Long time dynamics of random data nonlinear wave and dispersive equations.

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Analysis and Beyond: Celebrating Jean Bourgain's work and its impact.

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Plan

1. **Bourgain**'s approach (94') to invariant Gibbs measures and almost sure global well-posedness for Hamiltonian dispersive PDE
2. Some further ideas in recent probabilistic well posedness results for the nonlinear Schrödinger equation (NLS) on $\mathbb{T}^d$ ($d = 3, 2, 1$)
   - **joint with G. Staffilani.**
3. Existence and uniqueness of non-equilibrium invariant measures associated to the NLS.
   - **joint with Z. Hani, J. Mattingly, L. Rey Bellet and G. Staffilani.**

The Nonlinear Schrödinger Equation

(NLS) $$\begin{cases} i\,u_t + \Delta u = \pm |u|^{p-1} u, \\ u(0,x) = u_0(x) \in H^s, \qquad x \in \mathcal{M}^d \end{cases}$$

where $u : \mathbb{R} \times \mathcal{M}^d \to \mathbb{C}$.

$\mathcal{M}^d = \mathbb{R}^d$, $\mathbb{T}^d$ or any compact manifold.

Two conserved quantities (constant in time):

Mass: $M(u) := \int |u(t,x)|^2 \, dx$

Hamiltonian: $H(u) := \frac{1}{2} \int |\nabla u(t,x)|^2 \, dx \pm \frac{1}{p+1} \int |u(t,x)|^{p+1} dx$

Time reversibility: $u_0(x) \to \overline{u_0(x)}, \quad u(t,x) \to \overline{u(-t,x)}$.

Scaling on $\mathbb{R}^d$

If u solves NLS, then

$$u_\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right) \qquad \lambda > 0$$

solves the same equation with initial datum $u_{0,\lambda} = \lambda^{-\frac{2}{p-1}} u_0(\frac{x}{\lambda})$.

$$\|u_{0,\lambda}\|_{\dot{H}^s} \sim \lambda^{s_c - s} \|u_0\|_{\dot{H}^s}$$

Scale invariant problem:

$$s_c = \frac{d}{2} - \frac{2}{p-1} \qquad (\textit{critical exponent.})$$

So we can classify the difficulty of the NLS problem above in terms of s_c:

- If $s > s_c$ the space H^s is subcritical. Scaling help us.
- If $s = s_c$ the space $\dot{H}^s$ is critical. Scale invariant.
- If $s < s_c$ the space H^s is supercritical.Scaling is against us.

- Characteristic feature of dispersive equations is the fact that different frequencies travel at different velocities.
- On $\mathbb{R}^d$ **dispersion** $\rightarrow$ decay of the solution.
- Strichartz estimates $\longrightarrow$ local well-posedness in subcritical and small data global well posedness in certain critical regimes (mass, energy).

The question of long time well-posedness or blow up is far harder, more complex!

In last 20 years, lots of progress to prove **large data** global well-posedness and scattering for the defocusing NLS on $\mathbb{R}^d$:

In certain subcritical regimes:

- High-low method (Bourgain)
- I-Method of almost conservation laws (Colliander-Keel-Staffilani-Takaoka-Tao [CKSTT])

At the level of the energy (H^1) or mass (L^2) norms when they are **critical** spaces:

- Grillakis, Bourgain, CKSTT, Kenig-Merle, Tao, Killip-Visan, Zhang, Dodson, ...
  - concentrated compactness/rigidity method + new interaction Morawetz estimates (CKSTT, Kenig-Merle (focusing also)).

Yet there remain some important open questions: eg. in certain critical regimes, in supercritical case:

- Defocusing case: does blow up occur? Unknown despite strong ill-posedness results (*"norm explosion"*) by Christ, Colliander and Tao.

Strichartz estimates and well posedness on $\mathbb{T}^d$:

- On compact domains, wave packets have no escape from interacting indefinitely with each other.
- Dispersion does not necessarily translate to decay.
- We have limited dispersion. In fact, only a limited number of Strichartz estimates survive: **Bourgain (93'); Bourgain-Demeter (14')**
- Bourgain showed that dispersion is indeed weaker in the periodic setting:
  - For ex. the L^6_{xt} in 1D or L^4_{xt} in 2D necessarily have an ϵ-derivative-loss.
- This ϵ-loss prevents us to close the estimates in $L^2(\mathbb{T}^d)$ for the quintic NLS in 1D and the cubic in 2D. Need $s > 0$.

