A Complete Dichotomy Rises from the Capture of Vanishing Signatures

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Three Frameworks for Counting Problems

The following three frameworks are in increasing order of strength.

1. **Graph Homomorphisms**
2. **Constraint Satisfaction Problems (#CSP)**
3. **Holant Problems**

In each framework, there has been remarkable progress in the classification program.

Holant

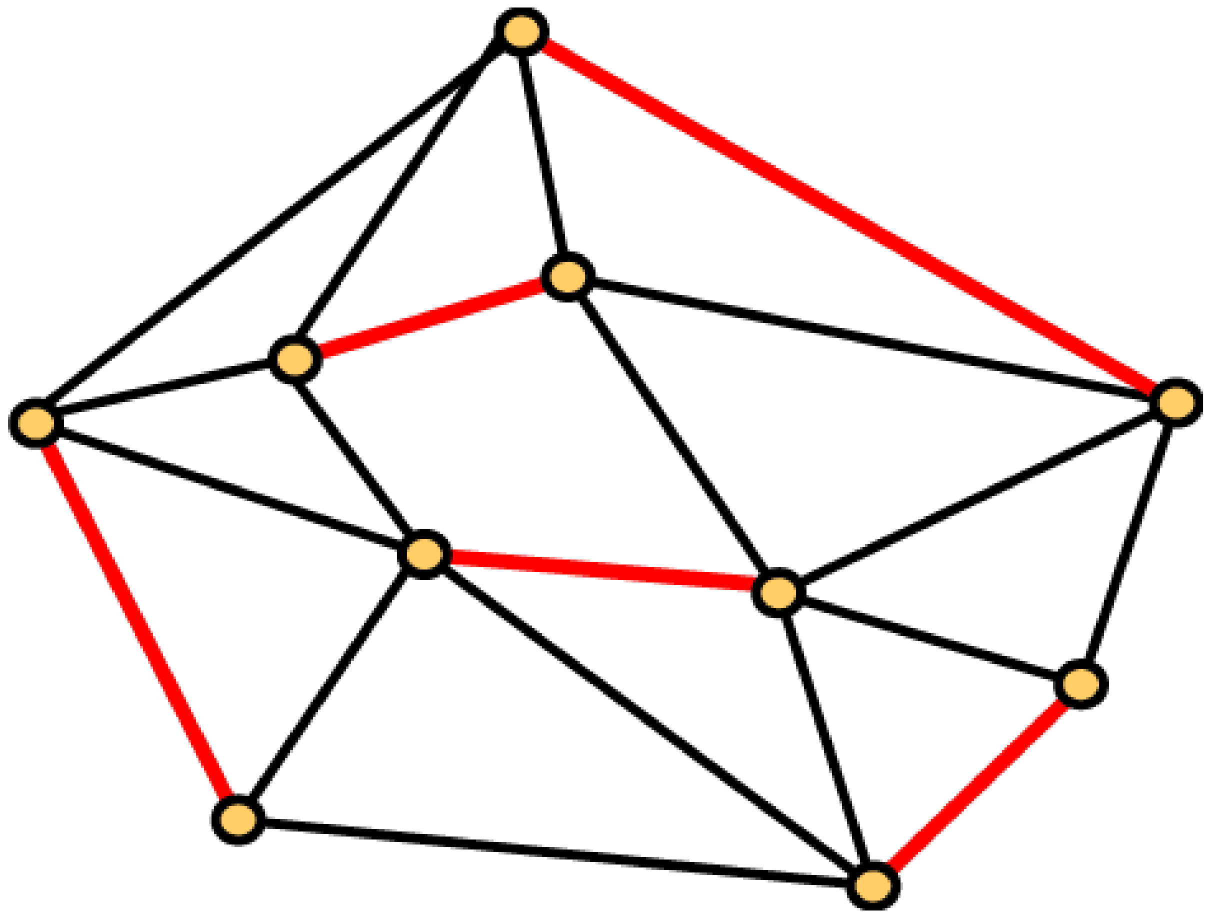
A signature grid $\Omega = (G, \mathcal{F}, \pi)$ **is a tuple, where** $G = (V, E)$ **is a graph,** π **labels each** $v \in V$ **with a function** $f_v \in \mathcal{F}$**, and** f_v **maps** $\{0,1\}^{\deg(v)}$ **to** $\mathbb{C}$**.**

$$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v \left(\sigma \mid_{E(v)}\right).$$

where

- $E(v)$ **denotes the incident edges of** v
- $\sigma \mid_{E(v)}$ **denotes the restriction of** σ **to** $E(v)$**.**

Perfect Matchings



Matching as Holant

$$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v\left(\sigma\,|_{E(v)}\right).$$

The problem of counting Perfect Matchings**s on** G **corresponds to attaching the Exact-One function at every vertex of** G**.**

The problem of counting all Matchings**s on** G **is to attach the At-Most-One function at every vertex of** G**.**

Graph Homomorphisms

If instead we consider all vertex assignments $\xi : V \to [q]$, and there is a single binary constraint function assigned to each edge, then this is Graph Homomorphism

$$Z_{\mathbf{A}}(G) = \sum_{\xi:V\to[q]} \prod_{(u,v)\in E} A_{\xi(u),\xi(v)}.$$

E.g., VERTEX COVERS

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is the OR function.

Three-Colorings

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is the Disequality **function.**

Even-Odd Induced Subgraphs

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$Z_{\mathbf{H}}(G)$ **computes the number of induced subgraphs of** G **with an even (or odd) number of edges.**

$$\Big(2^n - Z_{\mathbf{H}}(G)\Big)\Big/2$$

is the number of induced subgraphs of G **with an odd number of edges.**

Graph Homomorphism

Lovász first studied Graph homomorphisms.

L. Lovász: Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

`http://www.cs.elte.hu/~lovasz/hom-paper.html`

Getting Acquainted

Consider the constraint function

$$f : \{0,1\}^4 \to \mathbb{C},$$

where if the input (x_1, x_2, x_3, x_4) **has Hamming weight** w**, then** $f(x_1, x_2, x_3, x_4) = 3, 0, 1, 0, 3$**, if** $w = 0, 1, 2, 3, 4$**, resp.**

We denote this function by $f = [3, 0, 1, 0, 3]$**.**

What is the counting problem defined by the Holant sum?

$$\text{Holant}_\Omega = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f\left(\sigma\mid_{E(v)}\right).$$

What's that problem?

On 4-regular graphs, $\sum_\sigma \prod_{v \in V} f\left(\sigma \mid_{E(v)}\right)$ is a sum over all 0-1 edge assignments σ of products of local evaluations.

We only sum over assignments which assign an even number of 1's to the incident edges of each vertex, since

$$f = [3, 0, 1, 0, 3]$$

Thus $f = 0$ for $w = 1$ and 3.

Then each vertex contributes a factor 3 if the 4 incident edges are assigned all 0 or all 1, and contributes a factor 1 if exactly two incident edges are assigned 1.

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Before anyone thinks that this problem is artificial, let's consider a holographic transformation.

