Monotone Lower Bounds via Fourier Analysis

Princeton CCI

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$$\mathsf{NC}^1\subseteq \mathsf{L}\subseteq \mathsf{NL}\subseteq \mathsf{NC}^2\subseteq \mathsf{NC}^3\subseteq \cdots\subseteq \mathsf{NC}\subseteq \mathsf{P}\subseteq \mathsf{NP}$$

$$mNC^1 \subsetneq mL \subsetneq mNL \subseteq mNC^2 \subsetneq mNC^3 \subsetneq \cdots \subsetneq mNC \subsetneq mP \subsetneq mNP$$

$$\begin{array}{lll} \textit{K-Clique} & \textit{m-Circuit Depth} & \textit{m-SN-Size} \\ & [Razborov '85] & \Omega \left(\log^2 n\right) & 2^{\Omega(\log n)} \\ & [Alon-Boppana '87] & n^{\Omega(1)} & 2^{n^{\Omega(1)}} \\ & [Haken '95] & n^{\Omega(1)} & 2^{n^{\Omega(1)}} \\ & [Raz-McKenzie '99] & \Omega(k \log n) & 2^{\sqrt{\Omega(k \log n)}} & _{k \leq n^{O(1)}} \\ & [C-Potechin '12] & \Omega(k \log n) & _{k \leq n^{O(1)}} \end{array}$$

$$mNC^1 \subsetneq mL \subsetneq mNL \subseteq mNC^2 \subsetneq mNC^3 \subsetneq \cdots \subsetneq mNC \subsetneq mP \subsetneq mNP$$

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\begin{array}{lll} & k\text{-}\mathsf{Clique} & \text{$m$-}\mathsf{Circuit} \; \mathsf{Depth} & \text{$m$-}\mathsf{SN} \; \mathsf{Size} \\ & [\mathsf{Razborov} \; "85] & \Omega(\log^2 n) & 2^{\Omega(\log n)} \\ & [\mathsf{Alon-Boppana} \; "87] & n^{\Omega(1)} & 2^{n^{\Omega(1)}} \\ & [\mathsf{Haken} \; "95] & n^{\Omega(1)} & 2^{n^{\Omega(1)}} \\ & [\mathsf{Raz-McKenzie} \; "99] & \Omega(k \; \log n) & 2^{\sqrt{\Omega(k \; \log n)}} & {}_{k \; \leq \; n^{\Omega(1)}} \\ & [\mathsf{C.-Potechin} \; "12] & \Omega(k \; \log n) & {}_{k \; \leq \; n^{\Omega(1)}} \end{array}
```

$$\mathit{m}\mathsf{NC}^1 \subsetneq \mathit{m}\mathsf{L} \subsetneq \mathit{m}\mathsf{NL} \subseteq \mathit{m}\mathsf{NC}^2 \subsetneq \mathit{m}\mathsf{NC}^3 \subsetneq \cdots \subsetneq \mathit{m}\mathsf{NC} \subsetneq \mathit{m}\mathsf{P} \subsetneq \mathit{m}\mathsf{NP}$$

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k-Clique m-Circuit Depth [Razborov '85] \Omega(\log^2 n)
```

$$mNC^1 \subsetneq mL \subsetneq mNL \subseteq mNC^2 \subsetneq mNC^3 \subsetneq \cdots \subsetneq mNC \subsetneq mP \subsetneq mNP$$

 $(C-Potential) = O(k \log n)$ $k < n^{O(1)}$

$$\mathit{m}\mathsf{NC}^1 \subsetneq \mathit{m}\mathsf{L} \subsetneq \mathit{m}\mathsf{NL} \subseteq \mathit{m}\mathsf{NC}^2 \subsetneq \mathit{m}\mathsf{NC}^3 \subsetneq \cdots \subsetneq \mathit{m}\mathsf{NC} \subsetneq \mathit{m}\mathsf{P} \subsetneq \mathit{m}\mathsf{NP}$$