Well-posedness for NLS on $\mathbb{T}^d$. (**Bourgain 93'**)

Sample results. In both cases $s_c = 0$, results are in subcritical regime:

Theorem ($d = 1, p = 5$)

The quintic NLS on $\mathbb{T}$ (either focusing or defocusing) is LWP in $H^s(\mathbb{T})$, $\mathbf{s > 0}$

Theorem ($d = 2, p = 3$)

The defocusing cubic NLS on $\mathbb{T}^2$ is LWP in $H^s(\mathbb{T}^2)$, $\mathbf{s > 0}$.

One gets GWP for $s \geq 1$.

Bourgain (93') introduced two fundamental tools:

- Strichartz estimates on tori.
- Fourier restriction $X^{s,b}$ spaces/method in the context of NLS to perform the iteration/fixed point.

Unlike the $\mathbb{R}^d$ case the only **deterministic** results for NLS on $\mathbb{T}^d$ at **critical** regularity are:

- Global well-posedness in H^1:
  - Quintic defocusing NLS on $\mathbb{T}^3$: Ionescu-Pausader 12' (large data); and previously Herr-Tzvetkov-Tataru 11' (small data).
  - Cubic defocusing NLS on $\mathbb{T}^4$: H. Yue 16' (large data).

…

Not even a local-wellposedness at the level of L^2 for cubic NLS on $\mathbb{T}^2$.

Deterministic $\longrightarrow$ Nondeterministic

Invariant Measures and almost sure GWP on $\mathbb{T}^d$.

Starting point Bourgain (94'-96'): longtime dynamics of periodic NLS (d=1,2) in the almost sure sense:

He proved that the equation is globally well-posed for a **set of data of full** Gibbs measure and that the (Gibbs) measure is invariant under the flow.

Glimm-Jaffe (ϕ^4); Lebowitz-Rose-Speer; Zhidkov (infinite dim. Hamiltonian systems)

Why does it work?:

The invariance of the Gibbs measure, just like the usual conserved quantities, controls the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely.

- The virtue of the Gibbs measure is that it captures generic behavior of the flow.

Challenges/limitations of this approach to a.s. GWP for dispersive PDE:

- The actual existence of an invariant Gibbs (or weighted Wiener measures) under the flow and the almost sure global well-posedness.
- Measures are easier to construct on bounded domains.
- Not available in higher dimensions.
  - For $d = 2, 3$ and defocusing **cubic** NLS renormalization needed. (Glimm-Jaffe (70's))
  - No Gibbs measure for defocusing NLS if $d = 3$ and $p = 5$.
  - No Gibbs measure for focusing NLS on 2D (Brydges-Slade)
  - No Gibbs measure for $d \geq 4$. No Gibbs measure for focusing NLS on 2D (Brydges-Slade)
- If the PDE is not Hamiltonian: not always clear....

We have Gibbs measures for defocusing NLS

- on $\mathbb{T}^2$ and $p = 5$
- on $\mathbb{T}^3$ and $p = 3$.

But the support lives in very rough spaces where there is not a well defined flow/available estimates.

Proving their invariance and a.s. GWP is a challenge!

Let us review Bourgain's (94) work on the invariance of Gibbs measure and a.s GWP for the **focusing** quintic NLS on $\mathbb{T}$.

$$iu_t + u_{xx} + |u|^4 u = 0$$

with Hamiltonian

$$H(u) := \frac{1}{2}\int |u_x|^2 dx - \frac{1}{6}\int |u|^6 dx = \frac{1}{2}\int |u_x|^2 dx - \mathcal{N}(u).$$

L^2 is conserved $\rightarrow$ modify:

$$\begin{aligned} H(u) &:= \frac{1}{2}\int (|u_x|^2 + |u|^2) dx - \frac{1}{6}\int |u|^6 dx \\ &= \frac{1}{2}\int (|u_x|^2 + |u|^2) dx - \mathcal{N}(u). \end{aligned}$$

- Think of $u(t)$ as the infinite dimension vector given by its Fourier coefficients $(a_n(t) + ib_n(t))_{n\in\mathbb{Z}}$.
- The equation then becomes an infinite dimensional Hamiltonian system for $(a_n(t), b_n(t))_{n\in\mathbb{Z}}$.