An Equivalent Bipartite Formulation

Let

$$I(G) = (E(G), V(G), \{(e, v) \mid v \textbf{ is incident to } e \textbf{ in } G\})$$

be the edge-vertex incidence graph of G**.**

Holant $(=_2 \mid f)$ **on** $I(G)$**:**

Each $e \in E(G)$ **is attached** $=_2$ **(binary** Equality**).**

The truth table of $=_2$ **is** $(1, 0, 0, 1)$ **indexed by** $\{0, 1\}^2$**.**

A Holographic Transformation

Apply

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},$$

to

$$\text{Holant}\,(=_2 \mid f) = \text{Holant}\,\big((=_2)Z^{\otimes 2} \mid (Z^{-1})^{\otimes 4} f\big)$$

Here $(=_2)Z^{\otimes 2}$ **is a row vector indexed by** $\{0,1\}^2$ **denoting the transformed function under** Z **from** $(=_2) = (1, 0, 0, 1)$**, and** $(Z^{-1})^{\otimes 4} f$ **is the column vector indexed by** $\{0,1\}^4$ **denoting the transformed function under** Z^{-1} **from** f**.**

A Holographic Transformation

$Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ **transforms** $=_2$ **to the binary** DISEQUALITY**:**

$$
\begin{aligned}
(=_2) Z^{\otimes 2} &= (1\ 0\ 0\ 1) Z^{\otimes 2} \\
&= \left\{ (1\ 0)^{\otimes 2} + (0\ 1)^{\otimes 2} \right\} Z^{\otimes 2} \\
&= \tfrac{1}{2} \left\{ (1\ 1)^{\otimes 2} + (i\ \ -i)^{\otimes 2} \right\} \\
&= (0\ 1\ 1\ 0) \\
&= [0, 1, 0] \\
&= (\neq_2).
\end{aligned}
$$

A Holographic Transformation

Let $\hat{f} = [0, 0, 1, 0, 0]$ **be the** Exact-Two **function on** $\{0, 1\}^4$**.** **Consider** $Z^{\otimes 4}\hat{f}$**, where**

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},$$

$$\begin{aligned} & Z^{\otimes 4} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \\ = \; & \frac{1}{4} \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} + \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} + \cdots + \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} \\ = \; & \frac{1}{2} [3, 0, 1, 0, 3] \\ = \; & \frac{1}{2} f \end{aligned}$$

Hence $(Z^{-1})^{\otimes 4} f = 2\hat{f}$**.**

A Holographic Transformation

Let $\hat{f} = [0, 0, 1, 0, 0]$ **be the** Exact-Two **function on** $\{0, 1\}^4$**.**
Consider $Z^{\otimes 4}\hat{f}$**, where**

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},$$

$$
\begin{aligned}
& Z^{\otimes 4} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \\
= \quad & \frac{1}{4} \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} + \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} + \cdots + \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} \\
= \quad & \frac{1}{2} [3, 0, 1, 0, 3] \\
= \quad & \frac{1}{2} f
\end{aligned}
$$

Hence $(Z^{-1})^{\otimes 4} f = 2\hat{f}$**.**

What's Natural and What's Artificial?

$$\text{Holant}\,(=_2 \mid f) = \text{Holant}\left((=_2)Z^{\otimes 2} \mid (Z^{-1})^{\otimes 4} f\right) = \text{Holant}\,(\neq_2 \mid 2[0,0,1,0,0])$$

Hence, up to a global constant factor of 2^n on a graph with n vertices, the Holant problem with $[3,0,1,0,3]$ is exactly the same as $\text{Holant}\,(\neq_2 \mid [0,0,1,0,0])$.

A moment's reflection shows that $\text{Holant}\,(\neq_2 \mid [0,0,1,0,0])$ is counting the number of Eulerian orientations on 4-regular graphs, an eminently natural problem!

Our goal is to classify all of them.

Vanishing

For any $a, b \in \mathbb{C}$**,**

$$\text{Holant}(\neq_2 \mid [0,0,1,0,0]) = \text{Holant}\left(\neq_2 \mid [a,b,1,0,0]\right)$$

This is related to a phenomenon we call *vanishing*.

Definition

A set of signatures $\mathcal{F}$ **is called vanishing if the value** $\text{Holant}_\Omega(\mathcal{F})$ **is zero for every signature grid** Ω**. A signature** f **is called vanishing if the singleton set** $\{f\}$ **is vanishing.**

E.g.,

$$\text{Holant}(Z^{\otimes 4}[a,b,0,0,0]) = \text{Holant}\left(\neq_2 \mid [a,b,0,0,0]\right) = 0$$

Symmetrization

Definition

For positive integers t and n with $t \leq n$ and for unary signatures $v, v_1, \ldots, v_{n-t}$, we define

$$\mathrm{Sym}_n^t(v; v_1, \ldots, v_{n-t}) = \sum_{\pi \in S_n} \bigotimes_{k=1}^{n} u_{\pi(k)},$$

where the ordered sequence

$$(u_1, u_2, \ldots, u_n) = (\underbrace{v, \ldots, v}_{t \textbf{ copies}}, v_1, \ldots, v_{n-t}).$$

Vanishing Degrees

Definition

A nonzero signature f of arity n has vanishing degree $\mathrm{vd}^+(f) = k$, if $1 \le k \le n$ is the largest positive integer such that for some $n-k$ unary $v_1, \ldots, v_{n-k}$

$$f = \mathrm{Sym}_n^k([1, i]; v_1, \ldots, v_{n-k}).$$

If f cannot be expressed as such, we define $\mathrm{vd}^+(f) = 0$.

If f is the all zero signature, we define $\mathrm{vd}^+(f) = n + 1$.

Similarly we define $\mathrm{vd}^-(f)$, using $-i$ instead of i.

E.g.,

$$\begin{aligned} & \mathrm{Sym}_3^2([1,i];[a,b]) \\ = \quad & 2\{[a,b]\otimes[1,i]\otimes[1,i]+[1,i]\otimes[a,b]\otimes[1,i]+[1,i]\otimes[1,i]\otimes[a,b]\} \\ = \quad & 2[3a, 2ia+b, -a+2ib, -3b]. \end{aligned}$$

E.g., for $f = [1,0,1]$

$$\mathrm{vd}^+(f) = \mathrm{vd}^-(f) = 1.$$

The Vanishing Class

Definition

For $\sigma \in \{+, -\}$, we define

$$\mathscr{V}^\sigma = \{f \mid 2\,\mathrm{vd}^\sigma(f) > \mathrm{arity}(f)\}.$$

Lemma

For a set of symmetric signatures $\mathcal{F}$, if $\mathcal{F} \subseteq \mathscr{V}^+$ or $\mathcal{F} \subseteq \mathscr{V}^-$, then $\mathcal{F}$ is vanishing.

Characterizing Vanishing Signatures

Lemma

Let $f_+ \in \mathscr{V}^+$ and $f_- \in \mathscr{V}^-$. If neither f_+ nor f_- is the zero signature, then the signature set $\{f_+, f_-\}$ is not vanishing.

Lemma

Every symmetric vanishing signature is in $\mathscr{V}^+ \cup \mathscr{V}^-$.

Theorem

Let $\mathcal{F}$ be a set of symmetric signatures. Then $\mathcal{F}$ is vanishing if and only if $\mathcal{F} \subseteq \mathscr{V}^+$ or $\mathcal{F} \subseteq \mathscr{V}^-$.

A Dual Characterization

Definition

A signature $f = [f_0, f_1, \ldots, f_n]$ is in $\mathscr{R}_t^+$ for a nonnegative integer $t \geq 0$ if

- $t > n$**; or**
- **For any $0 \leq k \leq n - t$, $f_k, \ldots, f_{k+t}$ satisfy the recurrence relation**

$$\binom{t}{t} i^t f_{k+t} + \binom{t}{t-1} i^{t-1} f_{k+t-1} + \cdots + \binom{t}{0} i^0 f_k = 0.$$

We define $\mathscr{R}_t^-$ similarly but with $-i$ replacing i.

