Some inter-relations between random matrix ensembles

Peter Forrester,

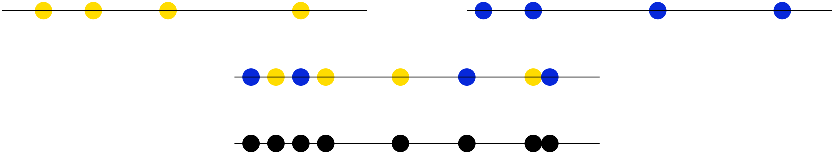
M&S, University of Melbourne

Outline

- Superpositions and decimations
- Averages of characteristic polynomials
- Structure function
- Moments and resolvent

Superimposed spectra

$$H = \begin{bmatrix} H_1 & \\ & H_2 \end{bmatrix}$$



$$p(x_1, \ldots, x_N) = \frac{1}{C} \prod_{l=1}^{N} e^{-x_l^2/2} \prod_{1 \le j < k \le N} |x_k - x_j|$$

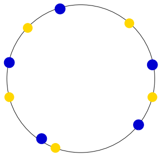
Superimposed spectra (cont)

Label the $2N$ points $x_1 < x_2 < \cdots < x_{2N}$. Must compute

$$\sum_{\substack{S \subset \{1,\ldots,2N\} \\ |S|=N}} p(x_S)\, p(x_{\{1,\ldots,2N\}-S})$$

With $\Delta(\theta_S) = \prod_{1 \le j < k \le N} \sin((\theta_{s_k} - \theta_{s_j})/2)$ it was proved by Gunson that

$$\sum_{\substack{S \subset \{1,\ldots,2N\} \\ |S|=N}} \Delta(\theta_S)\Delta(\theta_{\{1,\ldots,2N\}-S}) = 2^N \Delta(\theta_{\{1,3,\ldots,2N-1\}})\Delta(\theta_{\{2,4,\ldots,2N\}})$$



Superimposed spectra (cont)

Suggests that the distribution of every second eigenvalue is special. Integrate $\{\theta_2, \theta_2, \dots, \theta_{2N}\}$ over the region

$$R_N = \theta_1 < \theta_2 < \theta_3 < \theta_4 < \cdots < \theta_{2N-1} < \theta_{2N} < 2\pi + \theta_1$$

Using the Vandermonde identity

$$\prod_{1 \le j < k \le N} (x_j - x_k) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_N & x_{N-1} & x_{N-2} & \cdots & x_1 \\ x_N^2 & x_{N-1}^2 & x_{N-2}^2 & \cdots & x_1^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ x_N^{N-1} & x_{N-1}^{N-1} & x_{N-2}^{N-1} & \cdots & x_1^{N-1} \end{vmatrix}$$

can compute

$$\int_{R_N} d\theta_2 \cdots d\theta_{2N} \, \Delta(\theta_{\{2,4,\dots,2N\}}) \propto \Delta(\theta_{\{1,3,\dots,2N-1\}})$$

Dyson (1962) ex-conjecture

Let alt denote the operation of integration over every second eigenvalue.
Let $\cup$ denote the operation of random superposition.
We have

$$\mathrm{alt}\Big(\mathrm{COE}_N \cup \mathrm{COE}_N\Big) = \mathrm{CUE}_N$$

Consequence for gap probabilities

Let $E_N^{\mathrm{ME}}(0,J)$ denote the probability that there are no eigenvalues in the interval J of the matrix ensemble ME consisting of N eigenvalues. We have

$$E_N^{\mathrm{CUE}}(0;(-\theta,\theta)) =$$

$$E_N^{\mathrm{COE}}(0;(-\theta,\theta))\Big(E_N^{\mathrm{COE}}(0;(-\theta,\theta)) + E_N^{\mathrm{COE}}(1;(-\theta,\theta))\Big)$$

F & Rains (2001) (cont)

Question: For matrix ensembles with orthogonal symmetry, eigenvalue PDF of the form

$$\frac{1}{C_N}\prod_{l=1}^{N} f(x_l) \prod_{1\le j<k\le N} |x_k - x_j| =: \mathrm{OE}_N(f)$$

for what choices of f does

$$\mathrm{even}\left(\mathrm{OE}_N(f) \cup \mathrm{OE}_{N+1}(f)\right) = \mathrm{UE}_N(g)$$

for some g?