• Lebowitz, Rose and Speer (88') considered the Gibbs measure *formally* given by

$$`d\mu = Z^{-1} \exp\left(-\beta H(u)\right) \prod_{x \in \mathbb{T}} du(x)'$$

and showed that μ is a well-defined probability measure on $H^s(\mathbb{T}), s < \frac{1}{2}$ but not for $s = \frac{1}{2}$.

- The definition of μ above - although suggestive – is a purely formal expression.
  - It is impossible to define the Lebesgue measure as a countably additive measure on an infinite-dimensional space.
  - In fact, as written, all factors are a.s. infinite

How is μ defined?

Idea.

One uses a Gaussian measure as reference measure.

Then the weighted measure μ is constructed in **two steps**:

- First one constructs a Gaussian measure ρ on H^s as the weak limit of **finite-dimensional** Gaussian measures on $\mathbb{R}^{4N+2}$ given by

$$d\rho_N = Z_N^{-1} \exp\Big(-\frac{\beta}{2} \sum_{|n| \leq N} (1 + |n|^2)|\widehat{\phi}_n|^2 \Big) \prod_{|n| \leq N} da_n db_n$$

  where $\widehat{\phi}_n = a_n + ib_n$.
- View ρ_N as an induced probability measure on $\mathbb{C}^{2N+1} \equiv \mathbb{R}^{4N+2}$ under the map

$$\omega \longmapsto \left\{ \frac{g_n(\omega)}{\sqrt{1 + |n|^2}} \right\}_{|n| \leq N}.$$

where $\{g_n(\omega)\}_{|n| \leq N}$ are i.i.d complex Gaussian random variables (centered) on a probability space $(\Omega, \mathcal{F}, P)$.

Then ρ makes sense as a **countably additive** probability measure:

- On $H^s(\mathbb{T})$ for any $s < 1/2$, but **not** for $s \geq 1/2$.
- On $H^s(\mathbb{T}^2)$ for any $s < 0$, but **not** for $s \geq 0$.
- On $H^s(\mathbb{T}^3)$ for any $s < -\frac{1}{2}$, but **not** for $s \geq -\frac{1}{2}$.

  ...

ρ yields for ϕ the distribution of a random (Fourier) series

$$\phi = \phi^\omega = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\sqrt{1 + |n|^2}} e^{in \cdot x}.$$

which defines almost surely a function ($d = 1$) or a distribution.

One also has tail estimates:

$$\rho\big(\{\|\phi\|_{H^s} > K\}\big) < e^{-cK^2}.$$

- Once ρ has been constructed one constructs μ as a measure which is **absolutely continuous** with respect to ρ and whose Radon-Nikodym derivative is

$$\frac{d\mu}{d\rho} = \tilde{Z}^{-1} \chi_{\{\|\phi\|_L^2 \leq B\}} e^{\frac{\beta}{2}\mathcal{N}(\phi)} \in L^1(d\rho)$$

$\mathcal{N}(\phi)$ is the potential energy.

- For this measure to be normalizable one needs an L^2-cutoff $\chi_{\|\phi\|_{L^2} \leq B}$, $B > 0$ suff. small. **[LRS]**

Theorem (Bourgain 94')

Consider the Cauchy problem

$$\begin{cases} iu_t + u_{xx} + |u|^4 u = 0 \\ u(0,x) = \phi^\omega(x), \quad \textit{where } x \in \mathbb{T}. \end{cases}$$

where

$$\phi^\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{inx}$$

There exists $\Omega \subset H^{1/2-}$, such that $\mu(\Omega) = 1$ and for any $\phi^\omega \in \Omega$ the IVP is globally well-posed. Moreover μ is ***invariant****.*

What is important about Bourgain's result?

- Deterministically can only prove LWP in $H^s, s > 0$ and GWP in H^1 from conservation of Hamiltonian (small L^2 in focusing).
- Here, Bourgain uses the invariance of the measure to extend *almost surely* local solutions to global when $s = 1/2-$ and one has **no** conservation laws (small L^2 in focusing)

How does Bourgain prove a.s GWP of the flow?

Let P_N be the Fourier/Dirichlet projection onto the spatial frequencies $\leq N$.

Consider the finite dimensional approximation to NLS: :

$$\text{(FDA)} \qquad \begin{cases} iu_t^N + \Delta u^N + P_N(|u^N|^4 u^N) = 0 \\ u^N(0,x) = P_N \phi^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{inx}, \end{cases} \qquad x \in \mathbb{T}.$$

- By the **deterministic** local theory (1D) we have that (FDA) is LWP in $H^{1/2-}$ with time of existence **independent of** N.