A Dual Characterization

Definition

For a nonzero symmetric signature f of arity n, it is of *positive* (resp. *negative*) recurrence degree $t \leq n$, denoted by $\operatorname{rd}^+(f) = t$ (resp. $\operatorname{rd}^-(f) = t$), if and only if $f \in \mathscr{R}^+_{t+1} - \mathscr{R}^+_t$ (resp. $f \in \mathscr{R}^-_{t+1} - \mathscr{R}^-_t$).

If f is the all zero signature, define $\operatorname{rd}^+(f) = \operatorname{rd}^-(f) = -1$.

Lemma

Suppose f is a symmetric signature and $\sigma \in \{+, -\}$. Then $\operatorname{vd}^\sigma(f) + \operatorname{rd}^\sigma(f) = \operatorname{arity}(f)$, and

$$\mathscr{V}^\sigma = \{f \mid 2\,\operatorname{rd}^\sigma(f) < \operatorname{arity}(f)\}.$$

Some Tractable Function Families

The following three families of functions

$$\begin{aligned}\mathcal{F}_1 &= \{\ \lambda([1,0]^{\otimes k} + i^r[0,\ 1]^{\otimes k})\ \mid\ \lambda \in \mathbb{C}, k \geq 1,\ r = 0,1,2,3\};\\ \mathcal{F}_2 &= \{\ \lambda([1,1]^{\otimes k} + i^r[1,-1]^{\otimes k})\ \mid\ \lambda \in \mathbb{C}, k \geq 1,\ r = 0,1,2,3\};\\ \mathcal{F}_3 &= \{\ \lambda([1,\ i]^{\otimes k} + i^r[1,\ -i]^{\otimes k})\ \mid\ \lambda \in \mathbb{C}, k \geq 1,\ r = 0,1,2,3\}.\end{aligned}$$

give rise to tractable problems.

Theorem [C., Lu, Xia]
For every graph G where $V(G)$ is labeled by $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, Holant_G is computable in P.

$\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$

1. $[1, 0, 0, \ldots, 0, \pm 1]$; $(\mathcal{F}_1, r = 0, 2)$

2. $[1, 0, 0, \ldots, 0, \pm i]$; $(\mathcal{F}_1, r = 1, 3)$

3. $[1, 0, 1, 0, \ldots, 0$ **or** $1]$; $(\mathcal{F}_2, r = 0)$

4. $[0, 1, 0, 1, \ldots, 0$ **or** $1]$; $(\mathcal{F}_2, r = 2)$

5. $[1, i, 1, i, \ldots, i$ **or** $1]$; $(\mathcal{F}_2, r = 3)$

6. $[1, -i, 1, -i, \ldots, (-i)$ **or** $1]$; $(\mathcal{F}_2, r = 1)$

7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0$ **or** 1 **or** $(-1)]$; $(\mathcal{F}_3, r = 0)$

8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1$ **or** $(-1)]$; $(\mathcal{F}_3, r = 1)$

9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0$ **or** 1 **or** $(-1)]$; $(\mathcal{F}_3, r = 2)$

10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1$ **or** $(-1)]$. $(\mathcal{F}_3, r = 3)$

$\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$

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6. $[1, -i, 1, -i, \ldots, (-i)$ **or** $1]$; $(\mathcal{F}_2, r = 1)$

7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0$ **or** 1 **or** $(-1)]$; $(\mathcal{F}_3, r = 0)$

8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1$ **or** $(-1)]$; $(\mathcal{F}_3, r = 1)$

9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0$ **or** 1 **or** $(-1)]$; $(\mathcal{F}_3, r = 2)$

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$\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$

1. $[1, 0, 0, \ldots, 0, \pm 1]$; $(\mathcal{F}_1, r = 0, 2)$

2. $[1, 0, 0, \ldots, 0, \pm i]$; $(\mathcal{F}_1, r = 1, 3)$

3. $[1, 0, 1, 0, \ldots, 0$ **or** $1]$; $(\mathcal{F}_2, r = 0)$

4. $[0, 1, 0, 1, \ldots, 0$ **or** $1]$; $(\mathcal{F}_2, r = 2)$

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8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1$ **or** $(-1)]$; $(\mathcal{F}_3, r = 1)$

9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0$ **or** 1 **or** $(-1)]$; $(\mathcal{F}_3, r = 2)$

10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1$ **or** $(-1)]$. $(\mathcal{F}_3, r = 3)$

$Z_{\mathbf{H}}(G)$ and $\mathrm{Holant}(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$

Recall

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$Z_{\mathbf{H}}(G)$ **is a special case of** Holant **with** $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$**:**

H is included in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$**.**

As a binary function $\mathbf{H} = [1, 1, -1]$**.**

We can take $r = 1$**,** $k = 2$ **and** $\lambda = (1 + i)^{-1}$ **in** $\mathcal{F}_3$**, to get H.**

If we take $r = 0$**,** $\lambda = 1$ **in** $\mathcal{F}_1$**, we get the** Equality **function** $[1, 0, \ldots, 0, 1]$ **on** k **bits.**

So $Z_{\mathbf{H}}(\cdot)$ **is computable in P.**

Affine Signatures

Definition

A k-ary function $f(x_1, \ldots, x_k)$ is affine if it has the form

$$\lambda \cdot \chi_{Ax=0} \cdot \sqrt{-1}^{\sum_{j=1}^{n} \langle \alpha_j, x \rangle},$$

where $\lambda \in \mathbb{C}$, $x = (x_1, x_2, \ldots, x_k, 1)^{\mathtt{T}} \in \mathbb{F}_2^{k+1}$, A is a matrix over $\mathbb{F}_2$, α_j is a vector over $\mathbb{F}_2$, and χ is a 0-1 indicator function such that $\chi_{Ax=0}$ is 1 iff $Ax = 0$.

The exponent of $i = \sqrt{-1}$ is a mod 4 sum of mod 2 sums.

We use $\mathscr{A}$ to denote the set of all affine functions.

Dichotomy Theorem

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then $\operatorname{Holant}(\mathcal{F})$ is #P-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in P:

1. **All non-degenerate signatures in $\mathcal{F}$ have arity ≤ 2;**
2. **$\mathcal{F}$ is $\mathscr{A}$-transformable;**
3. **$\mathcal{F}$ is $\mathscr{P}$-transformable;**
4. **$\mathcal{F} \subseteq \mathscr{V}^\sigma \cup \{f \in \mathscr{R}_2^\sigma \mid \operatorname{arity}(f) = 2\}$ for $\sigma \in \{+, -\}$;**
5. **All non-degenerate signatures in $\mathcal{F}$ are in $\mathscr{R}_2^\sigma$ for $\sigma \in \{+, -\}$.**

Some Sample Problems

Vertex Cover

Input: An undirected graph G.

