The Graph Regularity Method

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Marston Morse Lecture Series
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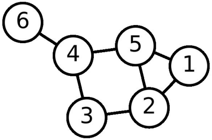
Graphs

Definition (graph)

A *graph* $G = (V, E)$ has a vertex set V and a set E of edges, which are pairs of vertices.



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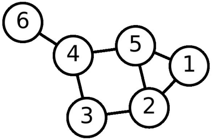

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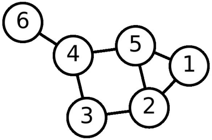

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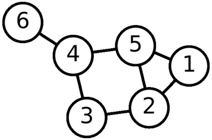

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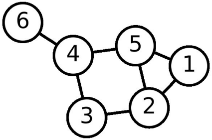

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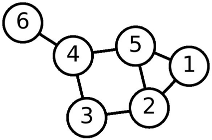

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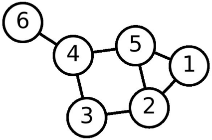

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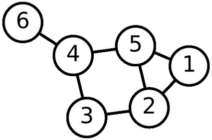

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From *Large Networks and Graph Limits* by Lovász

Szemerédi's Regularity Lemma

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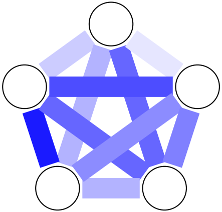
Roughly speaking, every graph can be partitioned into a bounded number of roughly equally-sized parts so that the graph is random-like between almost all pairs of parts.



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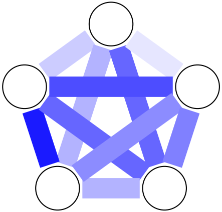

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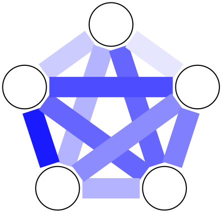

- Rough structural result for all graphs.
- One of the most powerful tool in combinatorics.

Regularity

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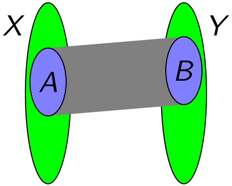
$\mathrm{irreg}(X, Y)$ is the maximum over all $A \subset X$ and $B \subset Y$ of

$$\Big| e(A, B) - d(X, Y)|A||B| \Big|$$



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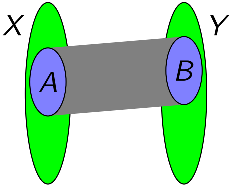

(X, Y) is *ε-regular* if $\operatorname{irreg}(X, Y) \leq \varepsilon |X||Y|$.

Szemerédi's regularity lemma

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Szemerédi's regularity lemma

For every $\varepsilon > 0$, there is an $M(\varepsilon)$ so that every graph G has an ε-regular vertex partition P with at most $M(\varepsilon)$ parts.

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Theorem (F.-L. M. Lovász 2016+)

$$M(\varepsilon) = T(\Theta(\varepsilon^{-2}))$$

Szemerédi's regularity lemma: proof idea

Definition (Mean square density)

For a vertex partition $P : V = V_1 \cup \ldots \cup V_k$,

$$q(P) = \sum_{i,j} p_i p_j d(V_i, V_j)^2$$

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If P with $|P| = k$ is not ε-regular, then there is a refinement P' into at most $k2^{k+1}$ parts such that $q(P') \geq q(P) + \varepsilon^2$.

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At most ε^{-2} iterations before obtaining an ε-regular partition.

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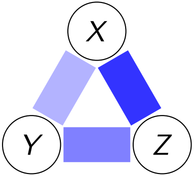
Triangle counting lemma

If each pair of parts is ε-regular, the number of triangles across the three parts is

$$\approx d(X,Y)d(X,Z)d(Y,Z)\,|X|\,|Y|\,|Z|\,.$$



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Triangle removal lemma (Ruzsa-Szemerédi 1976)

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Proof idea:

1. Apply Szemerédi's regularity lemma.
2. Delete edges between pairs which are irregular or sparse.
3. If there is a remaining triangle, then its edges go between pairs which are both dense and regular. The counting lemma then implies that there are more than δn^3 triangles.