Crucial Fact:

- FDA is still a **Hamiltonian system:** $iu_t^N = \frac{dH(u^N)}{d\bar{u}^N}$ with Hamiltonian

$$H(u^N)(t) = \frac{1}{2}\int |u_x^N|^2 dx - \frac{1}{6}\int |u^N|^6 dx$$

which is still **conserved** under the FDA flow.

- By Liouville's theorem and the conservation of $H(u^N)$ we have that the finite dimensional:

$$d\mu_N = \tilde{Z}_N^{-1} e^{-H(u^N)} \prod_{|n| \leq N} da_n db_n$$

is an invariant measure under the flow of (FDA); $\widehat{u^N}(n) = a_n + ib_n$.

- **It is the invariance of** μ_N what allow us to extend local in time solutions to global ones. Provided the data is drawn from a **good set of initial data**, the solutions u^N to the FDA extend 'globally' in time.

- By an **Approximation Lemma:** $\|u - u^N\|_{H^{s-}} \to 0$, one uses u^N to walk u to its side pass the local time τ and up to time T.
  - Not entirely trivial: cannot compare u and u^N directly on $[0, T]$: after the first step on $[0, \tau]$, u and u^N may in principle start becoming apart: we have no a priori bound on u on $[0, T]$.

- One still needs to prove that μ_N converges weakly to μ and that μ is an invariant measure on $H^{1/2-}(\mathbb{T})$.

Cubic NLS on $\mathbb{T}^2$: Nondeterministic Approach to LWP

Theorem (Bourgain 96')

$$\begin{cases} iu_t + \Delta u = |u|^2 u \\ \end{cases}$$

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Theorem (Bourgain 96')

$$\begin{cases} iu_t + \Delta u = |u|^2 u - 2(\int |u|^2 dx)\, u \\ u(x,0) = \phi^\omega(x) := \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x} \quad x \in \mathbb{T}^2, \end{cases}$$

is a.s. globally well-posed in $H^{-\varepsilon}$ and its associated Gibbs measure is invariant

- In 2D, $e^{-\int_{\mathbb{T}^2} |\phi|^4\, dx}$ is unbounded a.s. $\longrightarrow$ Wick order $|\phi|^4$ in the Hamiltonian.
- This renormalization crucially removes certain resonant terms.

This is a **supercritical** well posedness result.

Additional dificulty

Bourgain(93') had proved LWP for $s > 0$ and GWP in $H^1(\mathbb{T}^2)$ for cubic NLS.

Unlike the quintic in 1D, here there is **no deterministic** LWP available for data below $L^2(\mathbb{T}^2)$.

Bourgain's Point:

1. Enough to prove a probabilistic local well posedness; i.e. for 'typical elements' in the support of the measure. That is, for random data ϕ^ω in $H^{-\varepsilon}(\mathbb{T}^2)$ as above.

2. Once this is place, proceed as before to prove the invariance Gibbs measure and a.s. GWP.

On Randomized Data

Randomization does **not** improve regularity in terms of derivatives.

The improvement is with respect to L^p spaces *almost surely* which in turn, imply better estimates than the deterministic ones.

Classical results of **Rademacher, Kolmogorov**, **Paley** and **Zygmund** show that random series enjoy better L^p bounds than deterministic ones.

Bourgain's strategy:

- Look for solutions of the form $u = S(t)\phi^\omega + w$
- Use the fact that $S(t)\phi^\omega$ has a.s. better L^p estimates than ϕ to show that

$$w = u - S(t)\phi^\omega$$

  solves a **difference equation** that lives in a smoother space.

$$iw_t + \Delta w = \mathcal{N}(S(t)\phi^\omega + w)$$

- Obtain for w a *deterministic* local well-posedness in the smoother space.

As a consequence Bourgain showed that a.s. in ω the nonlinear part of the solution

$$w := u - S(t)\phi^\omega$$

is smoother than the linear part.

