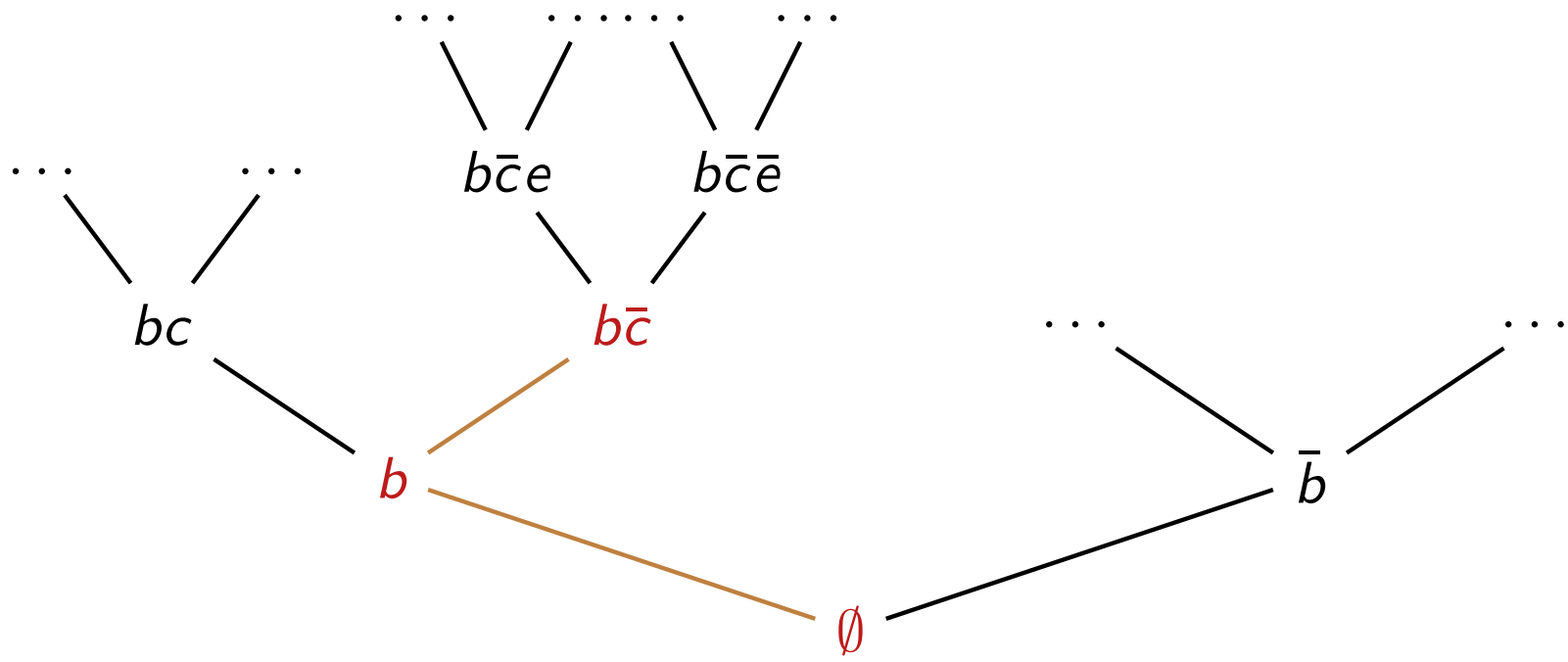

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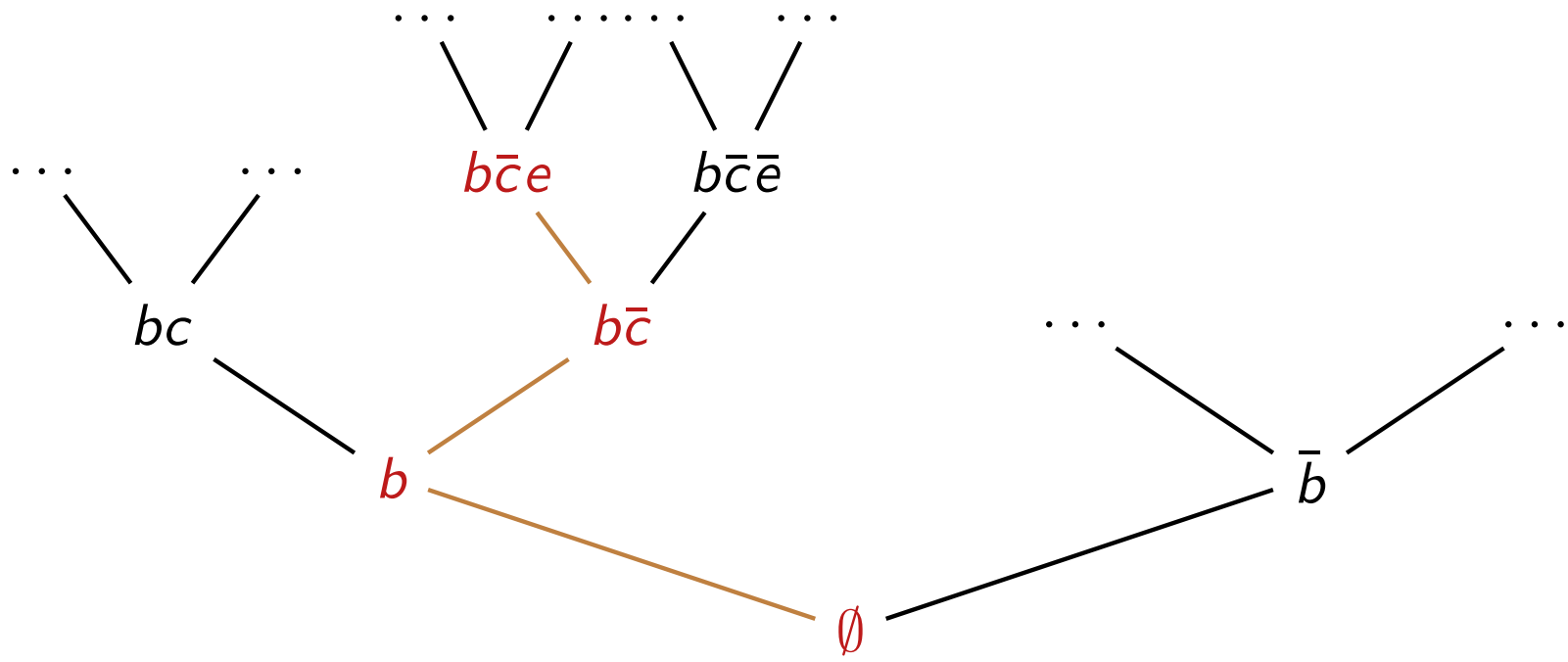