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k-Clique m-Circuit Depth [Razborov '85] \Omega(\log^2 n) [Alon-Boppana '87] n^{\Omega(1)} [Haken '95] n^{\Omega(1)} [Raz-McKenzie '99] \Omega(k \log n) k \leq n^{O(1)}
```

$$m\mathsf{NC}^1\subsetneq m\mathsf{L}\subsetneq m\mathsf{NL}\subseteq m\mathsf{NC}^2\subsetneq m\mathsf{NC}^3\subsetneq\cdots\subsetneq m\mathsf{NC}\subsetneq m\mathsf{P}\subsetneq m\mathsf{NP}$$

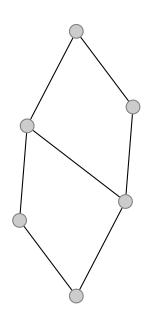
$$\begin{array}{lll} \textit{k-Clique} & \textit{m-Circuit Depth} & \textit{m-SN Size} \\ & [\text{Razborov '85}] & \Omega(\log^2 n) & 2^{\Omega(\log n)} \\ & [\text{Alon-Boppana '87}] & n^{\Omega(1)} & 2^{n^{\Omega(1)}} \\ & & [\text{Haken '95}] & n^{\Omega(1)} & 2^{n^{\Omega(1)}} \\ & & [\text{Raz-McKenzie '99}] & \Omega(k \log n) & 2^{\sqrt{\Omega(k \log n)}} & {}_{k \leq n^{O(1)}} \end{array}$$

$$\mathit{m}\mathsf{NC}^1 \subsetneq \mathit{m}\mathsf{L} \subsetneq \mathit{m}\mathsf{NL} \subseteq \mathit{m}\mathsf{NC}^2 \subsetneq \mathit{m}\mathsf{NC}^3 \subsetneq \cdots \subsetneq \mathit{m}\mathsf{NC} \subsetneq \mathit{m}\mathsf{P} \subsetneq \mathit{m}\mathsf{NP}$$

```
k-Clique
                         m-Circuit Depth
                                               m-SN Size
                       \Omega(\log^2 n)
                                              2\Omega(\log n)
     [Razborov '85]
                                               2^{n^{\Omega(1)}}
                        n^{\Omega(1)}
[Alon-Boppana '87]
                                               2^{n^{\Omega(1)}}
                        n^{\Omega(1)}
        [Haken '95]
                        \Omega(k \log n) \quad 2^{\sqrt{\Omega(k \log n)}}
                                                                       k < n^{O(1)}
[Raz-McKenzie '99]
                                             2^{\Omega(k \log n)}
                       \Omega(k \log n)
                                                                       k < n^{O(1)}
  [C.-Potechin '12]
```

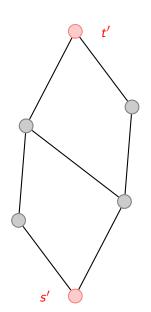
Problem

