TOPOLOGIES OF NODAL SETS OF RANDOM BAND LIMITED FUNCTIONS

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JOINT WITH IGOR WIGMAN.

- ONE REASON FOR STUDYING SUCH FUNCTIONS IS THAT ONE EXPECTS THE NODAL SETS OF HIGHLY EXCITED EIGEN-STATES OF THE QUANTIZATION OF A CHAOTIC HAMILTONIAN TO BEHAVE LIKE A RANDOM "MONOCHROMATIC WAVE".

- ANOTHER IS TO UNDERSTAND THE TOPOLOGY OF A RANDOM REAL VARIETY DEFINED BY EQUATIONS OF HIGH DEGREE.

SINGLE VARIABLE (KAC, RICE...)

$$f(x) = \sum_{j=1}^{t} a_j x^j \quad , \; a_j \in \mathbb{R}$$

$$V(f) := \{ x : f(x) = 0 \}$$

topology of $V(f)$ is the number of zeros $|V(f)|$.

$W_{1,t}$ the vector space of such f's.

WHAT IS RANDOM?

WE STICK TO CENTERED GAUSSIAN ENSEMBLES ON (FINITE) DIMENSIONAL VECTOR SPACES $\Longleftrightarrow$ GIVING AN INNER PRODUCT.

"NAIVE MODEL"

$W_{1,t}$, $$(f,g) = \sum_{j=0}^{t} a_j \cdot b_j$$

THIS IS THE SAME AS CHOOSING THE COEFFICIENTS a_j STANDARD GAUSSIANS AND INDEPENDENTLY. NOT NATURAL SINCE IT SINGLES OUT THE POINTS ± 1 IN $\mathbb{P}^1(\mathbb{R})$ AS WHERE MOST ZEROS WILL LOCATE.

REAL FUBINI STUDY ENSEMBLE

TURN $\mathbb{P}^1(\mathbb{R})$ INTO A HOMOGENEOUS SPACE SO THAT ALL POINTS ARE FAVORED EQUALLY

HOMOGENIZE:

$$f(x_0,x_1) = \sum_{j=0}^{t} a_j\, x_0^{j}\, x_1^{t-j}$$

$$(f,g) = \int_{\mathbb{R}^2} f(x)\,g(x)\,e^{-|x|^2/2}\,dx.$$

$$= * \int_{\substack{\mathbb{P}^1(\mathbb{R}) \\ \{|x|=1\}/\pm 1}} f(x)\,g(x)\,d\sigma(x).$$

Kac-Rice formula gives the expected number of zeros (and other statistics)

<u>Naive:</u> $\quad \mathrm{Exp}_f\,(|V(f)|) \sim \log t \quad$ as $\quad t \to \infty$

<u>Random Fubini Study:</u>

$$\mathrm{Exp}_f\,(|V(f)|) \sim \frac{t}{\sqrt{3}} \quad \text{as} \quad t \to \infty$$

(i.e. $\frac{1}{\sqrt{3}}$ of max).

⌈ O.n.b. for Random Fubini Study are trigonometric polynomials

$\sin(n\theta)$, $\cos(n\theta)$ ⌋

WORD ABOUT KAC-RICE :

$f(x) \in W_{1,t}$ for each x, $f(x)$ is a centered Gaussian so determined by covariance:

$$\mathrm{cov}(x,y) = \mathrm{Exp}_f\,(f(x)f(y)) := K_t(x,y)$$

IDEA TO COMPUTE $|V(f)|$:

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \left|\{x : |f(x)|<\varepsilon\}\right|$$

$$\to \sum_{a\in V(f)} \frac{1}{|f'(a)|}$$

So $\mathrm{Exp}_f\,(|V(f)|)$

$$= \mathrm{Exp}_f\left(\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{|f(y)|<\varepsilon} |f'(y)|\,dy \right)$$

SWITCH ORDERS ALLOWS TO COMPUTE IN TERM OF $K_t(x,y)$.