Must first obtain a Gunson type identity

$$\sum_{\substack{S\subset\{1,\dots,2N+1\}\\ |S|=N}} \Delta(x_S)\Delta(x_{\{1,\dots,2N+1\}-S}) = 2^N \Delta(x_{\{1,3,\dots,2N+1\}})\Delta(x_{\{2,4,\dots,2N\}})$$

where $\Delta(x_S) = \prod_{1\le j<k\le N}(x_{s_k} - x_{s_j})$.

F & Rains (2001) (cont)

Answer: (up to linear fractional transformation) the four classical weight functions:

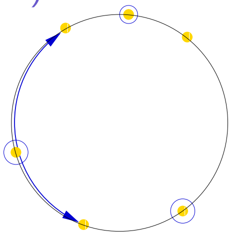
$$f(x) = \begin{cases} e^{-x^2/2}, & \text{Gaussian} \\ x^{(a-1)/2}e^{-x/2} \ (x>0), & \text{Laguerre} \\ (1-x)^{(a-1)/2}(1+x)^{(b-1)/2} \ (-1<x<1), & \text{Jacobi} \\ (1+ix)^{-(\alpha+1)/2}(1-ix)^{-(\bar{\alpha}+1)/2}, & \text{Cauchy} \end{cases}$$

$$g(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^{a}e^{-x} \ (x>0), & \text{Laguerre} \\ (1-x)^{a}(1+x)^{b} \ (-1<x<1), & \text{Jacobi} \\ (1+ix)^{-\alpha}(1-ix)^{-\bar{\alpha}}, & \text{Cauchy} \end{cases}$$

In particular

$$\text{even}\left(\text{GOE}_{N+1} \cup \text{GOE}_N\right) = \text{GUE}_N$$

Mehta and Dyson (1963)



Using direct integration, showed

$$\mathrm{alt}\,(\mathrm{COE}_{2N}) = \mathrm{CSE}_N$$

Consequence for gap probabilities

We have

$$\begin{aligned} E_N^{\mathrm{CSE}}&(0;(-\theta,\theta)) \\ &= E_{2N}^{\mathrm{COE}}(0;(-\theta,\theta)) + \frac{1}{2} E_{2N}^{\mathrm{COE}}(1;(-\theta,\theta)) \\ &= \frac{1}{2}\Big(E_{2N}^{\mathrm{COE}}(0;(-\theta,\theta)) + \frac{E_{2N}^{\mathrm{CUE}}(0;(-\theta,\theta))}{E_{2N}^{\mathrm{COE}}(0;(-\theta,\theta))}\Big) \end{aligned}$$

Further new question:

For what choice of f does

$$\text{even}\left(\text{OE}_{2N+1}(f)\right) = \text{SE}_N(g)$$

for some g?

Answer (FR 2001)

$$\text{even}\left(\text{OE}_{2N+1}(f)\right) = \text{SE}_N((g/f)^2) \Leftrightarrow$$

$$\text{even}\left(\text{OE}_N(f) \cup \text{OE}_{N+1}(f)\right) = \text{UE}_N(g)$$

In particular, with $(f, g) = (e^{-x^2/2}, e^{-x^2})$

$$\text{even}\left(\text{GOE}_{2N+1}\right) = \text{GSE}_N$$

A family of decimation relations (inspired by Bálint Virág)

Denote by $\mathrm{ME}_{\beta,N}(g(x))$ the PDF proportional to

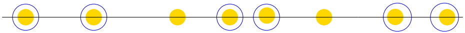
$$\prod_{l=1}^{N} g(x_l) \prod_{1 \le j < k \le N} |x_k - x_j|^{\beta}$$

and let D_r denote the distribution of every r-th eigenvalue.
For the Gaussian case we have (F. 2009)

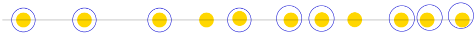
$$\mathrm{D}_{r+1}(\mathrm{ME}_{2/(r+1),(r+1)N+r}(e^{-x^2})) = \mathrm{ME}_{2(r+1),N}(e^{-(r+1)x^2})$$

e.g.