Important

The difference equation that w solves is not back to merely being at a 'smoother' level but rather it is a hybrid equation with nonlinearity $=$ supercritical (but random) $+$ deterministic (smoother).

$$iw_t + \Delta w = \mathcal{N}(S(t)\phi^\omega + w)$$

Large Deviation-type Estimates

One uses the following, where k would represent the number of random terms in a multilinear estimate at hand:

Proposition (Large Deviation-type)

Let $d \geq 1$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Let $\{(g_n)\}_{1 \leq n \leq d}$ as above. For $k \geq 1$ denote by $A(k,d) := \{(n_1, \ldots, n_k) \in \{1, \ldots, d\}^k,\ n_1 \leq \cdots \leq n_k\}$ and

$$F_k(\omega) = \sum_{A(k,d)} c(n_1, \ldots, n_k) g_{n_1}(\omega) \ldots g_{n_k}(\omega).$$

Then for $p \geq 2$

$$\|F_k\|_{L^p(\Omega)} \lesssim \sqrt{k+1}(p-1)^{\frac{k}{2}} \|F_k\|_{L^2(\Omega)}.$$

As a consequence from Chebyshev's inequality for every $\lambda > 0$,

$$\mathbb{P}(\{\omega \,:\, |F_k(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\,\lambda^{\frac{2}{k}}}{\|F_k(\omega)\|_{L^2(\Omega)}^{\frac{2}{k}}}\right).$$

Given $\delta > 0$, the large deviation result above with

$$\lambda = \delta^{-\frac{k}{2}}\|F_k(\omega)\|_{L^2(\Omega)}$$

says that in a set Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta}}$ we can replace:

$$|F_k(\omega)|^2 \quad \longrightarrow \|F_k(\omega)\|_{L^2(\Omega)}^2.$$

An Example from Bourgain's Work:

Take

$$\phi^{\omega}(x) = \sum_{n \neq 0} \frac{g_n(\omega)}{|n|} e^{ix \cdot n}$$

and look at the cubic Wick ordered nonlinearity, involving its free evolution $S(t)\phi^{\omega}(x)$, and that Bourgain had to estimate in L^2:

$$\|F_3(\omega)\|_{\ell^2_n \ell^2_m},$$

where

$$F_3(\omega) = \sum_{S_{n,m}} \frac{1}{|n_1|} \frac{1}{|n_2|} \frac{1}{|n_3|} g_{n_1}(\omega) \overline{g_{n_2}}(\omega) g_{n_3}(\omega),$$

$$S_{n,m} = \{(n_1, n_2, n_3) \,/\, n_1 - n_2 + n_3 = n;\; n_1, n_3 \neq n_2;\; m = |n_1|^2 - |n_2|^2 + |n_3|^2\}$$

Wick ordering $\longrightarrow \quad n_1, n_3 \neq n_2$

Naively we could just use C-S to estimate $\|F_3(\omega)\|^2_{\ell^2_n \ell^2_m}$ and obtain

$$\sum_{n,m}\left|\sum_{S_{n,m}}\frac{1}{|n_1|}\frac{1}{|n_2|}\frac{1}{|n_3|}g_{n_1}(\omega)\overline{g_{n_2}}(\omega)g_{n_3}(\omega)\right|^2 \lesssim \sum_{n,m}|S_{n,m}|\sum_{S_{n,m}}\frac{1}{|n_1|^2}\frac{1}{|n_2|^2}\frac{1}{|n_3|^2}$$

where $|S_{n,m}|$ = cardinality of $S_{n,m}$ which translates into a **loss** of derivatives.

Instead, by the Large Deviation Estimate, outside a small set of ω's we have:

$$\|F_3(\omega)\|^2_{\ell^2_n\ell^2_m} = \sum_{n,m}|F_3(\omega)|^2 \lesssim \delta^{-\mu}\sum_{n,m}\|F_3(\omega)\|^2_{L^2(\Omega)}$$

$$= \delta^{-\mu}\sum_{n,m}\sum_{S_{n,m}}\sum_{S'_{n,m}}\int_\Omega \frac{g_{n_1}}{|n_1|}\frac{\overline{g_{n_2}}}{|n_2|}\frac{g_{n_3}}{|n_3|}\frac{\overline{g_{n'_1}}}{|n'_1|}\frac{g_{n'_2}}{|n'_2|}\frac{\overline{g_{n'_3}}}{|n'_3|}\,d\omega$$

and by **independence** of the random variables the RHS contracts to

$$\|F_3(\omega)\|^2_{\ell^2_n\ell^2_m} \lesssim \delta^{-\mu}\sum_{n,m}\sum_{S_{n,m}}\frac{1}{|n_1|^2}\frac{1}{|n_2|^2}\frac{1}{|n_3|^2}$$

and this has good enough decay to absorb some regularity and close.