If $f(x) = \sum_{j=1}^{m} a_j \phi_j(x)$, $x\in[a,b]$
$v(x) = (\phi_1(x), \dots \phi_m(x))$, f i.i.d Gaussian
Then the expected density of zeros is

$$\frac{1}{\pi} \frac{\sqrt{\langle v,v\rangle\langle v',v'\rangle - \langle v,v'\rangle^2}}{\langle v,v\rangle}$$

(6)

SEVERAL VARIABLES:

$f(x_0, x_1, \dots, x_n)$ homogeneous degree t

$$V(f) = \{x : f(x) = 0\} \subset \mathbb{P}^n(\mathbb{R}). \qquad \boxed{W_{n,t}}$$

for typical f, $V(f)$ is a compact smooth $n-1$ dimensional manifold.

For any Gaussian Ensemble the Kac-Rice formula allows for the explicit computation of the expected values of *local* quantities.

eg: • $|V(f)|$ the induced $(n-1)$-volume of $V(f)$.

• Euler characteristic

• # of critical points

⋮

The question (global) of the topology of $V(f)$ is much more difficult.

Ensembles on $W_{n,t}$

(i) Naive $\quad (f,g) = \sum_{|J|=t} a_J b_J$

$$f(x) = \sum_{|J|=t} a_J x^J$$

$J = (j_0, \ldots, j_n)$

$|J| = \sum j_k$

(ii) COMPLEX FUBINI STUDY ("BOMBIERI NORM")

$$(f,g) = \int_{\mathbb{P}^n(\mathbb{C})} f(z)\overline{g(z)}\, d\sigma(z) = * \sum_{|J|=t} \binom{n}{J} a_J b_J$$

(iii) REAL FUBINI STUDY (RFS)

$$(f,g) = \int_{\mathbb{P}^n(\mathbb{R})} f(x) g(x)\, d\sigma(x)$$

BY THE WAY FOR $n=1$ CFS (ii)

$\mathrm{EXP}_f(|V(f)|) = \sqrt{t}$ (KAC-RICE)

RANDOM BAND LIMITED FUNCTIONS

M A COMPACT RIEMANNIAN (SMOOTH) n-DIMENSIONAL MANIFOLD.

ϕ_j, $j = 0, 1, 2, \ldots$

AN ORTHONORMAL BASIS OF EIGENFUNCTIONS OF THE LAPLACIAN Δ

$$\Delta \phi_j + t_j^2 \phi_j = 0 .$$

FIX α, $0 \leq \alpha \leq 1$

THE α-BAND LIMITED ENSEMBLE

$\mathcal{E}_{M,\alpha}(T)$ IS THE GAUSSIAN OF FUNCTIONS

$$f(x) = \sum_{\alpha T \leq t_j \leq T} c_j \phi_j(x)$$

c_j i.i.d STANDARD GAUSSIANS

IF $\alpha = 1$ WE MEAN THE SUM OVER $T - \eta(T) \leq t_j \leq T$ WHERE $\eta(T) = o(T)$ AND $\eta(T) \to \infty$ WITH T, "MONOCHROMATIC WAVE"

NB: (a) If (M, ds) is $(\mathbb{P}^n(\mathbb{R}), \sigma)$ THEN THE ϕ_j's ARE HOMOGENEOUS POLYNOMIALS (SPHERICAL HARMONICS), SO $\alpha = 0$ IS THE REAL FUBINI STUDY ENSEMBLE.

(b) $\alpha = 1$ IS THE MONOCHROMATIC WAVE ENSEMBLE.

$V(f)$ THE ZERO (NODAL) SET.

- LET $\mathcal{C}(f)$ DENOTE THE CONNECTED COMPONENTS OF $V(f)$

$$V(f) = \bigsqcup_{c \in \mathcal{C}(f)} c$$

- $$M \setminus V(f) = \bigsqcup_{\omega \in \Omega(f)} \omega \quad , \quad \omega \text{ the}$$

connected components or "NODAL DOMAINS".

- OUR INTEREST IS IN THE TOPOLOGIES OF THE c's IN $\mathcal{C}(f)$ AND ω's IN $\Omega(f)$.