Output: The number of vertex covers in G.

$$\text{Holant}\left([0,1,1] \mid \mathcal{EQ}\right)$$

The degree two vertices assigned $\textsc{Or} = [0,1,1]$ **are edges between its neighboring vertices.**

Holographic transformation by

$$T = \left[\begin{smallmatrix} 0 & -i \\ 1 & i \end{smallmatrix}\right].$$

Why?

$$
\begin{aligned}
[0,1,1] &= (0\ 1\ 1\ 1) \\
&= \left\{[1,1]^{\otimes 2} + [i,0]^{\otimes 2}\right\} \\
&= \left\{[1,0]^{\otimes 2} + [0,1]^{\otimes 2}\right\} \left[\begin{smallmatrix} 1 & 1 \\ i & 0 \end{smallmatrix}\right]^{\otimes 2} \\
&= (1\ 0\ 0\ 1)(T^{-1})^{\otimes 2} \\
&= (=_2)(T^{-1})^{\otimes 2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\text{Holant}\,([0,1,1] \mid \mathcal{EQ}) &\equiv_T \text{Holant}\left([0,1,1]T^{\otimes 2} \mid T^{-1}\mathcal{EQ}\right) \\
&\equiv_T \text{Holant}\left(=_2 \mid T^{-1}\mathcal{EQ}\right) \\
&\equiv_T \text{Holant}(T^{-1}\mathcal{EQ}).
\end{aligned}
$$

T^{-1} **transforms** $=_k$ **to**

$$\begin{aligned} f_{(k)} &= (T^{-1})^{\otimes k}(=_k) \\ &= \left[\begin{smallmatrix} 1 & 1 \\ i & 0 \end{smallmatrix}\right]^{\otimes k} \left\{ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\otimes k} + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^{\otimes k} \right\} \\ &= \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^{\otimes k} + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\otimes k} \\ &= [2, i, -1, -i, 1, i, -1, -i, 1, i, \ldots]^{\mathtt{T}} \end{aligned}$$

of length $k+1$**.**

By our dichotomy, $\mathrm{Holant}(f_{(k)})$ **(the restriction to** k**-regular graphs) is #P-hard, for every** $k \geq 3$**.**

λ-VertexCover

Input: An undirected graph G.

Output: $\displaystyle\sum_{C \in \mathcal{C}(G)} \lambda^{e(C)}$,

where $\mathcal{C}(G)$ denotes the set of all vertex covers of G, and $e(C)$ is the number of edges with both endpoints in C.

$$\text{Holant}\left([0, 1, \lambda] \mid \mathcal{EQ}\right)$$

Suppose $\lambda \neq 0$. On regular graphs, this problem is equivalent to the so-called hardcore gas model,

$$\sum_{I \in \mathcal{I}(G)} \mu^{|I|}.$$

$$\text{Holant}\left([1, 1, 0] \mid [1, 0, \ldots, 0, \mu]\right).$$

NoSinkOrientation

Input: An undirected graph G.

Output: The number of orientations of G such that each vertex has at least one outgoing edge.

$$\text{Holant}\left([0,1,0] \mid \mathcal{F}\right)$$

where $\mathcal{F}$ consists of $f_{(k)} = [0, 1, \ldots, 1, 1]$ for any arity k.

Pick $T = \frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right] = \frac{1}{\sqrt{2}} Z^{-1}$, with $T^{-1} = \sqrt{2} Z = \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ and get

$$\begin{aligned} \text{Holant}([0,1,0] \mid f_{(k)}) \quad &\equiv_T \quad \text{Holant}\left([0,1,0] T^{\otimes 2} \mid (T^{-1})^{\otimes k} f_{(k)}\right) \\ &\equiv_T \quad \text{Holant}\left(\tfrac{1}{2}[1,0,1] \mid \widehat{f_{(k)}}\right) \\ &\equiv_T \quad \text{Holant}(\widehat{f_{(k)}}), \end{aligned}$$

where $\widehat{f_{(k)}} = [2^k - 1, -i, 1, i, -1, \ldots]$.

This is actually a special case (consider $-\widehat{f_{(k)}}$) of the #λ-VertexCover problem with $\lambda = 2e^{\pi i/k}$.

It is #P-hard by our dichotomy, but mod 2^k, $\widehat{f_{(k)}}$ becomes $[-1, -i, 1, \ldots]$, and it is one of the tractable cases in our dichotomy.

NoSinkNoSourceOrientation

Input: An undirected graph G.

Output: The number of orientations of G such that each vertex has at least one incoming and one outgoing edge.

$$\text{Holant}\left([0,1,0] \mid \mathcal{F}\right)$$

where $\mathcal{F}$ consists of $f_{(k)} = [0,1,\ldots,1,0]$ for any arity k.

$$\begin{aligned}
\text{Holant}([0,1,0] \mid f_{(k)}) &\equiv_T \text{Holant}\left([0,1,0]T^{\otimes 2} \mid (T^{-1})^{\otimes k} f_{(k)}\right) \\
&\equiv_T \text{Holant}\left(\tfrac{1}{2}[1,0,1] \mid \widehat{f_{(k)}}\right) \\
&\equiv_T \text{Holant}(\widehat{f_{(k)}}),
\end{aligned}$$

where $\widehat{f_{(k)}} = [2^k - 2, 0, 2, 0, -2, \ldots]$.

Here we transform from one real weighted Holant problem to another real weighted Holant problem via a complex weighted transformation.

#NoSinkNoSourceOrientation is #P-hard, but it is tractable modulo 2^d, where d is the minimum degree of the input graph.

One-In-Or-One-Out-Orientation

Input: An undirected graph G.

Output: The number of orientations of G such that each vertex has exactly 1 incoming or exactly 1 outgoing edge.

$$\text{Holant}\left([0,1,0] \mid \mathcal{F}\right)$$

where the set $\mathcal{F} = \{f, g\}$ and

$$f = [0,1,0,\ldots,0], \quad \textbf{and} \quad g = [0,\ldots,0,1,0].$$

After a holographic transformation,
$\hat{f} = [k, (k-2)i, -(k-4), \ldots]$ **and** $\hat{g} = [k, -(k-2)i, -(k-4), \ldots]$ **of arity k.**

Each individually is tractable (because they are vanishing), but together the problem is #P-hard.

λ-WeightedMatching

Input: An undirected graph G.

Output: $\displaystyle\sum_{M \in \mathcal{M}(G)} \lambda^{v(M)}$,

where $\mathcal{M}(G)$ is the set of all matchings in G and $v(M)$ is the number of unmatched vertices in the matching M.

$$\text{Holant}(\mathcal{F})$$

where $\mathcal{F}$ consists of signatures of the form $[\lambda, 1, 0, \ldots, 0]$.

When $\lambda = 0$, this problem counts perfect matchings, which is #P-hard, but tractable over planar graphs.

When $\lambda = 1$, this problem counts all (not necessarily perfect) matchings.

Vadhan (2001) proved that counting general matchings is #P-hard over k-regular graphs for $k \geq 5$, but left open the question for $k \leq 4$.

The dichotomy shows that they are also #P-hard for all $k \geq 3$.

The scope of our dichotomy theorem is such that it gives a sweeping classification for *all* such problems; the open case for $k = 4$ from [Vad01] is merely a single *point* in the problem space.