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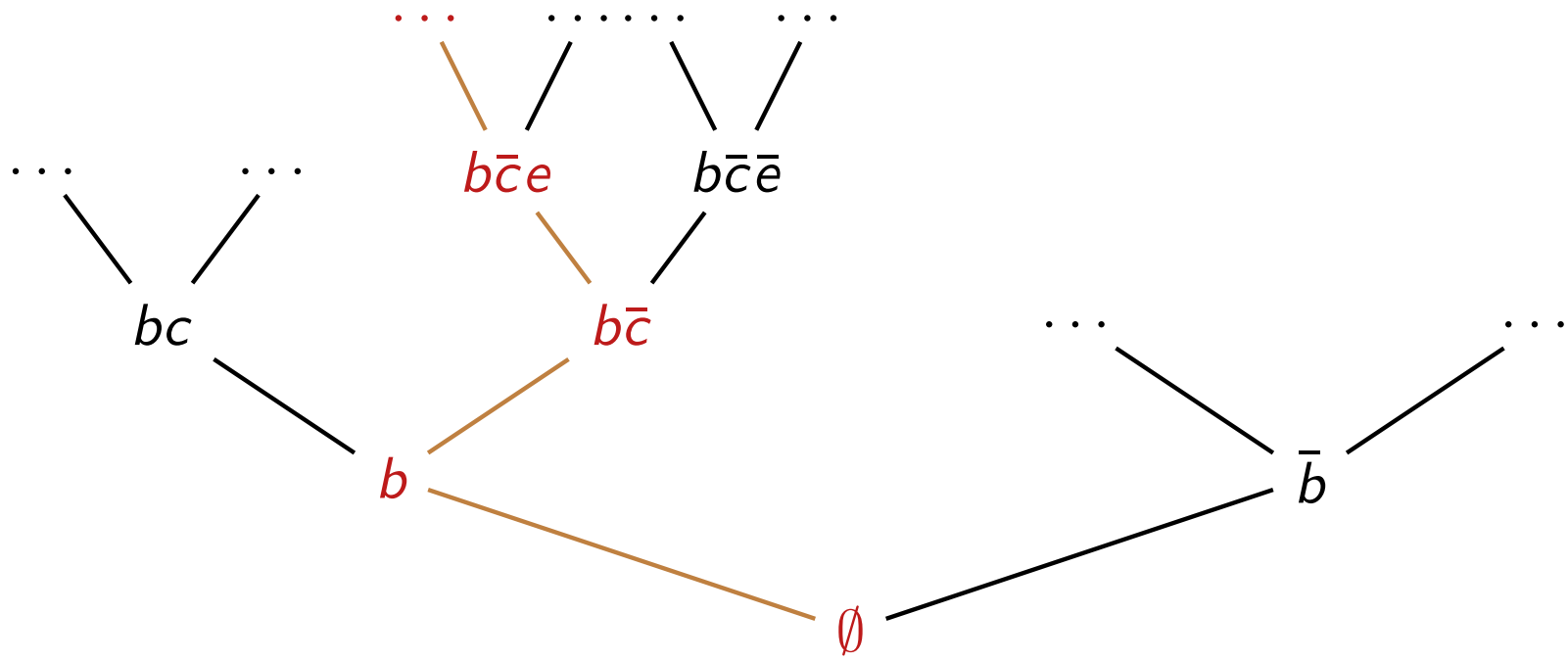