NODAL PORTRAIT
SUM OF SPHERICAL HARMONICS (RANDOM)
OF DEGREE ≤ 80 ($\alpha=0$ RANDOM
FUBINI-STUDY ENSEMBLE)

A. BARNETT.

rpw_sphere_nodal_1000_deg80.jpg

NODAL PORTRAIT
RANDOM SPHERICAL HARMONIC
OF DEGREE 80 ($\alpha = 1$ MODEL)
A. BARNETT

NAZOROV AND SODIN HAVE INTRODUCED SOME POWERFUL (SOFT) TECHNIQUES TO STUDY THIS PROBLEM. THE FOLLOWING CAN BE DEDUCED FROM THEIR WORK

<u>THEOREM (NAZAROV-SODIN)</u>

THERE ARE POSITIVE CONSTANTS $\beta_{n,\alpha}$ DEPENDING ON n AND α ONLY SUCH THAT

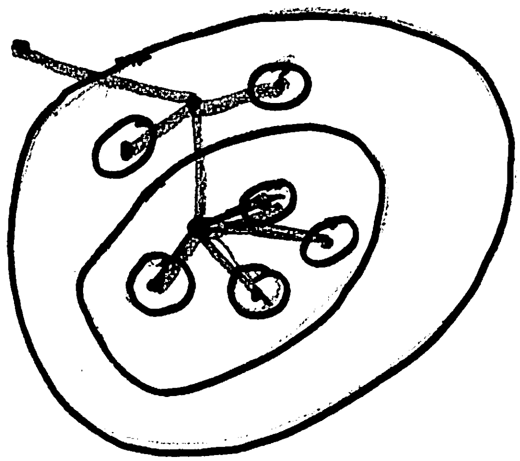
$$|\mathcal{C}(f)| \sim \beta_{n,\alpha} T^{n} \quad \text{AS } T \to \infty$$

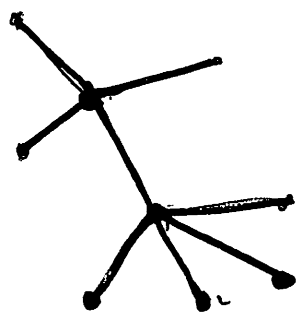
FOR MOST f's. IN $\mathcal{E}_{M,\alpha}(T)$ (I.E. WITH PROBABILITY TENDING TO 1 AS $T \to \infty$)

SO THERE MANY COMPONENTS AND ONE CAN ASK ABOUT THE DISTRIBUTION OF THE TOPOLOGIES OF $c \in \mathcal{C}(f)$ IN THE DISCRETE SPACE OF DIFFEOMORPHISM TYPES OF CONNECTED $(n-1)$ COMPACT MANIFOLDS, $H(n-1)$, AND OF $\omega \in \Omega(f)$ IN THE SPACE OF TYPES OF n MANIFOLDS WITH BOUNDARY $B(n)$.

NESTING GRAPH $X(f)$:

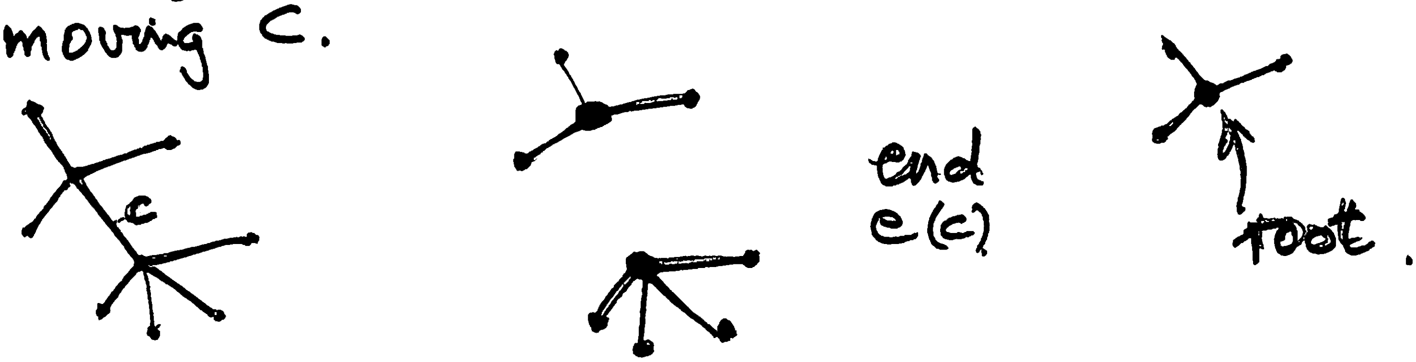
THE VERTICES OF $X(f)$ ARE THE NODAL COMPONENTS $\omega \in \Omega(f)$ AND THE EDGES ARE $c \in \mathcal{C}(f)$ YIELDING THE CONTIGUOUS ω's TO c AS ADJACENT.