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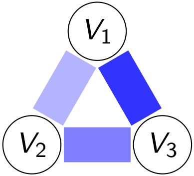
Best known lower bound on δ^{-1} is only $\varepsilon^{-c \log \varepsilon^{-1}}$.

Triangle removal lemma: new proof idea



Key lemma

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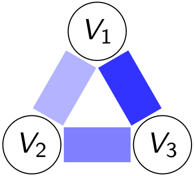

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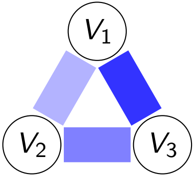

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Definition: Mean entropy density

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$$h(P) = \sum_{i,j} p_i p_j d(V_i, V_j) \log d(V_i, V_j)$$

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Testable graph properties

A graph property P is *testable* if for each $\varepsilon > 0$ there is a randomized algorithm which in constant time (depending on ε and P) distinguishes with probability at least $2/3$ between graphs having P and graphs which are ε-far from having P.

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Theorem (Alon-Fischer-Krivelevich-Szegedy 2000)

Induced H-freeness is testable.

Theorem (Alon-Shapira 2008)

Every hereditary graph property is testable.

Induced graph removal lemma

Lemma (Alon-Fischer-Krivelevich-Szegedy 2000)

For each $\varepsilon > 0$ and graph H on h vertices there is $\delta > 0$ such that every graph on n vertices with at most δn^h induced copies of H can be made induced H-free by adding or removing εn^2 edges.

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Problem (Alon)

Estimate the dependence of δ on ε.

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Proof developed the *strong regularity lemma*, which is proved by repeated use of Szemerédi's regularity lemma and gives a wowzer-type bound.

Wowzer function is given by $W(1) = 2$ and $W(n) = T(W(n-1))$.

Problem (Alon)

Estimate the dependence of δ on ε.

In particular, can the wowzer-type bound be improved?

Wowzer and Tower

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Further, a tower-type bound holds for testing hereditary properties.

Algorithmic regularity lemma

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Theorem (Kohayakawa-Rödl-Thoma)

Can be done in $O(n^2)$ time.

Counting cliques

Algorithmic problem

Count the number of cliques of order k in a graph on n vertices.

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How fast can we approximate the count within an additive εn^k?

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Can use the regularity method!

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Theorem (F.-L. M. Lovász-Zhao 2016)

Can be done in time $O(\varepsilon^{-k^2} n + n^{\omega})$.

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Corollary

We can approximate the count of K_{1000} in a graph on n vertices within an additive $n^{1000-10^{-6}}$ in time $O(n^{2.4})$.

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Theorem (Conlon-F.-Zhao)

A sparse counting lemma in graphs and hypergraphs.

Green–Tao Theorem (2008)

The primes contain arbitrarily long arithmetic progressions.

Examples:

- 3, 5, 7
- 5, 11, 17, 23, 29
- 7, 37, 67, 97, 127, 157
- Longest known: 26 terms

Green–Tao Theorem (2008)

The primes contain arbitrarily long arithmetic progressions (AP).

Szemerédi's Theorem (1975)

Every positive density subset of contains arbitrarily long APs.

(upper) density of $A \subset$ is $\limsup_{N\to\infty} \dfrac{|A \cap [N]|}{N}$

$[N] := \{1, 2, \ldots, N\}$

$P =$ prime numbers

Prime number theorem: $\dfrac{|P \cap [N]|}{N} \sim \dfrac{1}{\log N}$

Proof strategy of the Green–Tao theorem

Step 1:

Relative Szemerédi theorem (informally)

If $S \subset$ satisfies certain pseudorandomness conditions, then every subset of S of relative positive density contains long APs.

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$P =$ prime numbers, $Q =$ "almost primes"

$P \subseteq Q$ with relative positive density, i.e., $\dfrac{|P \cap [N]|}{|Q \cap [N]|} > \delta$

Relative Szemerédi theorem (informally)

If $S \subset$ satisfies certain pseudorandomness conditions, then every subset of S of positive density contains long APs.

What pseudorandomness conditions?

Green–Tao:

1. Linear forms condition
2. Correlation condition

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Does a relative Szemerédi theorem hold with weaker and more natural hypotheses?

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1. Linear forms condition
2. Correlation condition $\leftarrow$ no longer needed

A natural question (asked by Green, Gowers, . . .)