To summarize so far:

- When deterministic statements about existence, uniqueness and stability of solutions to certain evolution equations are **not** feasible/available:

  $\rightarrow$ turn to a more probabilistic point of view and investigate these problems from a nondeterministic viewpoint; e.g. for random data.

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- After Bourgain's work (mid 90's), this approach of almost sure well-posedness was re-taken again in 07-08'. Lots of activity.

- Schrödinger Equations: Bourgain, Tzvetkov, Thomann-Tzvetkov, A.N.-Oh-Rey-Bellet-Staffilani, A.N.-Rey-Bellet- Sheffield-Staffilani, Colliander-Oh, Burq-Thomann-Tzevtkov, Y. Deng, Burq-Lebeau, Bourgain-Bulut, A.N.- Staffilani, Poiret-Robert-Thomann, H. Yue, Bényi- Oh- Pocovnicu (conditional), . . .

- KdV Equations: Bourgain, Oh, Richards.

- NLW/NLKG Equations: Burq-Tzvetkov, de Suzzoni, Bourgain-Bulut, Luehrmann-Mendelson, S. Xu, Pocovnicu, Oh-Pocovincu, Mendelson.

- Benjamin-Ono Equations: Y. Deng, Tzvetkov-Visciglia. and Y. Deng-Tzvetkov-Visciglia.

- Navier-Stokes Equations: A.N.-Pavlovic-Staffilani: infinite 'energy' global (weak) sols in $\mathbb{T}^2, \mathbb{T}^3$, global energy bounds, uniqueness in $\mathbb{T}^2$. Also work by Deng-Cui and Zhang-Fang

. . . .

Further Probabilistic results

- A.s. local well-posedness and large data long time existence with positive probability to the 3D quintic NLS for **supercritical** data in $H^{1-\epsilon}(\mathbb{T}^3)$. (A.N.- Staffilani)
  - Local result is the analogue of Bourgain's cubic NLS result on $\mathbb{T}^2$.
  - **Differences** Local: quintic nonlinearity, integer lattice counting in 3D, removal of resonances (**no** Wick ordering helps).
  - **Differences** Global: No measure, no conserved quantity for the difference equation

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  - **Differences** Local: quintic nonlinearity, integer lattice counting in 3D, removal of resonances (**no** Wick ordering helps).
  - **Differences** Global: No measure, no conserved quantity for the difference equation
- Probabilistic propagation of regularity result (A.N.- Staffilani)
  - A.s global well-posedness for 2D, cubic defocusing NLS in $H^s(\mathbb{T}^2)$, $s > 0$.
  - A.s global well-posedness for 1D, quintic (small mass) focusing NLS in $H^s(\mathbb{T})$, $s > 1/2$.

These results close an important gap between the a.s GWP of Bourgain and the known deterministic GWP

Probabilistic propagation of regularity

What Was Known in 2D:

- Deterministic methods: LWP for $s > 0$ (Bourgain) and GWP $s > 2/3$ (Bourgain; De Silva-Pavlovic-Staffilani-Tzirakis)
- Data randomization and invariant Gibbs measure μ: a.s. GWP in $H^{-\epsilon}$, (Bourgain)

Remark: Our theorem is not trivial: any $\Sigma \subset H^s, s > 0$, is such that for the Gibbs measure μ one has $\mu(\Sigma) = 0$.

- Our key idea instead is to suitably decompose the data into a term that is close to the support of the invariant measure in the rougher topology, and a smoother remainder term to which deterministic arguments can be applied.
- Then, a nondeterministic perturbation argument is used to conclude.
- Argument is general given an a.s. GWP proved using an invariant or almost invariant measure.

Transfer of Energy and Out of Equilibrium Dynamics

- On $\mathbb{R}^d$ we have by now several results that prove that dispersion sets in and after a time long enough solutions settle into a purely linear behavior (scattering/asymptotic stability).
  - For linear solutions, the energy (kinetic or mass) while remaining conserved - does **not** move its concentration zones from low to high frequencies or viceversa. That is there is no forward / backward cascade.
- As a consequence of scattering, nonlinear solutions in $\mathbb{R}^d$ also will avoid these cascades.
- On compact domains, asymptotic stability results around equilibrium solutions (e.g. zero solution) **are lost:**

  $\longrightarrow$ Out of equilibrium dynamics are expected.