$X(f)$

For $c \in \mathcal{C}(f)$ an edge of $X(f)$ define the end of $X(f)$ corresponding to c, $e(c)$ to be the smaller of the two (typically) rooted graphs remaining after removing c.



FOR f IN $\mathcal{E}_{M,\alpha}(T)$ (GENERIC) SET

$$\cdot\ \mu_{e(f)} := \frac{1}{|e(f)|} \sum_{c \in e(f)} \delta_{t(c)}$$

WHERE $t(c)$ IS THE TOPOLOGICAL TYPE OF c IN $\widetilde{H(n-1)}$, SO $\mu_{e(f)}$ IS THE DISTRIBUTION OF TOPOLOGIES OF c's.

$$\cdot\ \mu_{\Omega(f)} := \frac{1}{|\Lambda(f)|} \sum_{\omega \in \Lambda(f)} \delta_{t(\omega)}$$

WHERE $t(\omega)$ IS THE TOPOLOGICAL TYPE OF ω IN $\widetilde{B(n)}$, SO $\mu_{\Omega(f)}$ IS THE DISTRIBUTION OF ω's

$$\cdot\ \mu_{X(f)} := \frac{1}{|e(f)|} \sum_{c \in e(f)} \delta_{e(c)}$$

DISTRIBUTION OF ENDS OF $X(f)$ IN THE ~~???~~ SET OF FINITE CONNECTED ROOTED GRAPHS $\widetilde{\mathcal{C}}$.

theorem (WIGMAN - S 2014)*

THERE ARE PROBABILITY MEASURES

$\mu_{e,n,\alpha}$ ON $H(n-1)$ THE SET OF TOPOLOGICAL TYPES OF CONNECTED COMPACT $n-1$ MANIFOLDS THAT EMBED IN $\mathbb{R}^n$

$\mu_{\mathcal{R},n,\alpha}$ ON $B(n)$ THE SET OF TOPOLOGICAL TYPES WHICH EMBED IN $\mathbb{R}^n$

$\mu_{X,n,\alpha}$ ON $\mathcal{T}$ THE SET OF FINITE ROOTED TREES

SUCH THAT:

(i) FOR $\varepsilon > 0$, $\text{PROB}\{f \in \mathcal{E}_{M,\alpha}(T) : D(\mu_{e(f)}, \mu_{e,n,\alpha}) > \varepsilon\}$ TENDS TO 0 AS $T \to \infty$, WHERE D IS THE DISCREPANCY.

AND SIMILARLY FOR $\mu_{\mathcal{R}(f)}$ AND $\mu_{\mathcal{R},n,\alpha}$ AND $\mu_{X(f)}$ AN $\mu_{X,n,\alpha}$.

(ii)
$$\text{support } \mu_{e,n,\alpha} = H(n-1)$$
$$\text{support } \mu_{\mathcal{R},n,\alpha} = B(n).$$
$$\text{support } \mu_{X,n,\alpha} = \mathcal{T}.$$

THESE GIVE UNIVERSAL LAWS FOR THE DISTRIBUTION OF THE TOPOLOGIES OF THE COMPONENTS OF THE ZERO SETS OF RANDOM BAND LIMITED FUNCTIONS.

$\Longrightarrow$ UNIVERSAL LAWS FOR THE DISTRIBUTION OF THE VECTOR OF BETTI NUMBERS OF SUCH NODAL SETS.

UPPER AND LOWER BOUNDS FOR THE EXPECTED BETTI NUMBERS $\beta^j(V(f))$ HAVE BEEN GIVEN BY GAYET AND WELSCHINGER (2013) AND LERARIO AND LUNDBERG (2013).

NUMERICAL MONTE-CARLO $n=2$ A. BARNETT:

$n=2$, $H(1)$ IS THE CIRCLE ONE POINT.

$B(2)$: FINITELY CONNECTED PLANAR DOMAINS

$\omega \in B(2)$ $\quad t(\omega) =$ connectivity of $\omega \in \mathbb{N}$ IS THE ONLY INVARIANT

$B(2) \cong \mathbb{N}$, $\mu_{\Omega,2,\alpha}$ IS A MEASURE ON $\mathbb{N}$.