Does a relative Szemerédi theorem hold with weaker and more natural hypotheses?

Our main result

Yes! A weak linear forms condition suffices.

Roth's theorem

Roth's theorem (1952)

If $A \subseteq [N]$ is 3-AP-free, then $|A| = o(N)$.

$[N] := \{1, 2, \ldots, N\}$

3-AP = 3-term arithmetic progression

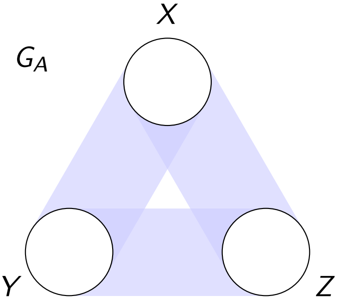
It'll be easier (and equivalent) to work in $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$.

Proof of Roth's theorem

Roth's theorem (1952)

If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Given A, construct tripartite graph G_A with vertex sets $X = Y = Z = \mathbb{Z}_N$.



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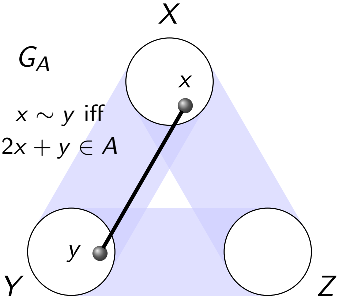

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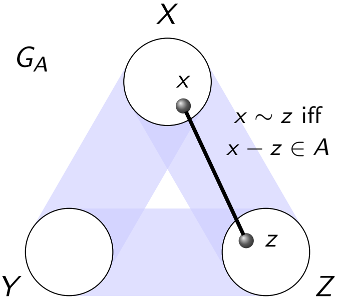

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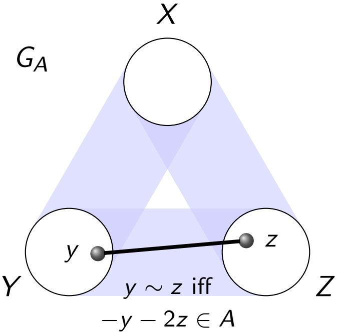

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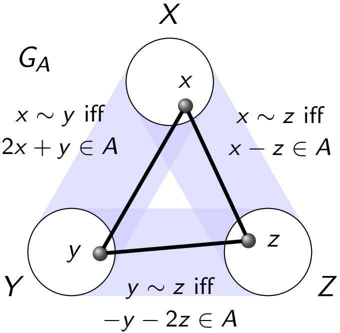

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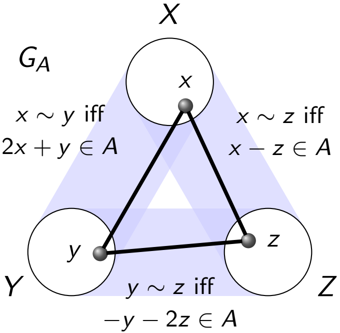

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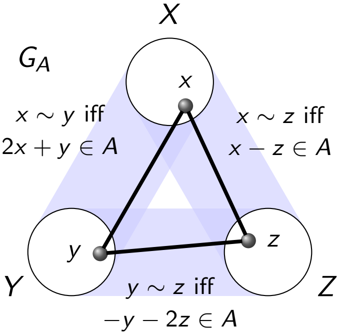

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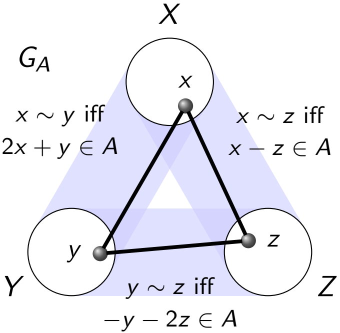

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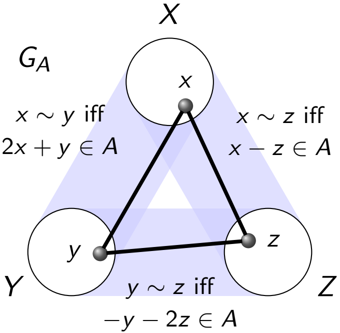

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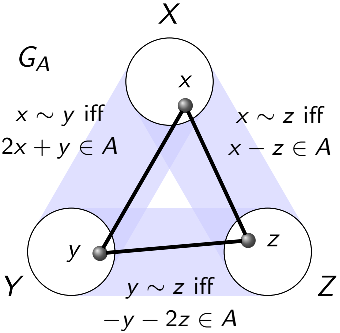

No triangles? Only triangles $\longleftrightarrow$ trivial 3-APs with diff 0.
Every edge of the graph is contained in exactly one triangle (the one with $x + y + z = 0$).