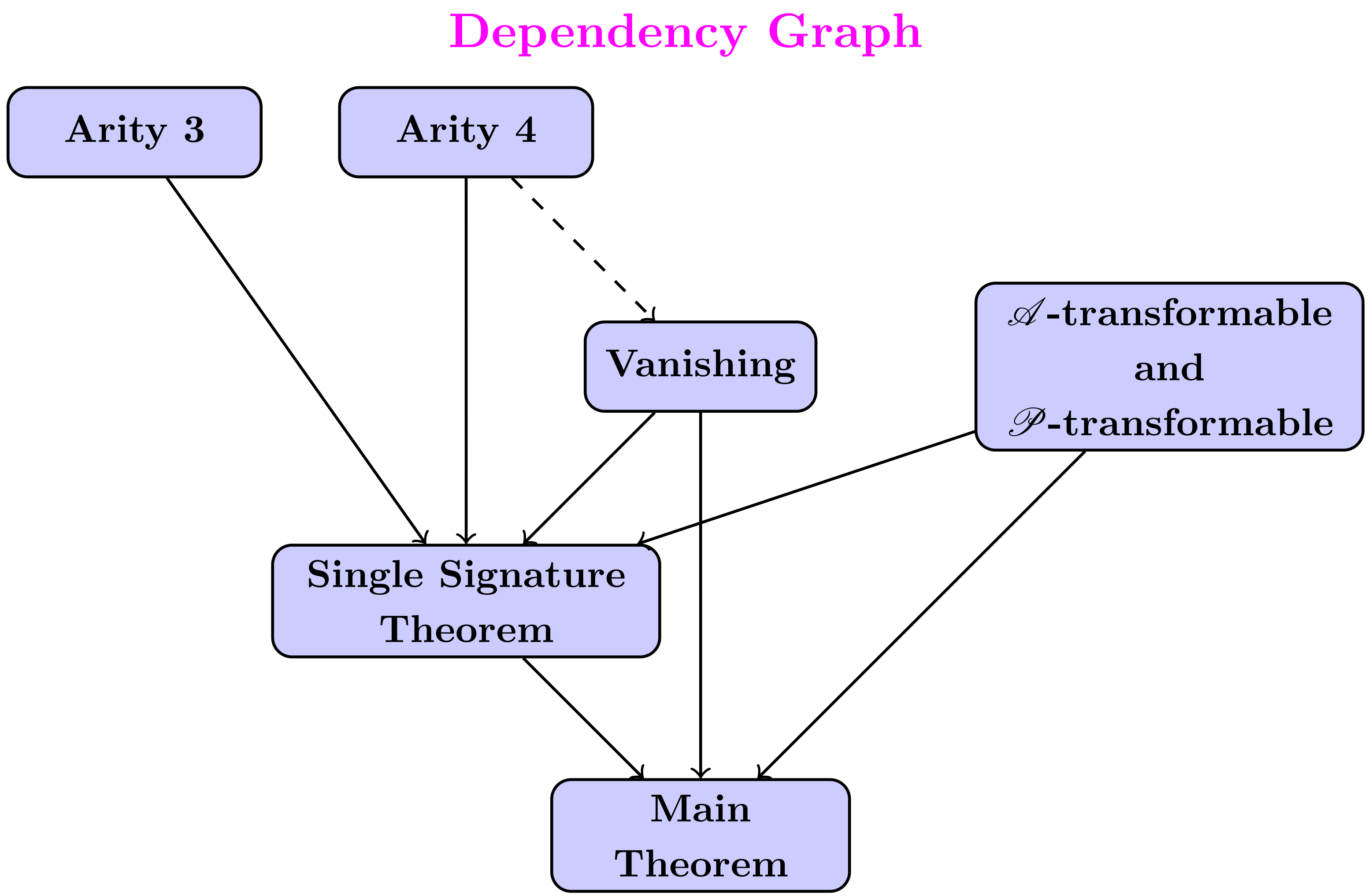


Figure 1: Dependency graph of key hardness results for our main dichotomy. Dependencies on previous dichotomies are not shown.

A Peek Under the Hood

"A mathematics lecture without a proof is like a movie without a love scene."

— Hendrik Lenstra

Arity 4 Signatures

Definition **The** **signature matrix** **of an 4-ary symmetric signature** $f = [f_0, f_1, f_2, f_3, f_4]$ **is**

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \\ f_1 & f_2 & f_2 & f_3 \\ f_2 & f_3 & f_3 & f_4 \end{bmatrix}.$$

For asymmetric signatures,

$$M_f = \begin{bmatrix} f^{0000}, f^{0010}, f^{0001}, f^{0011} \\ f^{0100}, f^{0110}, f^{0101}, f^{0111} \\ f^{1000}, f^{1010}, f^{1001}, f^{1011} \\ f^{1100}, f^{1110}, f^{1101}, f^{1111} \end{bmatrix}$$

indexed by $(x_1, x_2) \in \{0,1\}^2$ **and columns by** $(x_4, x_3) \in \{0,1\}^2$**.**

An Isomorphism

A 4-by-4 matrix is called redundant if it has identical middle two rows and identical middle two columns.

If f is symmetric then M_f is redundant. But not all redundant matrices correspond to a symmetric signature.

$\mathrm{RM}_4(\mathbb{C})$ **denotes redundant 4-by-4 matrices.**

There is a semi-group isomorphism

$$\varphi : \mathrm{RM}_4(\mathbb{C}) \to \mathbb{C}^{3\times 3}$$

defined by $\varphi(M) = AMB$, where $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Denote by $\psi = \varphi^{-1}$.

If M_f is redundant, we also define the compressed signature matrix of f as $\widetilde{M_f} = \varphi(M_f)$.

One Special Asymmetric Signature

$$M_g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\widetilde{M_g} = I_3.$$

Theorem

Holant(g) **is #P-hard.**

#P-hardness of g

We reduce from the Eulerian orientation problem

$$\text{Holant}([1, 0, 1/3, 0, 1])$$

We achieve this via an arbitrarily close approximation using the recursive construction.

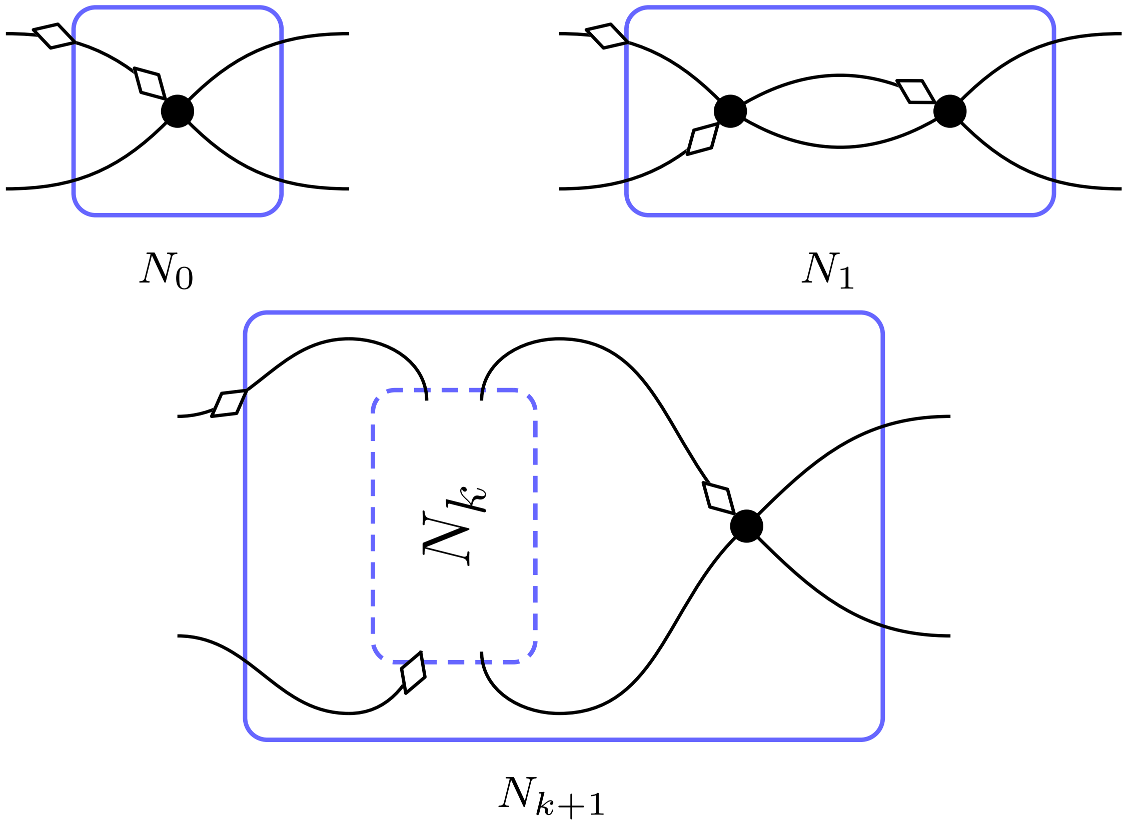


Figure 2: Recursive construction to approximate $[1, 0, 1/3, 0, 1]$. The vertices are assigned g.