$\alpha = 0$ RANDOM RFS PLANE OVAL

$\mu_{\Omega,2,0}$ IS THE DISTRIBUTION OF NODAL DOMAINS

m	1	2	3	4	5	6	7	8	9
$\mu_{\Omega,2,0}$	.937	.027	.009	.003	.002	.002	.001	.001	.0005

THE NAZAROV-SODIN CONSTANT $\beta_{2,0}$ IS SUCH THAT THE RANDOM OVAL IS ABOUT 4% HARNACK (I.E. HAS 4% OF THE MAXIMAL NO' OF OVALS).

(M. NASTASESCU)

$\alpha = 1$, THE RANDOM MONOCHROMATIC NODAL DOMAIN

m	1	2	3	4	5	6	7	8	9
$\mu_{\Omega,2,1}$	.906	.055	.010	.006	.003	.002	.001	.0008	.0004

SOME COMMENTS ABOUT THE PROOFS:

STEP 1: SEMI LOCALITY

COVARIANCE

$$\mathrm{Exp}_{f \in \mathcal{E}_{M,\alpha}(T)} [f(x) f(y)]$$

$$= \sum_{\alpha T \le t_j \le T} \phi_j(x) \phi_j(y) := K_\alpha(T; x, y)$$

SPECTRAL PROJECTOR.

AS $T \to \infty$ IS STUDIED BY PARAMETRIX TO WAVE EQN, FOURIER INTEGRAL OPERATORS (LAX, ...)

SAY $\mathrm{VOL}(M) = 1$, THEN.

$$\frac{K_\alpha(T; x, y)}{\dim \mathcal{E}_{M,\alpha}(T)} = \begin{cases} B_{n,\alpha}(T d(x,y)) + O(T^{-1}) \\ \quad \text{IF } T d(x,y) \ll 1 \\ O(T^{-1}) \quad \text{IF NOT.} \end{cases}$$

WHERE

$$B_{n,\alpha}(\omega) = B_{n,\alpha}(|\omega|) = \frac{1}{|\Omega_{n,\alpha}|} \int_{\Omega_\alpha} e(\langle \omega, \xi \rangle)\, d\xi$$

$\Omega_\alpha = \{\omega : \alpha \le |\omega| \le 1\}$, $d(x,y)$ = distance x to y.

FOLLOWING THE METHODS OF NAZAROV AND SODIN WE SHOW THAT OUR QUANTITIES ARE SEMI LOCAL IN NBH'S OF SIZE $\approx 1/T$ IN M (AND OTHERWISE INDEPENDENT).

AFTER SCALING $\Longrightarrow$

GAUSSIAN TRANSLATION INVARIANT, ISOTROPIC, INFINITE DIMENSIONAL FIELD $H_{n,\alpha}$ ON $\mathbb{R}^n$ (REPLACING $\mathcal{E}$ LOCALLY).

Let ψ_j BE AN ORTHONORMAL BASIS OF $L^2(\Omega_\alpha, d\nu)$, SET

$$H_{n,\alpha}: \quad f(x) = \sum_{j=1}^{\infty} c_j \widehat{\psi}_j(x) \quad \text{on } \mathbb{R}^n$$

c_j i.i.d. standard Gaussians.

<u>NB:</u> THE TYPICAL f IS ANALYTIC IN x (THANKS TO DECAY IN $\widehat{\psi}_j(x)$ FOR x IN A COMPACT).

THE EXISTENCE OF OUR LIMITING MEASURES AS WELL AS CONVERGENCE IN MEASURE FOLLOWS FROM SOFT ERGODIC THEORY (OF THE ACTION OF THE TRANSLATION GROUP $\mathbb{R}^n$) IN A SIMILAR FASHION TO THE NAZAROV-SODIN ASYMPTOTICS FOR CONNECTED COMPONENTS.

THE PROPERTIES OF μ'S, NAMELY THAT THE ARE PROBABILITY MEASURES & "NO ESCAPE OF TOPOLOGY" AND THAT THEY CHARGE EVERY ATOM, REQUIRE ANALYTIC, GEOMETRIC AND TOPOLOGICAL INPUT.