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If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Constructed a graph with

- $3N$ vertices
- $3N|A|$ edges
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Theorem (Ruzsa & Szemerédi '76)

If every edge in a graph $G = (V, E)$ is contained in exactly one triangle, then $|E| = o(|V|^2)$.

(a consequence of the *triangle removal lemma*)

So $3N|A| = o(N^2)$. Thus $|A| = o(N)$.

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If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Relative Roth theorem (Conlon, F., Zhao)

If $S \subseteq \mathbb{Z}_N$ satisfies the 3-linear forms condition,
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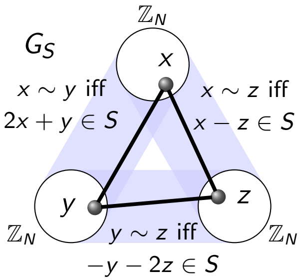

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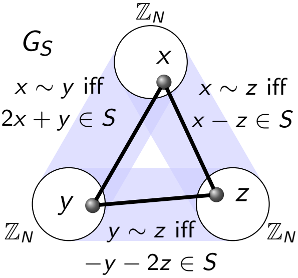
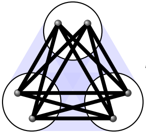
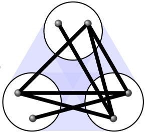


3-linear forms condition:
G_S has asymp. the expected number of embeddings of $K_{2,2,2}$ & its subgraphs (compared to random graph of same density)



$K_{2,2,2}$ & subgraphs, e.g.,



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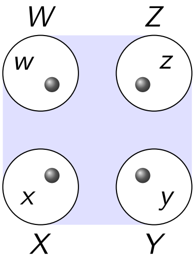
Fix $k \geq 3$. If $S \subseteq \mathbb{Z}_N$ satisfies the k-linear forms condition, and $A \subseteq S$ is k-AP-free, then $|A| = o(|S|)$.

$k = 4$: build a weighted 4-partite 3-uniform hypergraph

Vertex sets $W = X = Y = Z = \mathbb{Z}_N$

- $xyz \in E \iff 3w + 2x + y \in S$
- $wxz \in E \iff 2w + x - z \in S$
- $wyz \in E \iff w - y - 2z \in S$
- $xyz \in E \iff -x - 2y - 3z \in S$



common diff: $-w - x - y - z$

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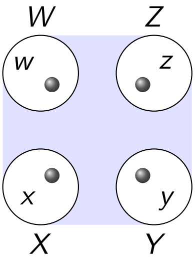

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4-linear forms condition: correct count of the 2-blow-up of the simplex $K_4^{(3)}$ (as well as its subgraphs)

Avoiding regularity

A major drawback of the regularity lemma is the tower dependence.

Problem

Find alternative proofs of the applications which avoid using the regularity lemma and give improved bounds.

Preview of the next lecture

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Green developed an arithmetic regularity lemma and used it to prove the following two extensions of Roth's theorem:

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For each $\varepsilon > 0$ there is $\delta > 0$ such that if G is an abelian group and $A, B, C \subset G$ with at most $\delta|G|^2$ triples $(a, b, c) \in A \times B \times C$ with $a + b + c = 0$, then we can delete $\varepsilon|G|$ elements from A, B, C and get rid of all solutions.

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Green's Roth theorem with popular differences

For each $\varepsilon > 0$ there is $N(\varepsilon)$ such that if G is an abelian group with $|G| \geq N(\varepsilon)$ and $A \subset G$ with $|A| = \alpha|G|$, then there is a nonzero $d \in G$ such that the density of three-term arithmetic progressions with common difference d in A is at least $\alpha^3 - \varepsilon$.

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Next lecture: tight quantitative bounds in vector spaces.