$$D_3(\mathrm{ME}_{2/3,3N+2}(e^{-x^2}) = \mathrm{ME}_{6,N}(e^{-3x^2})$$



$$D_4(\mathrm{ME}_{1/2,4N+3}(e^{-x^2}) = \mathrm{ME}_{8,N}(e^{-4x^2})$$



Consequences for asymptotic spacing distributions

Let $p_\beta^{\rm bulk,sp.}(n;s)$ denote the probability that in the bulk scaling limit there are n eigenvalues between 2 eigenvalues separated by distance s.
The decimation relations imply that for large s

$$E^{\rm bulk}_{2/(r+1)}((r+1)k+r;(r+1)s) \sim E^{\rm bulk}_{2(r+1)}(k;s).$$

A conjecture of Dyson, and of Fogler and Shklovskii (1995),

$$\log E_\beta^{\rm bulk}(n;(0,s)) \underset{s\to\infty}{\sim} -\beta\frac{(\pi s)^2}{16} + \Big(\beta n + \frac{\beta}{2} - 1\Big)\frac{\pi s}{2} + \Big\{\frac{n}{2}\Big(1-\frac{\beta}{2}-\frac{\beta n}{2}\Big) + \frac{1}{4}\Big(\frac{\beta}{2}+\frac{2}{\beta}-3\Big)\Big\}\log s$$

has this property.

Averages of characteristic polynomials

For the Gaussian β ensemble (Baker & F 1997)

$$\Big\langle \prod_{j=1}^{N} (c - \sqrt{\alpha} y_j)^n \Big\rangle_{\mathrm{ME}_{2/\alpha,N}(e^{-y^2})} = \Big\langle \prod_{j=1}^{n} (c - i y_j)^N \Big\rangle_{\mathrm{ME}_{2\alpha,n}(e^{-y^2})}.$$

Consequences

- The simplest case is $n = 1$. It tells us that the average of the characteristic polynomial for the Gaussian β ensemble is proportional to the Hermite polynomial $H_N(c)$.
- Suppose β is even. Then setting $n = \beta$ the LHS multiplied by $e^{-c^2/\alpha}$ is proportional to the eigenvalue density at $c/\sqrt{\alpha}$. Hence, for even β, this can be expressed as a β dimensional integral.
- Large N asymptotic analysis using the saddle point method gives oscillatory corrections to the Wigner semi-circle law, and the scaled density at the edge.

Explicit form of the scaled density at the edge

We have (Desrosiers & F (2006))

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}N^{1/6}} \rho_{(1)}\Big(\sqrt{2N} + \frac{x}{\sqrt{2}N^{1/6}}\Big) = \frac{\Gamma(1+\beta/2)}{2\pi}\Big(\frac{4\pi}{\beta}\Big)^{\beta/2} \prod_{j=1}^{\beta} \frac{\Gamma(1+2/\beta)}{\Gamma(1+2j/\beta)} K_{\beta,\beta}(x),$$

where

$$K_{n,\beta}(x) := -\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} dv_1 \cdots \int_{-i\infty}^{i\infty} dv_n \prod_{j=1}^{n} e^{v_j^3/3 - x v_j} \prod_{1\le k<l\le n} |v_k - v_l|^{4/\beta}.$$

Asymptotics of the edge density

$$\rho_{(1)}^{\mathrm{soft},\beta}(x) \underset{x\to\infty}{\sim} \frac{1}{\pi}\frac{\Gamma(1+\beta/2)}{(4\beta)^{\beta/2}}\frac{e^{-2\beta x^{3/2}/3}}{x^{3\beta/4-1/2}} + \mathrm{O}\left(\frac{1}{x^{3\beta/4+1}}\right),$$

$$\rho_{(1)}^{\mathrm{soft},\beta}(x) \underset{x\to-\infty}{\sim} \frac{\sqrt{|x|}}{\pi} - \frac{\Gamma(1+\beta/2)}{2^{6/\beta-1}|x|^{3/\beta-1/2}}\cos\left(\frac{4}{3}|x|^{3/2} - \frac{\pi}{2}\left(1-\frac{2}{\beta}\right)\right)$$