Energy cascade

- How to analytically describe this expected out-of-equilibrium behavior ?
- This sort of problem also goes under the name *weak turbulence*.
- **Bourgain**'s approach via growth of higher Sobolev norms $\rightarrow$ Migration of energy to high frequencies
  - $\|u\|_{H^s}$ weighs the higher frequencies more as s becomes larger.
  - Its growth gives us a quantitative estimate for how much of the support of $|\widehat{u}|^2$ has transferred from the low to the high frequencies while maintaining constant mass and energy (forward cascade.).
- Important progress by Bourgain (95'-97') and by Kuksin 97'. For ex.:
  - Bourgain constructed a perturbed 1D NLW with periodic bdry conditions exhibiting an energy transition from low to high Fourier modes and a power-like growth of higher derivatives in time

- **Bourgain Question (00'):** Does there exist global solutions to the cubic NLS whose $H^s(\mathbb{T}^d)$ norm (some $s >> 1$) grows indefinitely in time:

$$\limsup_{t\to\infty} \|u\|_{H^s} = +\infty?$$

(*unbounded orbits/infinite cascade conjecture*).

- Fundamental progress by Colliander-Keel-Staffilani-Takaoka-Tao, Hani, Gerard-Grellier and Guardia-Kaloshin (10'-12').
  - CKSTT constructed large but finite growth of the Sobolev norms:
  - For any $0 < \delta < 1$, and any $K > 1$ there exists a solution to the cubic NLS on $\mathbb{T}^2$ and a time T such that

$$\|u(0)\|_{H^s(\mathbb{T}^2)} \leq \delta, \qquad \text{and} \qquad \|u(T)\|_{H^s(\mathbb{T}^2)} \geq K.$$

- More recently, Hani-Pausader-Tzvetkov-Visciglia gave a positive answer to Bourgain's Q. for the cubic NLS on product domains $\mathbb{R} \times \mathbb{T}^d$ ($d \geq 2$).
  - Moreover, gave a rate.: there exists a sequence of times $t_k \to \infty$ s.t.

$$\|u(t_k)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)} \geq C \exp\left(c \log\log t_k\right)^{\frac{1}{2}}.$$

Non-Equilibrium Invariant Measures for resonant NLS

Z. Hani, J. Mattingly, A.N., L.Rey Bellet and G. Staffilani.

- An intermediate problem between:

  (1) The existence of (equilibrium) invariant Gibbs measures and

  (2) Bourgain's unbounded orbits conjecture/understanding of out-of-equilibrium dynamics for NLS

  is the study of the **existence** and uniqueness of **non-equilibrium invariant** measures.

- The latter has an interest in its own right.

- Even for stochastically forced systems, proving the existence and uniqueness of non-equilibrium invariant measures is very hard in the context of PDEs.

- However, we have recent developments in understanding analogous questions for some ODE systems modeling heat transfer in a chain of oscillators:

- Works of Eckmann, Pillet, and Rey-Bellet (99') up to more recent progress by Hairer and Mattingly (07') for a finite collection (3) of anharmonic oscillators with nearest neighbor couplings (classical Hamiltonian system) put into contact with two heat baths at **different** temperatures.
    - Interaction with heat baths is modeled by standard Langevin dynamics.

The stochastic ODE model

Our Point of departure: is the reduced *Toy model* first derived by [CKSTT] whose Hamiltonian is given by:

$$H(c) = \frac{1}{2}\Big(\sum_j |c_j|^2\Big)^2 - \frac{1}{4}\sum_j |c_j|^4 + \frac{1}{2}\sum_{j=1}^{n}\big(c_{j-1}^2 \overline{c_j}^2 + \overline{c_{j-1}}^2 c_j^2\big)$$

We will attach the first and the last modes c_1 and c_n to two heat baths at temperatures $T_1 < T_n$ respectively.

- This is a mechanism to stochastically add and dissipate energy from the system in a controlled way.