Approximation

We claim the matrix M_{N_k} of Gadget N_k is

$$\begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & a_{k+1} & a_{k+1} & 0 \\ 0 & a_{k+1} & a_{k+1} & 0 \\ a_k & 0 & 0 & 1 \end{bmatrix},$$

where $a_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k$.

This is true for N_0.

Inductively assume M_{N_k} has this form. Then the rotated form of the signature matrix for N_k in N_{k+1} is

$$\begin{bmatrix} 1 & 0 & 0 & a_{k+1} \\ 0 & a_k & a_{k+1} & 0 \\ 0 & a_{k+1} & a_k & 0 \\ a_{k+1} & 0 & 0 & 1 \end{bmatrix}.$$

This corresponds to the rotated placement of N_k in N_{k+1} in the Figure.

Approximation

The action of g on the far right side of N_{k+1} is to replace each of the middle two entries in the middle two rows of this matrix with their average, $(a_k + a_{k+1})/2 = a_{k+2}$. This gives $M_{N_{k+1}}$.

Since a_k approaches $1/3$ exponentially fast, we may approximate the signature $[1, 0, 1/3, 0, 1]$ sufficiently close after only polynomially many steps of gadget construction. This completes the proof.

A Reduction From g

Lemma

Let f be any arity 4 signature with complex weights. If M_f is redundant and $\widetilde{M_f}$ is nonsingular, then for any set $\mathcal{F}$ containing f, we have

$$\text{Holant}(\mathcal{F} \cup \{g\}) \leq_T \text{Holant}(\mathcal{F}).$$

A Construction

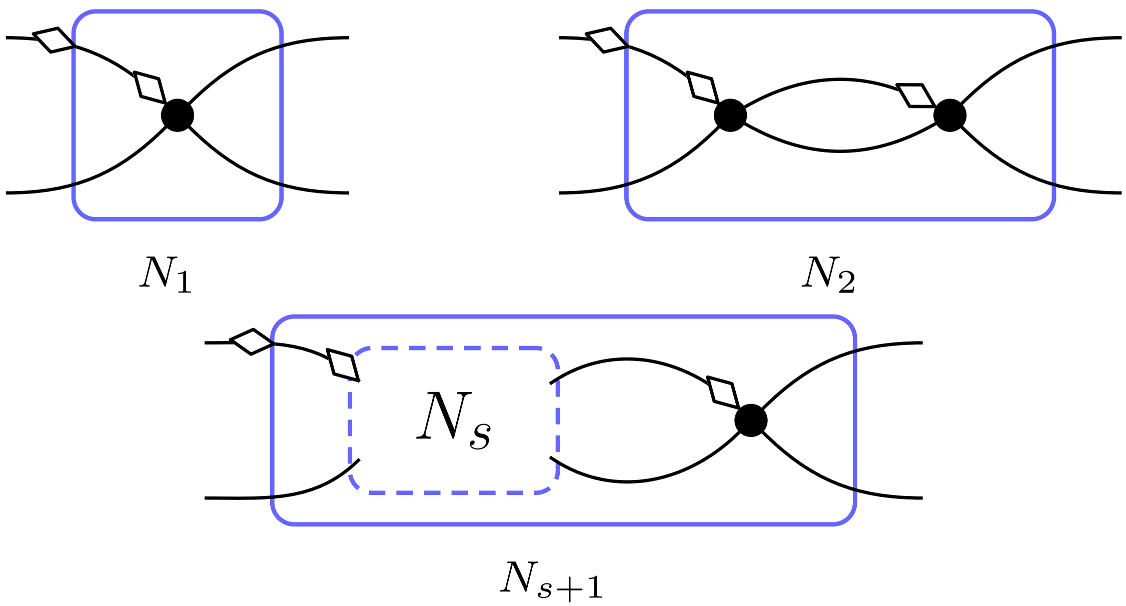


Figure 3: Recursive construction to interpolate g. The vertices are assigned f. Diamonds indicates the most significant bit and the bits are ordered in counter clockwise order.

Suppose g appears n times in an instance Ω of $\text{Holant}(\mathcal{F} \cup \{g\})$.

Construct instances Ω_s of $\text{Holant}(\mathcal{F})$ indexed by $s \geq 1$.

We obtain Ω_s from Ω by replacing each occurrence of g with N_s with f assigned to all vertices.

To obtain Ω_s from Ω, we effectively replace M_g with $M_{N_s} = (M_f)^s$.

Consider the Jordan normal form of $\widetilde{M_f}$

$$\widetilde{M_f} = T\Lambda T^{-1} = T \begin{bmatrix} \lambda_1 & b_1 & 0 \\ 0 & \lambda_2 & b_2 \\ 0 & 0 & \lambda_3 \end{bmatrix} T^{-1},$$

where $b_1, b_2 \in \{0, 1\}$.

$\det(\widetilde{M_f}) = \lambda_1\lambda_2\lambda_3 \neq 0$.

Now we use the isomorphism φ and its inverse ψ:

Since $\widetilde{M_g} = \varphi(M_g) = I_3$, and $T I_3 T^{-1} = I_3$, we have

$$\psi(T)M_g\psi(T^{-1}) = M_g.$$

We can view our construction of Ω_s as first replacing each M_g by $\psi(T)M_g\psi(T^{-1})$, which does not change the Holant value, and then replacing each new M_g with $\psi(\Lambda^s) = \psi(\Lambda)^s$ to obtain Ω_s. Observe that

$$\varphi(\psi(T)\psi(\Lambda^s)\psi(T^{-1})) = T\Lambda^s T^{-1} = (\widetilde{M_f})^s = \varphi((M_f)^s),$$

hence $\psi(T)\psi(\Lambda^s)\psi(T^{-1}) = M_{N_s}$.

Since $M_g = \psi(T)M_g\psi(T^{-1})$ and $M_{N_s} = \psi(T)\psi(\Lambda^s)\psi(T^{-1})$, replacing each M_g, sandwiched between $\psi(T)$ and $\psi(T^{-1})$, by $\psi(\Lambda^s)$ indeed transforms Ω to Ω_s.

Stratification

Now stratify all assignments in Ω_s based on the assignments to $\psi(\Lambda^s)$.

The inputs to each copy of $\psi(\Lambda^s)$ are from $\{0,1\}^2 \times \{0,1\}^2$. However, we can combine 01 and 10, since $\psi(\Lambda^s)$ is redundant.

Consider only the case $b_1 = b_2 = 1$.

We stratify all assignments to Λ^s according to:

- $(0,0)$ **or** $(2,2)$ i **many times,**
- $(1,1)$ j **many times,**
- $(0,1)$ k **many times,**
- $(1,2)$ ℓ **many times, and**
- $(0,2)$ m **many times**

All other assignments contribute a factor 0.

Stratification

Let $c_{ijk\ell m}$ **be the sum over all such assignments of the products of evaluations (including the contributions from** T **and** T^{-1}**) on** Ω_s**.**

$$\mathrm{Holant}_{\Omega} = \sum_{i+j=n} \frac{c_{ij000}}{2^j}.$$

$$\begin{aligned} \mathrm{Holant}_{\Omega_s} &= \sum_{i+j+k+\ell+m=n} \lambda^{(i+j)s} \left(s\lambda^{s-1}\right)^{k+\ell} \left(s(s-1)\lambda^{s-2}\right)^m \left(\frac{c_{ijk\ell m}}{2^{j+k+m}}\right) \\ &= \lambda^{ns} \sum_{i+j+k+\ell+m=n} s^{k+\ell+m}(s-1)^m \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m}2^{j+k+m}}\right). \end{aligned}$$

By reduction to $\mathrm{Holant}(\mathcal{F})$**, we have the values** $\mathrm{Holant}_{\Omega_s}$**.**
Will view this as a linear equation system.

