Dependent Random Choice

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The Probabilistic Method

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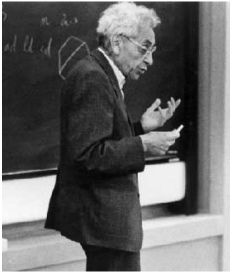

Example: Lower bound on Ramsey numbers

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The Ramsey number $r(n)$ is the minimum N such that every 2-edge-coloring of K_N contains a monochromatic K_n.

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Open Problem

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Progress: Frankl-Wilson, Barak-Rao-Shaltiel-Wigderson, Chattopadhyay-Zuckerman, Cohen

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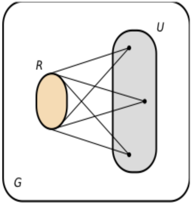

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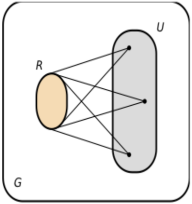

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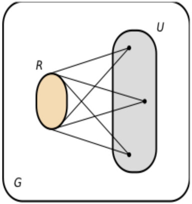
If d vertices have only *few* common neighbors, it is very unlikely that R will be chosen among these neighbors. Hence we do not expect U to contain any such d vertices.



Dependent Random Choice: Applications

Dependent Random Choice has many applications in Ramsey Theory, Extremal Graph Theory, Additive Combinatorics, and Combinatorial Geometry. Example we will cover include:

- Erdős problem on heavy monochromatic cliques
- Conjectures of Hajós and Erdős-Fajtlowicz
- Conjectures of Erdős-Simonovits and Sidorenko
- Burr-Erdős conjecture on Ramsey numbers

Heavy Monochromatic Cliques

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Theorem: (Rödl 2003)

$$c_1 \frac{\log \log \log \log N}{\log \log \log \log \log N} \leq f(N) \leq c_2 \log \log \log N.$$

Upper bound proof: $f(N) = O(\log \log \log N)$

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Color edges inside interval I_j with no monochromatic set of order $2\log 2^{2^j} = 2^{j+1}$. Then I_j contributes at most $2^{j+1}/\log 2^{2^{j-1}} = 4$ to the weight of any monochromatic clique.

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Consider a 2-edge-coloring c of the complete graph on $[s]$ with no monochromatic K_t with $t > 2\log s$. For $j \neq j'$, color the edges from I_j to $I_{j'}$ the color $c(j, j')$.

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Hence,

$$f(N) \leq 4 \cdot 2\log s = 8\log\log\log N.$$

Solved!

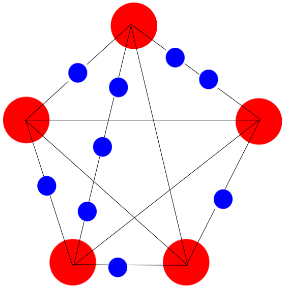
Theorem: (Conlon-F.-Sudakov)

$$f(N) = \Theta(\log \log \log N).$$

That is, every 2-edge-coloring of the complete graph on $\{2, \ldots, N\}$ contains a monochromatic clique S with

$$\sum_{i \in S} \frac{1}{\log i} = \Omega(\log \log \log N).$$

A *subdivision* of a graph is obtained by replacing edges by paths.



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Erdős and Fajtlowicz in 1981 showed that:

almost all graphs are counterexamples!

Erdős-Fajtlowicz theorem

$\sigma(G) =$ maximum t for which G contains a subdivision of K_t.

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The random graph $G = G(n, 1/2)$ almost surely satisfies

$$\chi(G) = \Theta(n/\log n) \qquad \text{and} \qquad \sigma(G) = \Theta(\sqrt{n}).$$

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They further conjectured that

the random graph is essentially the strongest counterexample!

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Theorem: (F.-Lee-Sudakov)

The Erdős-Fajtlowicz conjecture is true.

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(THOMASON 1989) FALSE FOR $H = K_4$.

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Has connections to matrix theory, Markov chains, graph limits, and quasi-randomness.

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Conjecture: (Sidorenko, Erdős-Simonovits 1980s)

For every bipartite graph H and every graph G,

$$t_H(G) \geq t_{K_2}(G)^{e(H)}.$$

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Remark: The expression on the left hand side is quite common. For example, Feynman integrals in quantum field theory, Mayer integrals in statistical mechanics, and multicenter integrals in quantum chemistry are of this form.

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Surprising fact

Quasirandomness follows from (1) for $H = K_2$ and $H = C_4$.

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Provides a stronger stability result for Sidorenko's conjecture.

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Theorem: (Conlon, F., Sudakov 2010)

The forcing conjecture holds for every bipartite H which has two vertices in one part complete to the other part.

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Sidorenko's conjecture holds for every bipartite graph H which has a vertex complete to the other part.

Remarks: Sidorenko's conjecture holds for a large class of graphs. An approximate version holds for all graphs.

THEOREM: (CONLON, F., SUDAKOV 2010)

The forcing conjecture holds for every bipartite H which has two vertices in one part complete to the other part.

Hence, the forcing conjecture holds for a large class of graphs.

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Question (Burr-Erdős 1975)

How large is $r(H)$ for a sparse graph H on n vertices?

Ramsey numbers of bounded degree graphs

Conjecture (Burr-Erdős 1975)

For every d there is a constant c_d such that if a graph H has n vertices and maximum degree d, then

$$r(H) \leq c_d n.$$

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5. (Conlon-F.-Sudakov 2012) $c_d \leq 2^{cd \log d}$.

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3. (Lee 2016+) The Burr-Erdős conjecture is true!

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Simpler proof develops dependent random choice for hypergraphs.