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For $B = (B_1, B_3)$ a two-dimensional Brownian motion, $\gamma > 0$,

$$\begin{aligned}
dc_1 &= i\frac{\partial H}{\partial \overline{c_1}}dt - \gamma\frac{\partial H}{\partial \overline{c_1}}dt + \sqrt{2\gamma T_1}dB_1 \\
dc_j &= i\left[2(\sum_k |c_k|^2)c_j - |c_j|^2 c_j + 2(c_{j-1}^2 + c_{j+1}^2)\overline{c_j}\right] dt \quad j = 2, \dots, n-1 \\
dc_n &= i\frac{\partial H}{\partial \overline{c_n}}dt - \gamma\frac{\partial H}{\partial \overline{c_n}}dt + \sqrt{2\gamma T_n}dB_n
\end{aligned}$$

- If $T_1 = T_n = T \to$ **equilibrium** and we prove $\exp(-H/2T)dc d\bar{c}$ is an **invariant Gibbs measure**.

- Interest in $T_1 < T_n$: does there exist a unique smooth ergodic nonequilibrium invariant measure?
  - One expects an initial distribution of system to converge to a (stationary) nonequilibrium state in which energy/matter is flowing.

Theorem [HMNRS16]

For $n = 3$ there exists a unique ergodic non-equilibrium invariant measure.

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Theorem [HMNRS16]

For $n = 3$ there exists a unique ergodic non-equilibrium invariant measure.

Steps involved:

Change coordinates $\quad I_j = |c_j|^2, \quad \varphi_j = \text{Arg}\, c_j.$

Let $M := I_1 + I_2 + I_3$ and $\mathcal{L}$ = the Fokker-Planck operator (generator of the transition semigroup T_t), we have:

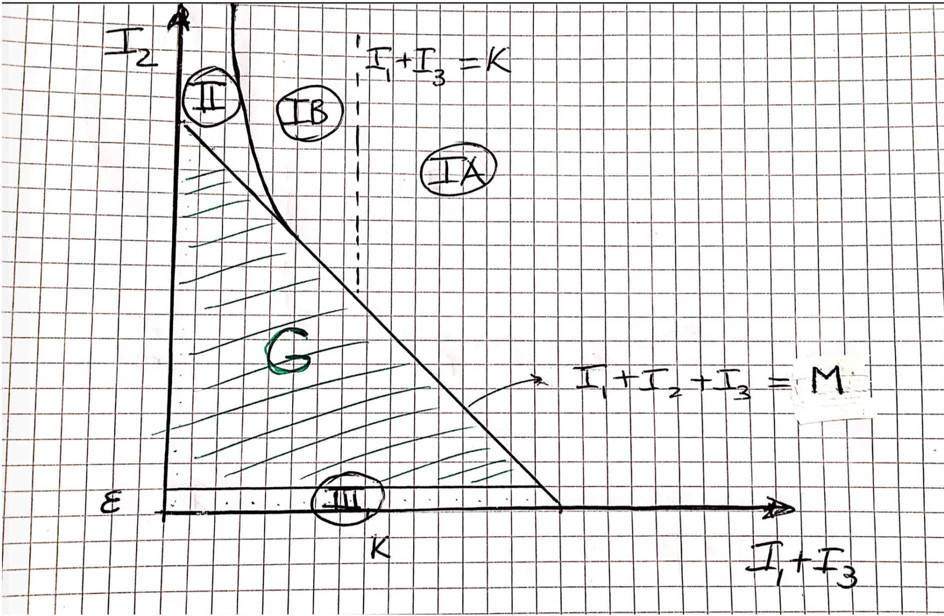
- Hypoellipticity on $X := (\mathbb{R}^3_+ \times \mathbb{T}^3) \setminus \{I_2 = 0\}$ (local smoothing)

- **Existence of Measure**: We construct a continuous and piecewise C^2 Lyapunov function V that penalizes the region I_2 small and high energy.
  - Such construction gives an upper bound on the hitting time of the good region G (compact set) where the dynamics spends most of time:

$$\mathcal{L}V \leq -cV + \kappa \mathbb{1}_G$$

  - Natural candidate is to use a coercive conserved quantity of the original Hamiltonian system such as $V = e^M$.
  - But this does not work in the whole space.
  - We need to split our phase space in 4 regions (**difficulty**).
    - ★ Need to solve suitable Poisson equations for V with e^M at the boundary.
    - ★ Need to study of the behavior of the phases and and proving that asymptotically they get locked.

- Uniqueness and Ergodicity of the invariant measure follow from a controllability lemma for the deterministic system showing one can access any region of phase space + Strook-Varadhan theorem.



I_2
II
IB
IA
$I_1+I_3=K$
G
$I_1+I_2+I_3=M$
$\mathcal{E}$
III
K
I_1+I_3