Inverse Image $\psi(\Lambda^s)$ in $\mathrm{RM}_4(\mathbb{C})$

$$\begin{aligned}\mathrm{Holant}_{\Omega_s} &= \sum_{i+j+k+\ell+m=n} \lambda^{(i+j)s} \left(s\lambda^{s-1}\right)^{k+\ell} \left(s(s-1)\lambda^{s-2}\right)^m \left(\frac{c_{ijk\ell m}}{2^{j+k+m}}\right) \\ &= \lambda^{ns} \sum_{i+j+k+\ell+m=n} s^{k+\ell+m}(s-1)^m \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m} 2^{j+k+m}}\right).\end{aligned}$$

comes from

$$\psi(\Lambda^s) = \psi \begin{bmatrix} \lambda^s & s\lambda^{s-1} & \binom{s}{2}\lambda^{s-2} \\ 0 & \lambda^s & s\lambda^{s-1} \\ 0 & 0 & \lambda^s \end{bmatrix} = \begin{bmatrix} \lambda^s & \frac{s\lambda^{s-1}}{2} & \frac{s\lambda^{s-1}}{2} & \binom{s}{2}\lambda^{s-2} \\ 0 & \frac{\lambda^s}{2} & \frac{\lambda^s}{2} & s\lambda^{s-1} \\ 0 & \frac{\lambda^s}{2} & \frac{\lambda^s}{2} & s\lambda^{s-1} \\ 0 & 0 & 0 & \lambda^s \end{bmatrix}$$

Rank Deficiency

Unfortunately this linear system is rank deficient.

We now define new unknowns for any $p, q, m \geq 0$ **and** $p + q + m = n$**,**

$$c'_{pqm} = \sum_{i+j=p,k+\ell=q} \frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m} 2^{j+k+m}}$$

Note that c'_{n00} **is precisely the desired value**

$$\text{Holant}_\Omega = \sum_{i+j=n} \frac{c_{ij000}}{2^j}$$

The new linear system is

$$\text{Holant}_{\Omega_s} = \lambda^{ns} \sum_{p+q+m=n} s^{q+m}(s-1)^m c'_{pqm}$$

Unfortunately this new and condensed linear system is still rank deficient.

We now index the columns by (q, m)**, where** $q \geq 0$**,** $m \geq 0$**, and** $q + m \leq n$**. Correspondingly, we rename the variables** $x_{q,m} = c'_{pqm}$**. Note that** $p = n - q + m$ **is determined by** (q, m)**.**

Observe that the column indexed by (q, m) **is the sum of the columns indexed by** $(q - 1, m)$ **and** $(q - 2, m + 1)$ **provided** $q - 2 \geq 0$**. Namely,**

$$s^{q+m}(s-1)^m = s^{q-1+m}(s-1)^m + s^{q-2+m+1}(s-1)^{m+1}.$$

We write the linear system as

$$\sum_{q\geq 0,\ m\geq 0,\ q+m\leq n} \alpha_{q,m} x_{q,m} = \frac{\text{Holant}_{\Omega_s}}{\lambda^{ns}},$$

where $\alpha_{q,m} = s^{q+m}(s-1)^m$ **are the coefficients. Hence** $\alpha_{q,m} x_{q,m} = \alpha_{q-1,m} x_{q,m} + \alpha_{q-2,m+1} x_{q,m}$**, and we define new variables**

$$\begin{aligned} x_{q-1,m} &\leftarrow x_{q,m} + x_{q-1,m} \\ x_{q-2,m+1} &\leftarrow x_{q,m} + x_{q-2,m+1} \end{aligned}$$

from $q = n$ **down to** 2**.**

We write the linear system as

$$\sum_{q\geq 0,\ m\geq 0,\ q+m\leq n} \alpha_{q,m} x_{q,m} = \frac{\text{Holant}_{\Omega_s}}{\lambda^{ns}},$$

where $\alpha_{q,m} = s^{q+m}(s-1)^m$ **are the coefficients. Hence** $\alpha_{q,m}x_{q,m} = \alpha_{q-1,m}x_{q,m} + \alpha_{q-2,m+1}x_{q,m}$**, and we define new variables**

$$x_{q-1,m} \leftarrow x_{q,m} + x_{q-1,m}$$

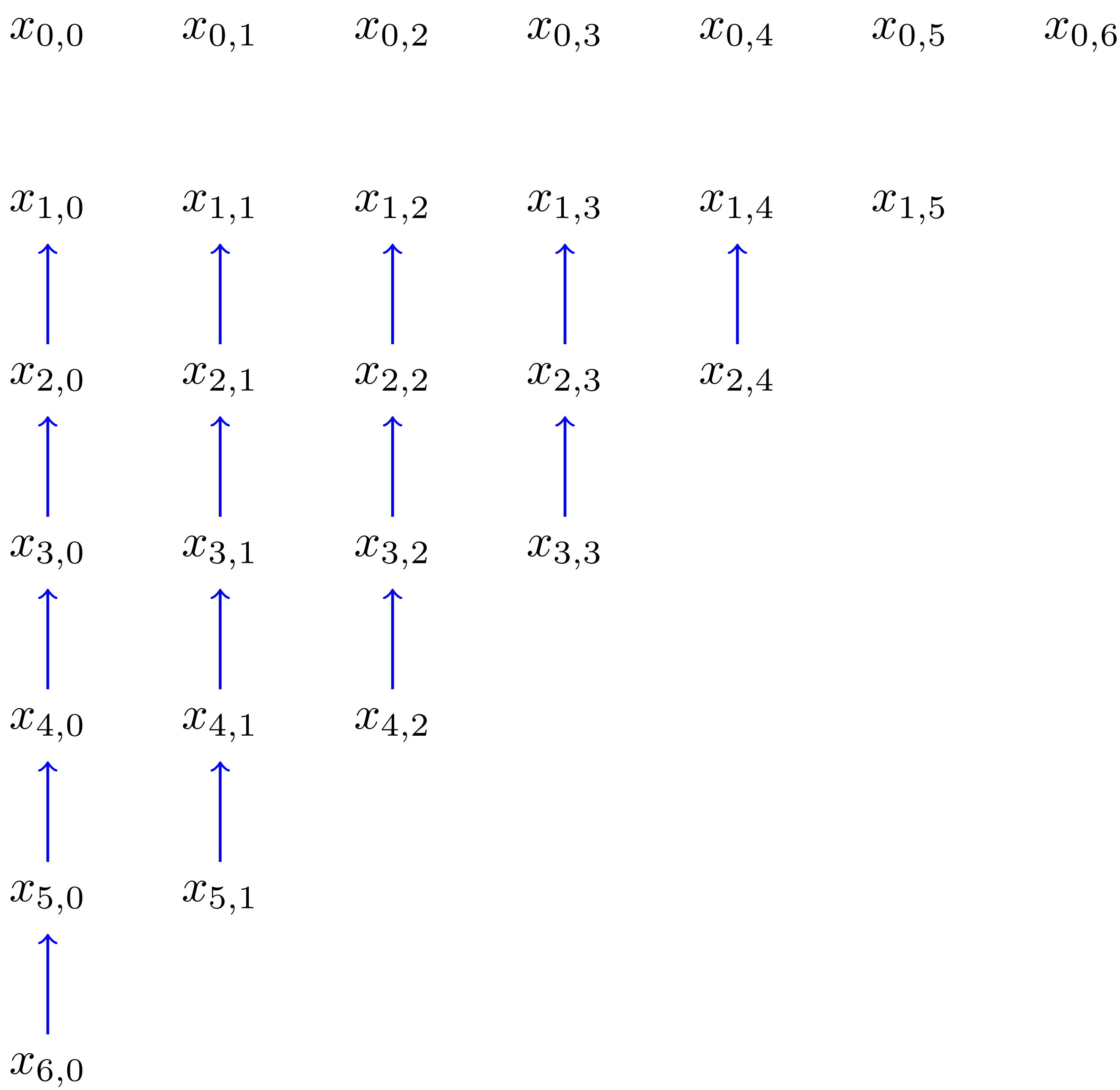
$$x_{q-2,m+1} \leftarrow x_{q,m} + x_{q-2,m+1}$$

