University of BRISTOL

GSE statistics without spin

joint work with

Chris Joyner and Martin Sieber

Sebastian Müller

Spectral statistics

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- background: discrete geometrical symmetries

Quantum graphs

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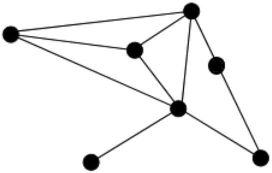

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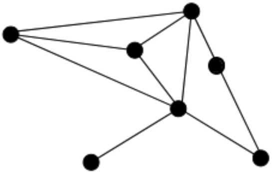

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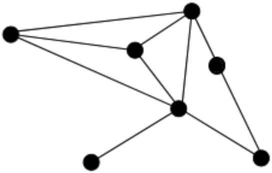

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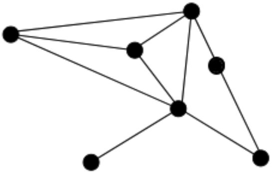

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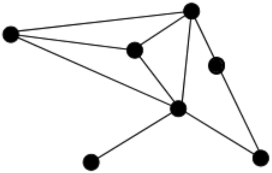

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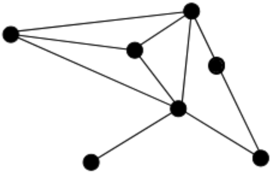

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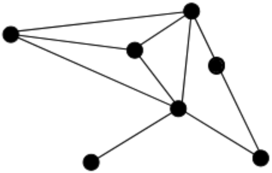

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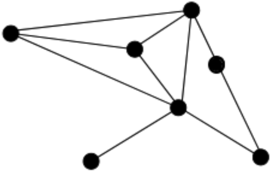

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- if Hamiltonian and vertex conditions symmetric w.r.t. complex conjugation: GOE

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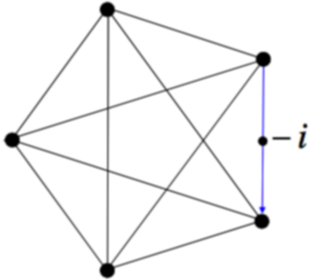
- here time-reversal invariance is broken by a complex phase factor: GUE



Quantum chaos

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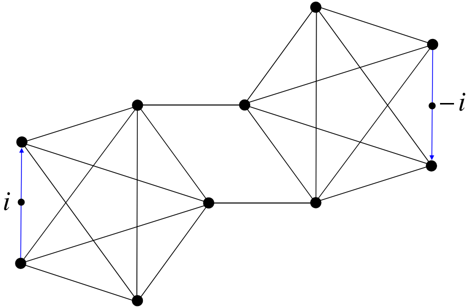

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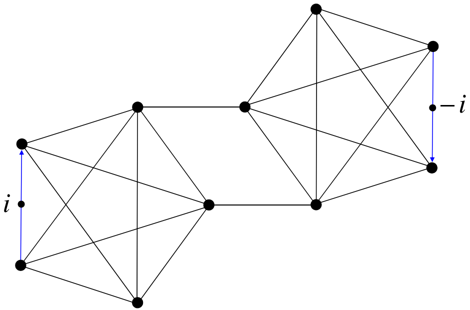

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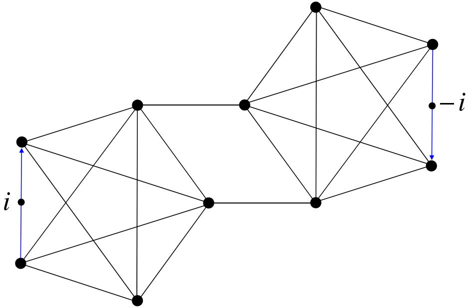

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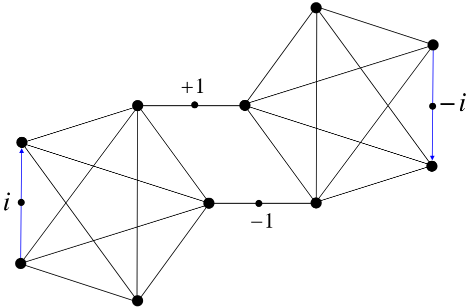