```
f: \{0,1\}^n \to \{0,1\}
input size n
variables x_i i \in [n]
literals x_i, \bar{x}_i i \in [n]
```



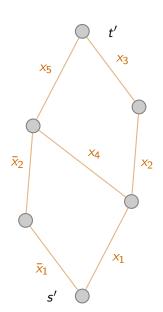
SN Model

 $\mathsf{undirected} \quad \mathit{G'} = (\mathit{V'}, \mathit{E'})$



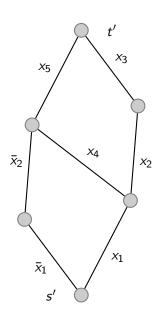
SN Model

undirected G' = (V', E')source $s' \in V'$ sink $t' \in V'$



SN Model

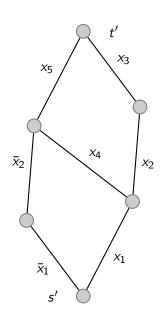
undirected G' = (V', E')source $s' \in V'$ sink $t' \in V'$ labeling $\lambda' : E' \rightarrow$ literals



Computation

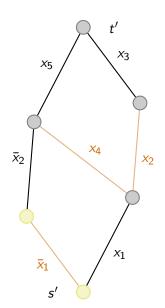
1) $x \in \{0,1\}^n$ reaches $a' \in V'$ \updownarrow

 \exists path P' connecting s' and a' labeled with literals in x.



Computation

- 1) $x \in \{0,1\}^n$ reaches $a' \in V'$ \updownarrow
 - \exists path P' connecting s' and a' labeled with literals in x.
- 2) $x \in \{0,1\}^n$ is accepted by G' $\downarrow \\ x \in \{0,1\}^n \text{ reaches } t'$



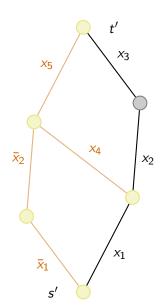
Computation

1) $x \in \{0,1\}^n$ reaches $a' \in V'$ \updownarrow

 \exists path P' connecting s' and a' labeled with literals in x.

Instance 1

x = 0 1 0 1 0 $x_1 = x_2 = x_3 = x_4 =$



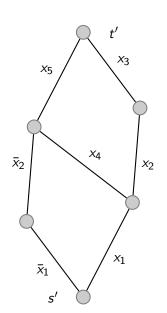
Computation

1) $x \in \{0,1\}^n$ reaches $a' \in V'$ \updownarrow

 \exists path P' connecting s' and a' labeled with literals in x.

Instance 2

$$x = 0$$
 0 0 1 1 $x_1 = x_2 = x_3 = x_4 = x_5$



Computation

1) $x \in \{0,1\}^n$ reaches $a' \in V'$ \$ \$ \$ path P' connecting s' and a'

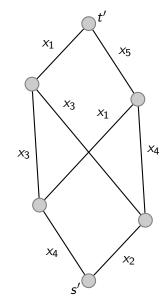
labeled with literals in x.

2) Reachability table $R_{a'}$ for $a' \in V'$ $R_{a'}(x) = \text{True} \text{ if } x \text{ reaches } a'.$ $R_{t'}(x) = f(x)$

Monotone SN

A switching network G' labeled only with *positive* literals. $\lambda' \colon E' \to \text{positive}$ literals.

⇒ Computes only monotone functions.

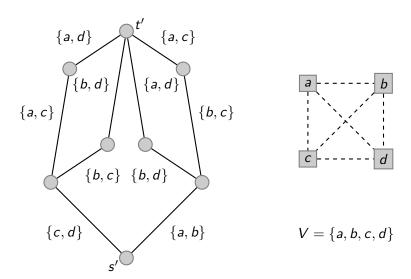


k-CLIQUE Problem

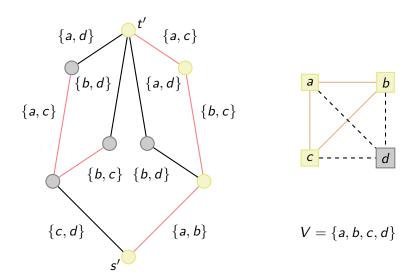
Input a set of edges $E \subseteq \binom{V}{2}$

Output if the graph G = (V, E) has a k-clique P

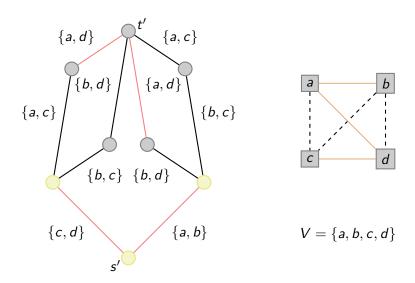
i.e. $P \subseteq V, |P| = k$ $\forall u \neq v \in P, \{u, v\} \in E.$



k = 3



k = 3



k = 3

Monotone SN for k-CLIQUE

has size $n^{O(k)}$ when $k \leq n^{O(1)}$

Monotone SN for k-CLIQUE

has size $n^{O(k)}$ and $n^{\Omega(k)}$ when $k \leq n^{O(1)}$

 β : k-cliques \rightarrow nodes on SN

$$\forall v' \text{ on SN} \quad |\beta^{-1}(v')| \leq \text{few } k\text{-cliques}$$

$$\# k$$
-cliques $= n^{\Theta(k)}$

$$\#$$
nodes on SN $\geq n^{\Theta(k)}$

 β : k-cliques \rightarrow nodes on SN

 $|\forall v' \text{ on SN} \quad |\beta^{-1}(v')| \leq \text{few } k\text{-cliques}$

$$\#k$$
-cliques = $n^{\Theta(k)}$

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 $\beta \colon k$ -cliques \to nodes on SN

$$\forall v'$$
 on SN $|\beta^{-1}(v')| \leq \text{few } k\text{-cliques}$