<u>EG</u>: SUPPORT OF $\mu_{X,2,1}$ IS ALL $\mathcal{T}$, REDUCES TO SHOWING THAT THERE IS AN $f \in H_{2,1}$ WITH NESTING ANY (FINITE) ROOTED TREE. $\Longleftrightarrow$ PRODUCING

$$f(x) = \sum_{j=1}^{L} a_j \, e(\langle x, \xi_j \rangle) \qquad (*)$$

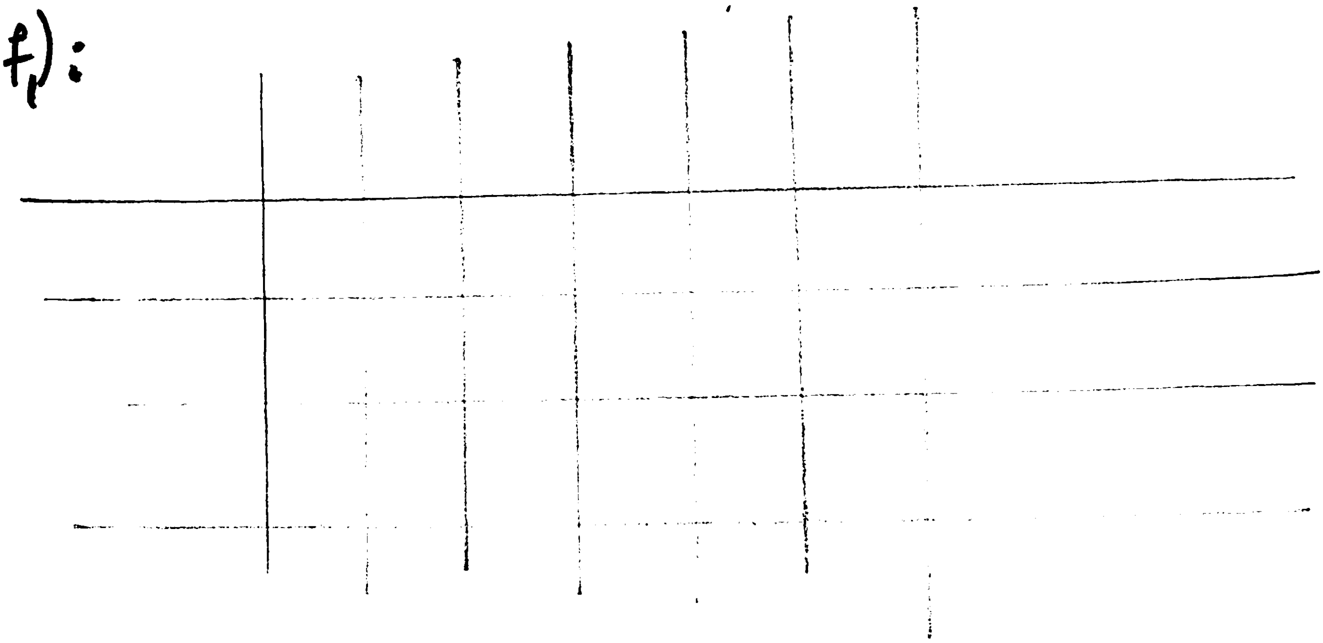
TRIG POLY WITH $|\xi_j| = 1$.

AND WITH $e(X(f))$ GIVEN.

START WITH

$$f_1(x_1, x_2) = \sin \pi x_1 \, \sin \pi x_2$$

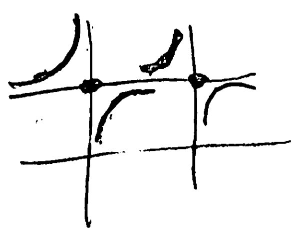
$V(f_1)$:



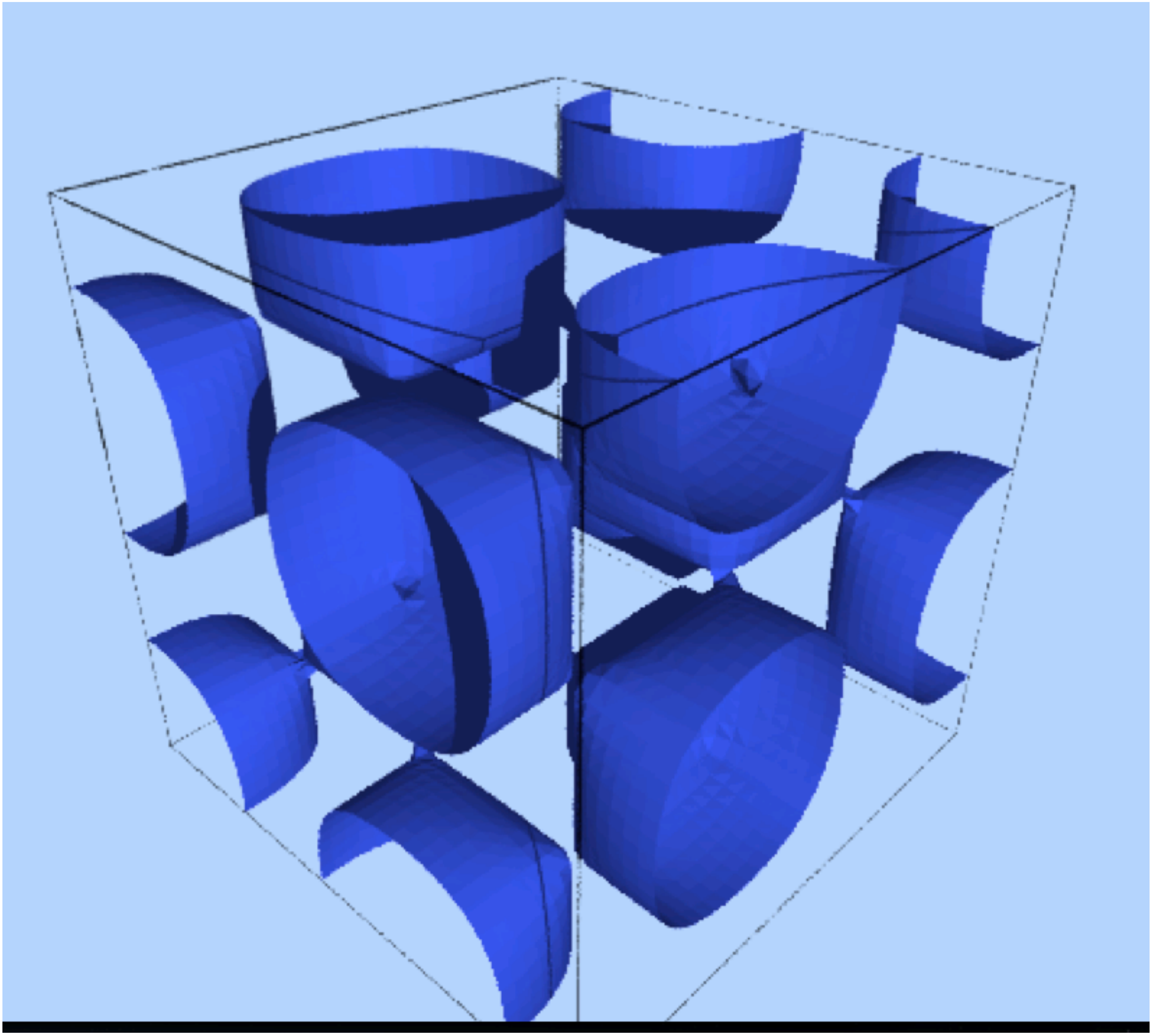
<u>LEMMA:</u> GIVEN $K \subset \mathbb{Z}^2$ FINITE AND $\varepsilon_k = \pm 1$ FOR $k \in K$, THERE IS A $\psi(x)$ OF THE FORM (*) SUCH THAT $\psi(k) = \varepsilon_k$, $k \in K$.

⌈TWO PROOFS: ONE USES $|\xi| = 1$ IE A CIRCLE FOR RESTRICTION, ONE MORE ALGEBRAIC USING AX'S SCHANUEL THEOREM FOR FUNCTION FIELDS ⌋

NOW SET $$f(x) = f_1(x) + \varepsilon \psi(x)$$



SHOW THIS IS RICH ENOUGH TO GET ANY $\tau \in \mathcal{T}$!



THE ANALOGUE OF f_1 IN THREE VARIABLES (IN ORDER TO SHOW THAT $\mu_{\mathcal{C},3,1}$ CHARGES EVERY $h \in H(2)$, IE EVERY SURFACE OF GENUS $g \geq 0$)

IS

$$f(x_1,x_2,x_3) = \sin(\pi x_1)\sin(\pi x_2) + \sin(\pi x_2)\sin(\pi x_3) + \sin(\pi x_1)\sin(\pi x_3).$$