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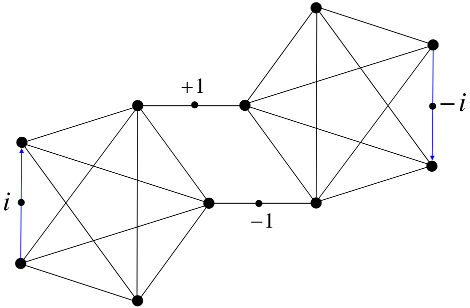

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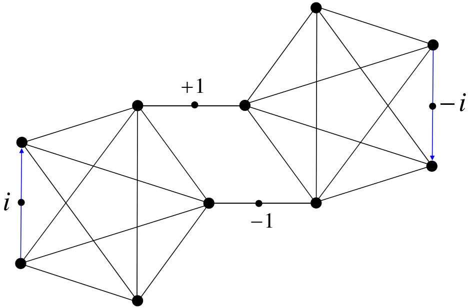

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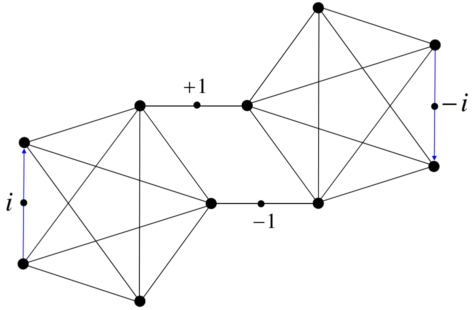

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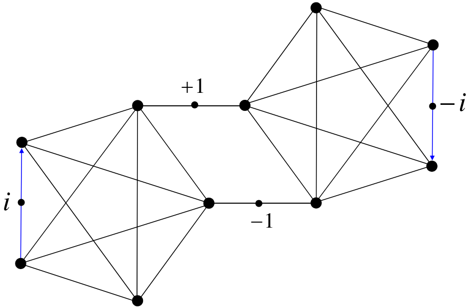

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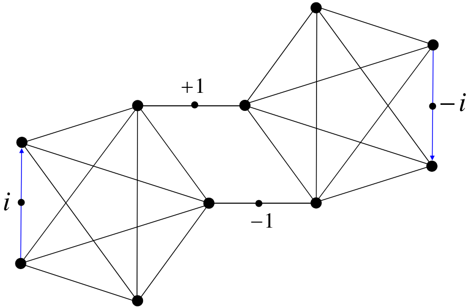

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General approach to symmetries

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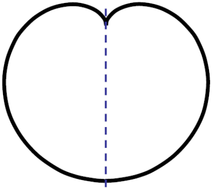

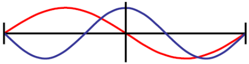

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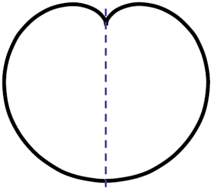

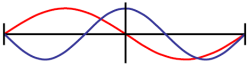

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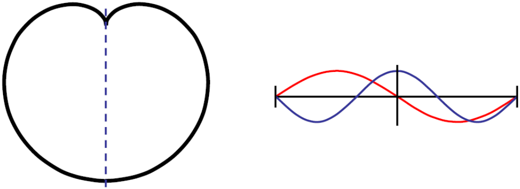

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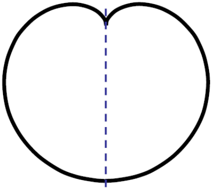

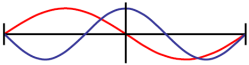

two subspectra:

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Spectral statistics in systems with (discrete) **geometric symmetries**?