$$\#k$$
-cliques = $n^{\Theta(k)}$

#nodes on $\mathsf{SN} \geq n^{\Theta(k)}$

$$\beta \colon k\text{-cliques} \to \mathsf{nodes} \ \mathsf{on} \ \mathsf{SN}$$

$$\forall v'$$
 on SN $|\beta^{-1}(v')| \leq \text{few } k\text{-cliques}$

$$\#k$$
-cliques = $n^{\Theta(k)}$

$$\#$$
nodes on $\mathsf{SN} \geq n^{\Theta(k)}$ \square

min terms

k-clique P $P \subseteq V . |P| = k$

No instances

structured

$$(k-1)$$
-colorable C
 $C: V \rightarrow [k-1]$

 $\{u, v\} \in \text{instance}(C)$ \updownarrow $C(u) \neq C(v)$

No instances

k-clique P $P \subseteq V, |P| = k$ (k-1)-colorable C $C: V \rightarrow [k-1]$

 $\{u, v\} \in \text{instance}(P)$ \emptyset $u \in P, v \in P$ $\{u,v\} \in \mathsf{instance}(C)$ \updownarrow $C(u) \neq C(v)$

k-clique P $P \subseteq V, |P| = k$

 $\{u,v\} \in \mathsf{instance}(P)$ \emptyset $u \in P, v \in P$

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min terms

$$k$$
-clique P
 $P \subseteq V, |P| = k$

$$\{u, v\} \in \mathsf{instance}(P)$$

$$\downarrow \downarrow u \in P, v \in P$$

No instances

$$(k-1)$$
-colorable C
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$$\{u, v\} \in \text{instance}(C)$$

$$\updownarrow$$

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min terms

$$k$$
-clique P
 $P \subseteq V, |P| = k$

$$\{u, v\} \in \mathsf{instance}(P)$$

$$\downarrow u \in P, v \in P$$

No instances

(k-1)-colorable C $C: V \rightarrow [k-1]$

$$\{u, v\} \in \text{instance}(C)$$

$$\updownarrow$$

$$C(u) \neq C(v)$$

min terms

$$k$$
-clique P
 $P \subseteq V, |P| = k$

$$\{u, v\} \in \mathsf{instance}(P)$$

$$\downarrow \downarrow u \in P, v \in P$$

No instances

$$(k-1)$$
-colorable C
 $C: V \rightarrow [k-1]$

 $\{u,v\} \in \operatorname{instance}(C)$ \updownarrow $C(u) \neq C(v)$

YES instances

min terms

No instances

k-clique P

structured

 $P \subset V, |P| = k$

(k-1)-colorable C $C: V \rightarrow [k-1]$

 $\{u,v\} \in \operatorname{instance}(P)$

 $u \in P$. $v \in P$

 $\{u,v\}\in \mathrm{instance}(C)$

 $C(u) \neq C(v)$

All
$$(k-1)$$
-colorings of V (\cong structured No instances) $\mathcal{C} \stackrel{\mathsf{def}}{=} \big\{ C \colon V \to [k-1] \big\} \cong [k-1]^V$

C-vector

$$g\colon \mathcal{C} o \mathbb{C}$$
 a complex vector indexed by \mathcal{C}

e.g.
$$R_{a'}$$
 is a C -vector for any a' on SN

Inner Product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \mathbb{E} \big[f(C) \overline{g}(C) \big] \stackrel{\text{def}}{=} \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} f(C) \overline{g}(C)$$

verage reachability over NO instances

All
$$(k-1)$$
-colorings of V (\cong structured No instances) $\mathcal{C} \stackrel{\mathsf{def}}{=} \big\{ \mathit{C} : V \to [k-1] \big\} \cong [k-1]^V$

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 $g\colon \mathcal{C} o \mathbb{C}$ a complex vector indexed by \mathcal{C}

 $g\in\mathbb{C}^{\mathcal{C}}$