This has consequence to the asymptotics of the right tail of the scaled distribution of the largest eigenvalue:

$$p_{\beta}^{\mathrm{soft}}(X) \underset{X\to\infty}{\sim} \rho_{(1)}^{\mathrm{soft},\beta}(X).$$

Averages of characteristic polynomials — circular ensemble

Let $\alpha = 2/\beta - 1$ and $\mu \in \mathbb{Z}^+$. We have

$$\Big\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \Big\rangle_{\mathrm{CE}_{\beta,N}} \propto \Big\langle \prod_{l=1}^{\mu} \Big(1 - (1 - |z|^2) x_l\Big)^N \Big\rangle_{\mathrm{ME}_{4/\beta,\mu}(x^\alpha (1-x)^\alpha)}.$$

This can be generalized to allow a factor $|z - e^{i\theta_l}|^{2\mu_1}$ in the product on the LHS.
Hence for even β the two-point function can be written as a β-dimensional integral. It's proportional to (F. (1994))

$$(2 \sin \pi(r_1 - r_2)/L)^\beta e^{-\pi i \beta N (r_1 - r_2)} \int_{[0,1]^\beta} du_1 \cdots du_\beta$$

$$\times \prod_{j=1}^{\beta} (1 - (1 - e^{2\pi i (r_1 - r_2)}) u_j)^N u_j^{-1+2/\beta} (1 - u_j)^{-1+2/\beta} \prod_{j<k} |u_k - u_j|^{4/\beta}.$$

- The large N bulk scaled limit can be taken immediately.
- Can analyze the large N global expansion (no scaling of variables)

$$\Big(\frac{2\pi}{N}\Big)^2 \rho_{(2)}(0,\theta) = 1 - \frac{1}{\beta(2N\sin\theta/2)^2} + \frac{3(\beta-2)^2}{2\beta^3(2N\sin\theta/2)^4} - \cdots$$

Not suited to computing the structure function. In the bulk, for $\beta = p/q$ have

$$S(k;\beta) = \frac{|k|}{\pi\beta} f(|k|;\beta),$$

where for $|k| < 2\pi$

$$f(k;\beta) \propto \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j \, Q_{p,q}^2 \hat{F}(q,p,\lambda|\{x_i,y_j\};k)\,\delta(1-Q_{p,q}),$$

with $\lambda = \beta/2$, $Q_{p,q} = 2\pi(\sum_{i=1}^q x_i + \sum_{j=1}^p y_j)$,

$$\hat{F}(q,p,\lambda|\{x_i,y_j\};k) = \frac{1}{\prod_{i=1}^q (x_i(1+kx_i/\lambda))^{1-\lambda} \prod_{j=1}^p (y_j(1-ky_j))^{1-1/\lambda}} \times \frac{\prod_{i<i'} |x_i - x_{i'}|^{2\lambda} \prod_{j<j'} |y_j - y_{j'}|^{2/\lambda}}{\prod_{i=1}^q \prod_{j=1}^p (x_i + \lambda y_j)^2}.$$

Functional equation for the structure function

From the exact form of $\rho_{(2)}^{\mathrm{bulk}}(0;x)$ have

$$S(k) = \begin{cases} \dfrac{|k|}{\pi} - \dfrac{|k|}{2\pi}\log\Big(1 + \dfrac{|k|}{\pi}\Big), & |k| \le 2\pi,\ (\beta = 1) \\ \dfrac{|k|}{2\pi}, & |k| \le 2\pi,\ (\beta = 2) \\ \dfrac{|k|}{4\pi} - \dfrac{|k|}{8\pi}\log\Big(1 - \dfrac{|k|}{2\pi}\Big), & |k| \le 4\pi,\ (\beta = 1) \end{cases}$$