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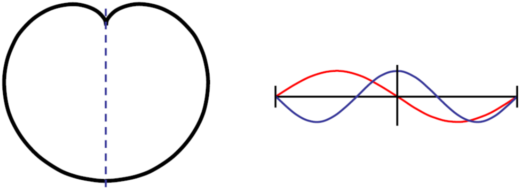

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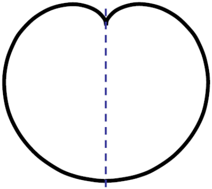

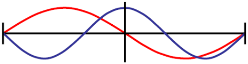

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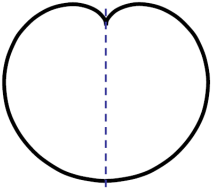

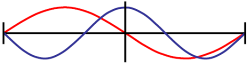

two subspectra:

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General discrete symmetries

- group of **classical** symmetry operations g

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Find a graph whose symmetry group has a pseudo-real representation.

Construction of a GSE quantum graph

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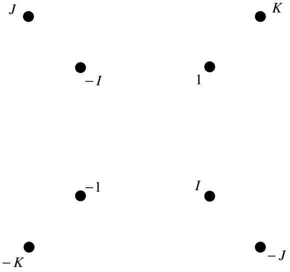

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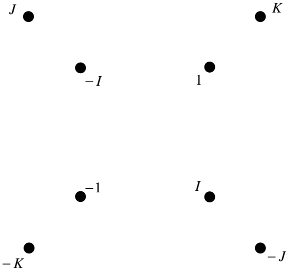

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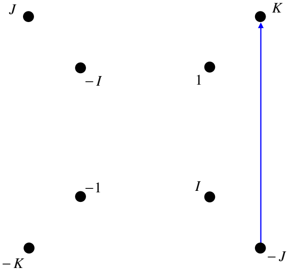

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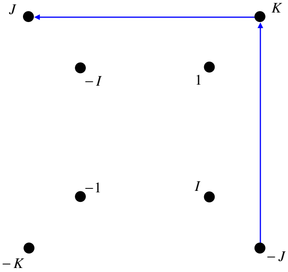

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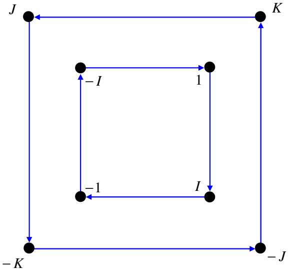

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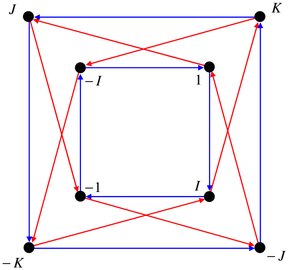

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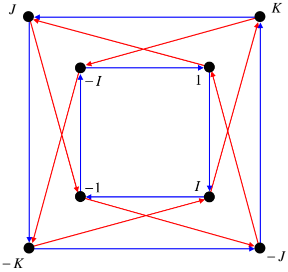

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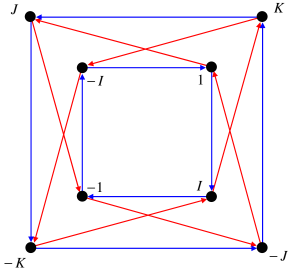

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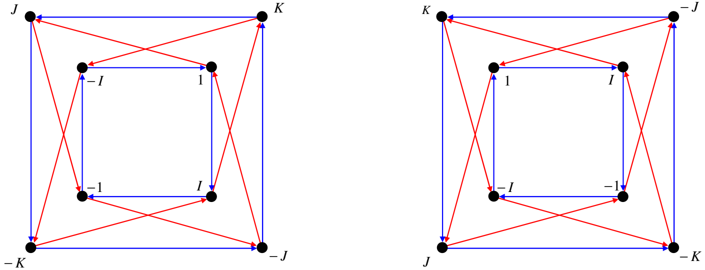