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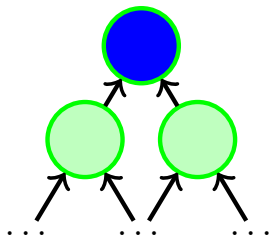

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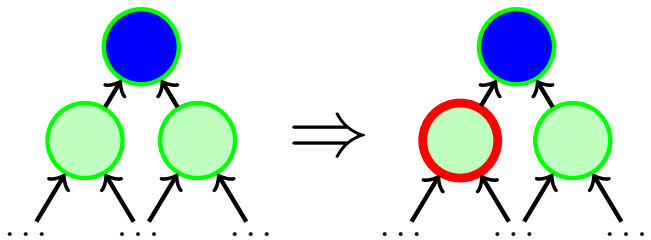

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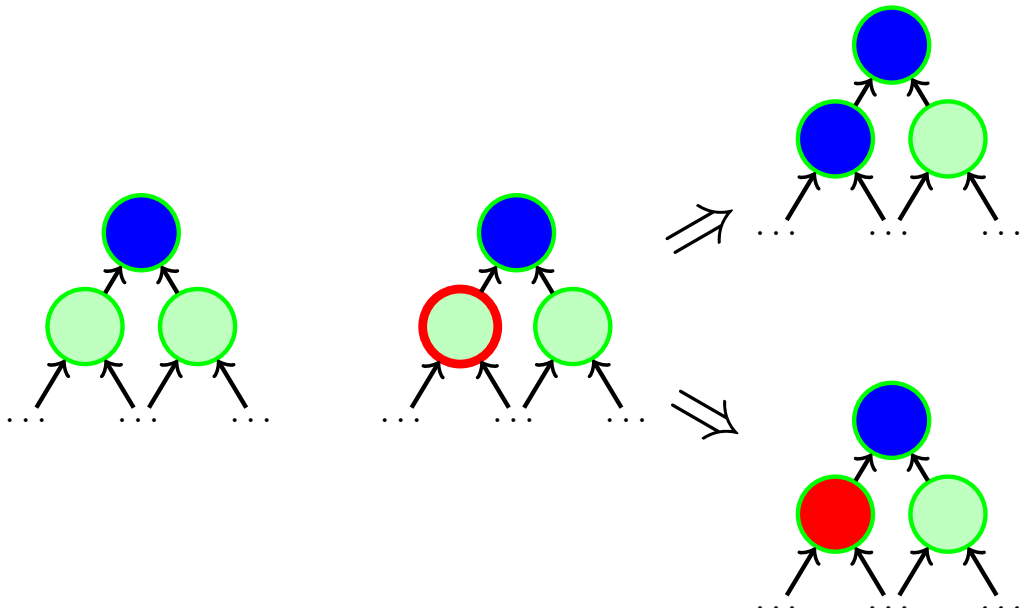

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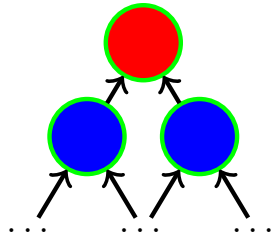

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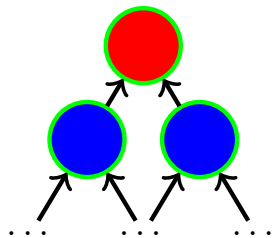

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- Add an initial set-up to make it more like Dymond–Tompa game.

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Dymond–Tompa pebble game

- Two players: **Pebbler** and **Challenger**, competitive
- Alternate to move, **Pebbler** moves first
- Initial Set-up:
  - **Pebbler** pebbles sink
  - **Challenger** challenges sink
- Each round:
  - **Pebbler** chooses a node to pebble
  - **Challenger** chooses to stay or jump
- **Pebbler** wins if, before she moves, the challenged node has all immediate predecessors pebbled
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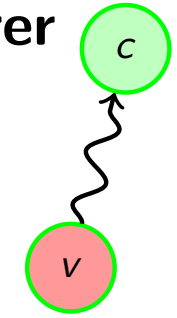

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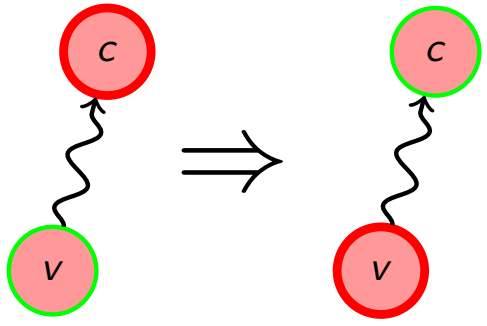

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Questions