From the exact form for $S(k)$ for β rational can check that with

$$f(k;\beta) = \frac{\pi\beta}{|k|} S(k;\beta), \qquad 0 < k < \min(2\pi, \pi\beta)$$

and f defined by analytic continuation for $k < 0$,

$$f(k;\beta) = f\Big(-\frac{2k}{\beta}; \frac{4}{\beta}\Big).$$

The simplest structure consistent with the functional equation is

$$\frac{\pi\beta}{|k|}S(k;\beta) = 1 + \sum_{j=1}^{\infty} p_j(\beta/2)\Big(\frac{|k|}{\pi\beta}\Big)^j, \qquad 0 < k < \min(2\pi, \pi\beta)$$

where $p_j(x)$ is a polynomial of degree j which satisfies the functional relation

$$p_j(1/x) = (-1)^j x^{-j} p_j(x).$$

Put $x = \beta/2$, $y = |k|/\pi\beta$. We have (F., Jancovici, McAnally (2000))

$$\frac{\pi\beta}{|k|}S(k;\beta) = 1 + (x-1)y + (x-1)^2y^2 + (x-1)(x^2 - \frac{11}{6}x + 1)y^3$$
$$+ (x-1)^2(x^2 - \frac{3}{2}x + 1)y^4 + (x-1)(x^4 - \frac{91}{30}x^3 + \frac{62}{15}x^2 - \frac{91}{30}x + 1)y^5 + \cdots$$

Moments of the density and loop equations

For the Gaussian β ensemble, with the eigenvalues scaled so that the leading support is $(-1, 1)$, and with $\lambda = \beta/2$, let

$$m_{2l}(N, \lambda) = \int_{-\infty}^{\infty} \lambda^{2l} \rho_{(1)}^{N}(x; \lambda)\, dx$$

It is known rigorously (Dumitriu and Edleman (2006)) that $m_{2l}(N, \lambda)$ is a polynomial of a degree $l + 1$ in N with constant term zero, satisfying

$$m_{2l}(N, \lambda) = (-1)^{l+1} \lambda^{-l-1} m_{2l}(-\lambda N, \lambda^{-1}).$$

$$\begin{aligned}
m_0 &= N \\
m_2 &= N^2 + N(-1 + \lambda^{-1}) \\
m_4 &= 2N^3 + 5N^2(-1 + \lambda^{-1}) + N(3 - 5\lambda^{-1} + 3\lambda^{-2}) \\
&\vdots
\end{aligned}$$

Consequences.
Let

$$W(x, N, \lambda) = \int_{-\infty}^{\infty} \frac{\rho_{(1)}^{N}(y; \lambda)}{x - y} \, dy$$

Then

$$W(x, N, \lambda) = -\lambda^{-1} W(x, -\lambda N, \lambda^{-1})$$

A linear differential equation of degree $2\lambda + 1$ for $\lambda \in \mathbb{Z}^{+}$ can be derived for $Y := \rho_{(1)}^{N}(y; \lambda)$, e.g. for $\beta = 2$ (Haagerup and Thorbjornsen (2003))

$$\frac{1}{4N^2} Y''' + (1 - y^2) Y' + yY = 0.$$

Can check that W satisfies an inhomogeneous form of the same equation. Hence must have that

$$\rho_{(1)}^{N}(x, \lambda) = -\lambda^{-1} \rho_{(1)}^{-\lambda N}(x, \lambda^{-1})$$

e.g. For $\beta = 1$ the density satisfies a 5th order homogeneous differential equation which is the same as that satisfied for $\beta = 4$ but with N replaced by $-N/2$.

On going research

- Linear differential equations for one-point functions/ averages of characteristic polynomials. e.g. What is the behaviour of

$$\Big\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \Big\rangle_{\mathrm{CE}_{\beta,N}}$$

as $z \to 1$ for $\mu < 0$?

- Can the loop equation formalism be used to systematically generate the expansion

$$\left(\frac{2\pi}{N}\right)^2 \rho_{(2)}(0,\theta) = 1 - \frac{1}{\beta(2N\sin\theta/2)^2} + \frac{3(\beta-2)^2}{2\beta^3(2N\sin\theta/2)^4} - \cdots$$

- What is the q, t generalization of the family of Dixon-Anderson integrals used to derive the decimation identities?
- Duality formulas for random matrix ensembles with a source (Desrosiers).