graph symmetric w.r.t. left multiplication with any group element

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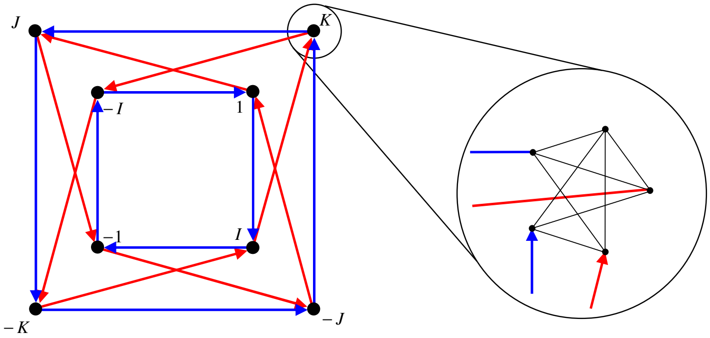

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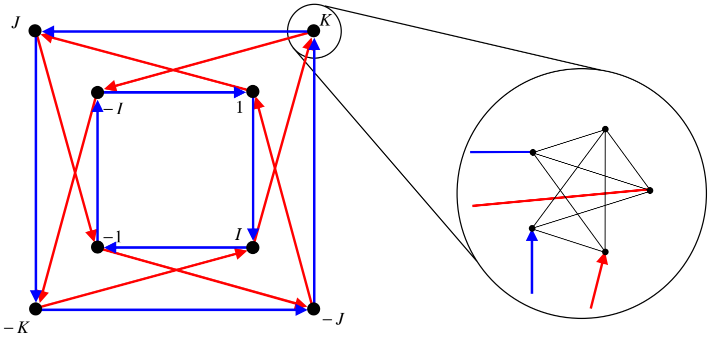

graph with GSE subspectrum

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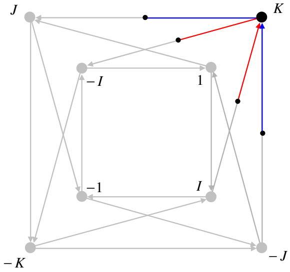

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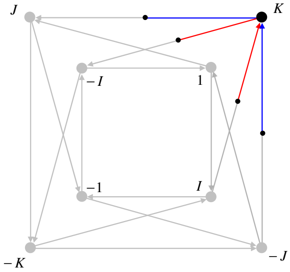

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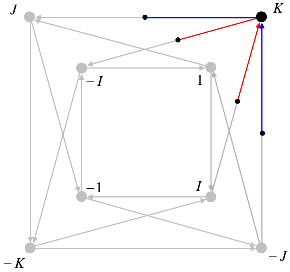

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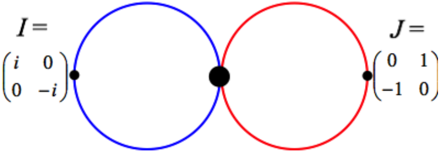

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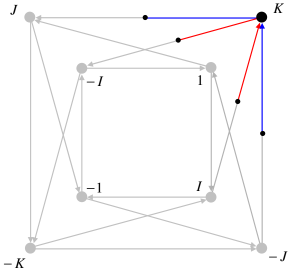

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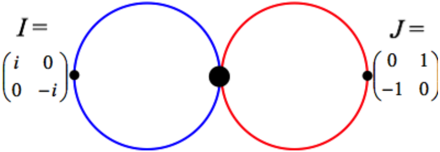

graph with pure GSE statistics

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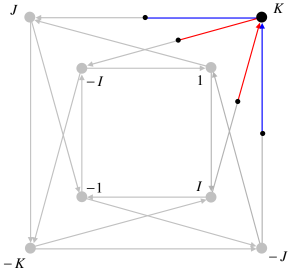