from $q = n$ **down to** 2**.**

A Triangular Table of Variables

$$
\begin{array}{ccccccc}
x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,n-2} & x_{0,n-1} & x_{0,n} \\
x_{1,0} & x_{1,1} & x_{1,2} & \cdots & x_{1,n-2} & x_{1,n-1} & \\
x_{2,0} & x_{2,1} & x_{2,2} & \cdots & x_{2,n-2} & & \\
\vdots & \vdots & \vdots & & & & \\
x_{n-2,0} & x_{n-2,1} & x_{n-2,2} & & & & \\
x_{n-1,0} & x_{n-1,1} & & & & & \\
x_{n,0} & & & & & &
\end{array}
$$

$x_{0,0}$	$x_{0,1}$	$x_{0,2}$	$x_{0,3}$	$x_{0,4}$	$x_{0,5}$	$x_{0,6}$
$x_{1,0}$	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	
$x_{2,0}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$		
$x_{3,0}$	$x_{3,1}$	$x_{3,2}$	$x_{3,3}$			
$x_{4,0}$	$x_{4,1}$	$x_{4,2}$				
$x_{5,0}$	$x_{5,1}$					
$x_{6,0}$						


$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
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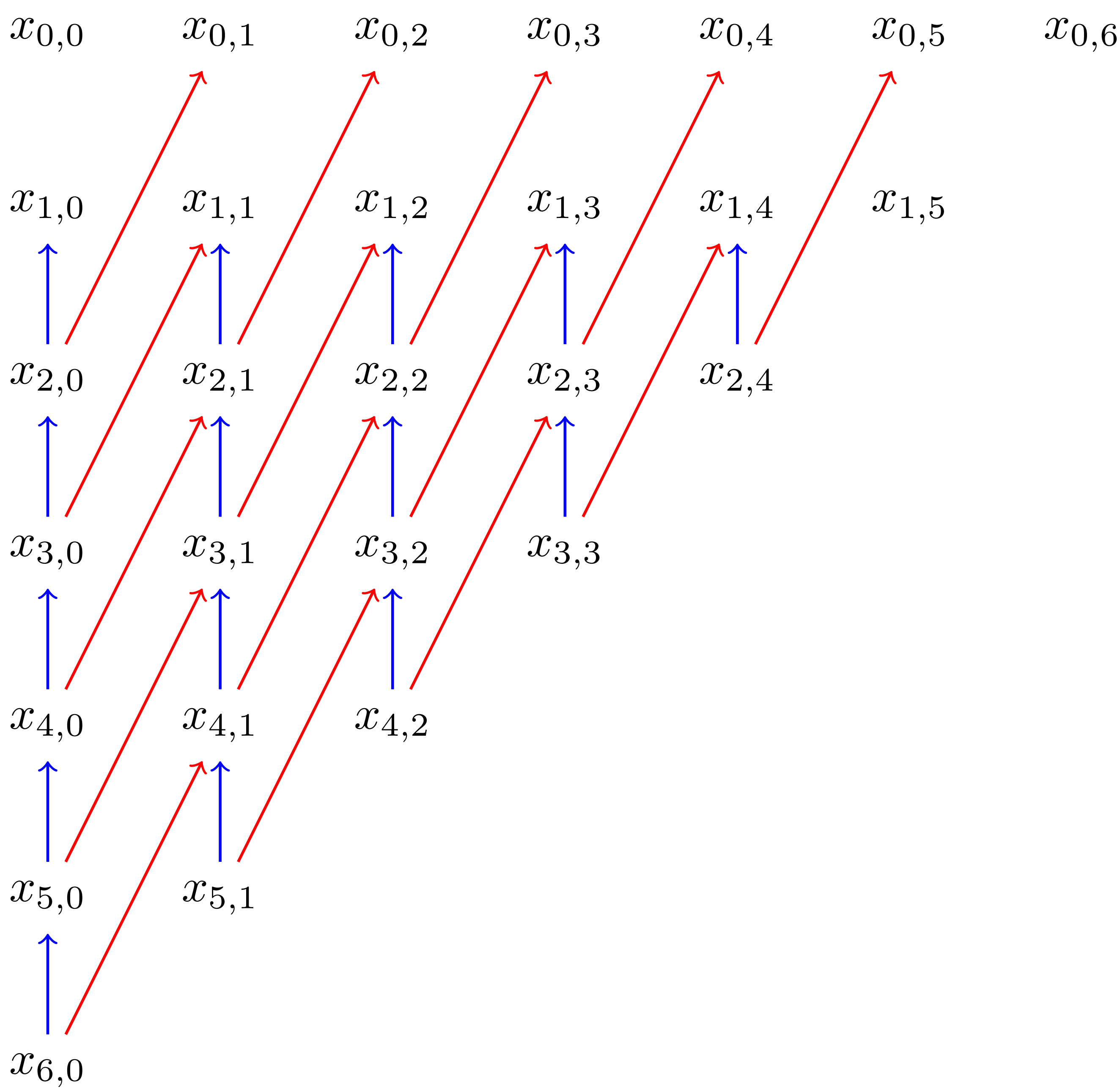
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$


$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
$x_{4,0}$ $x_{4,1}$ $x_{4,2}$
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$

A crucial observation is that the column indexed by $(0,0)$ is never updated.

Hence $x_{0,0} = c'_{n00}$ is still the Holant value on Ω.

The $2n+1$ unknowns that remain are

$$x_{0,0},\ x_{0,1},\ x_{0,2},\ \ldots, x_{0,n-1},\ x_{0,n}$$

$$x_{1,0},\ x_{1,1},\ x_{1,2},\ \ldots, x_{1,n-1}$$

and their coefficients in row s are

$$1, s, s(s-1), \ldots, s^{n-1}(s-1)^{n-1}, s^n(s-1)^{n-1}, s^n(s-1)^n.$$

It is clear that the κ-th entry in this row is a monic polynomial in s of degree κ, where $0 \le \kappa \le 2n$, and thus s^κ is a linear combination of the first κ entries.

Vandermonde

It follows that the coefficient matrix is a product of the standard Vandermonde matrix multiplied to its right by an upper triangular matrix with all 1's on the diagonal. Hence the matrix is nonsingular, and we can solve the linear system, in particular, to compute c'_{n00} in polynomial time.

Some References

A Complete Dichotomy Rises from the Capture of Vanishing Signatures

`http://arxiv.org/abs/1204.6445` **(53 pages)**

Some other papers can be found on my web site

`http://www.cs.wisc.edu/~jyc`

THANK YOU!