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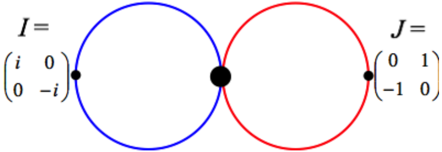
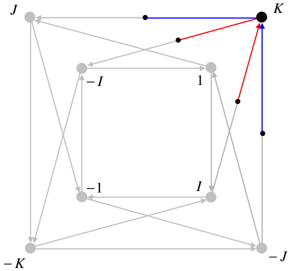


graph with pure GSE statistics

Construction of a GSE quantum graph

- take fundamental domain (eighth of graph)



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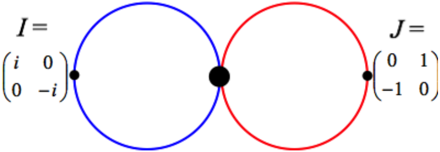

graph with pure GSE statistics

... but boundary conditions mix pairs of degenerate eigenfunctions

Construction of a GSE quantum graph

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- let each of the two eigenfunctions live on a separate copy of the graph

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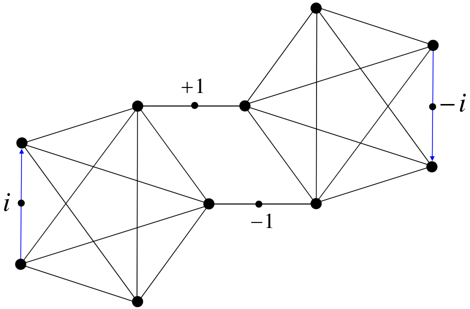

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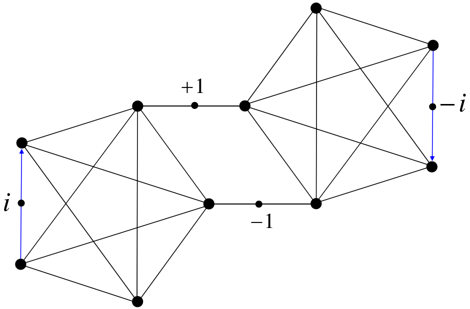

graph with a pure GSE spectrum and no resemblance of spin

Numerical Results

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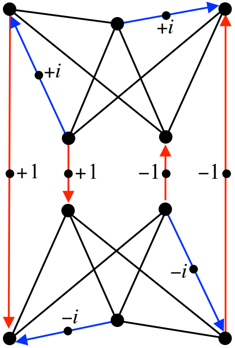
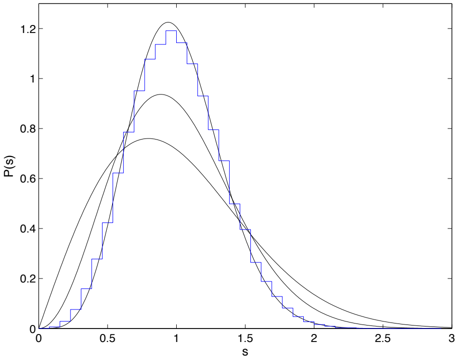
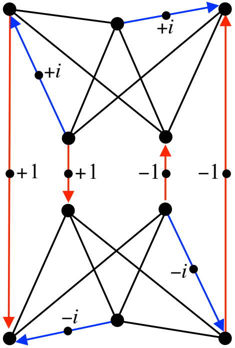
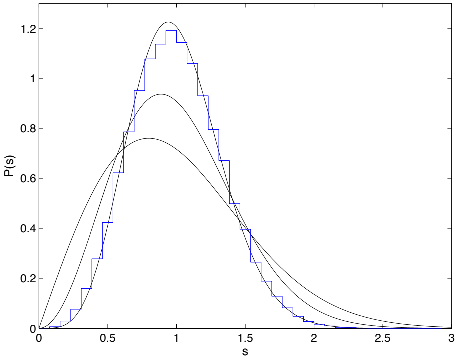




Numerical Results





Agreement with GSE ☺

Conclusions

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- Quantum graph with Q8 symmetry has GSE subspectrum, this can be isolated
- Generalisation to the ’tenfold way’?
- Experimental realisation?