e.g. $R_{a'}$ is a C-vector for any a' on SN

Inner Product



 $_{
m i}$ verage reachability over $_{
m NO}$ instances

All
$$(k-1)$$
-colorings of V (\cong structured No instances) $\mathcal{C} \stackrel{\mathsf{def}}{=} \big\{ \mathit{C} : V \to [k-1] \big\} \cong [k-1]^V$

C-vector

 $g\colon \mathcal{C} o \mathbb{C}$ a complex vector indexed by \mathcal{C}

e.g. $R_{a'}$ is a C-vector for any a' on SN

 $g\in\mathbb{C}^{\mathcal{C}}$

Inner Product

 $\langle f, g \rangle \stackrel{\text{def}}{=} \mathbb{E} \big[f(C) \overline{g}(C) \big] \stackrel{\text{def}}{=} \underbrace{\frac{1}{|C|} \sum_{C \in C} f(C) \overline{g}(C)}_{C}$

All
$$(k-1)$$
-colorings of V (\cong structured No instances) $\mathcal{C} \stackrel{\mathsf{def}}{=} \big\{ \mathit{C} \colon \mathit{V} \to [k-1] \big\} \cong [k-1]^{\mathit{V}}$

C-vector

$$g\colon \mathcal{C} o \mathbb{C}$$
 a complex vector indexed by \mathcal{C}

e.g.
$$R_{a'}$$
 is a $\mathcal{C}\text{-vector}$ for any a' on SN

Inner Product

$$\langle f, g \rangle \stackrel{\mathsf{def}}{=} \mathbb{E} \big[f(C) \overline{g}(C) \big]$$

 $g\in\mathbb{C}^{\mathcal{C}}$

$$f,g\in\mathbb{C}^{\mathcal{C}}$$

All
$$(k-1)$$
-colorings of V (\cong structured No instances) $\mathcal{C} \stackrel{\mathsf{def}}{=} \big\{ \mathit{C} \colon \mathit{V} \to [k-1] \big\} \cong [k-1]^{\mathit{V}}$

C-vector

 $g\colon \mathcal{C} o \mathbb{C}$ a complex vector indexed by \mathcal{C}

e.g. $R_{a'}$ is a C-vector for any a' on SN

Inner Product

$$\langle f, g \rangle \stackrel{\mathsf{def}}{=} \mathbb{E} \big[f(C) \overline{g}(C) \big] \stackrel{\mathsf{def}}{=} \underbrace{\frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} f(C) \overline{g}(C)}_{\mathsf{average reachability over No instances}}$$

 $g\in\mathbb{C}^{\mathcal{C}}$

 $f,g\in\mathbb{C}^{\mathcal{C}}$

Fourier Analysis over $\mathbb{C}^{\mathcal{C}}$

Fourier Basis $\chi_U(C) \stackrel{\text{def}}{=} \omega^{\sum_{v \in V} U(v)C(v)}$

$$\chi_U \in \mathbb{C}^{\mathcal{C}}$$
 for $U \in \mathcal{C}$

Primitive Root of Unity

$$\omega \stackrel{\text{def}}{=} e^{2\pi i/(k-1)}$$

$$(k-1)^{\rm st}$$
 root

$$\{\chi_{U}\}_{U \in \mathcal{C}} \text{ is Orthonormal}$$

$$\langle \chi_{U}, \chi_{W} \rangle = \begin{cases} 1 & \text{if } U = W \\ 0 & \text{if } U \neq W \end{cases} \text{ for } U, W \in \mathcal{C}$$

Fourier Analysis over $\mathbb{C}^{\mathcal{C}}$

Fourier Basis

$$\chi_U(C) \stackrel{\mathsf{def}}{=} \omega^{\sum_{v \in V} U(v)C(v)}$$

$$\chi_U \in \mathbb{C}^{\mathcal{C}}$$
 for $U \in \mathcal{C}$

Primitive Root of Unity

$$\omega \stackrel{\mathsf{def}}{=} e^{2\pi i/(k-1)}$$

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$$\{\chi_U\}_{U\in\mathcal{C}}$$
 is Orthonormal

$$\langle \chi_U, \chi_W \rangle = egin{cases} 1 & ext{if } U = W \\ 0 & ext{if } U
eq W \end{cases} \quad ext{for } U, W \in \mathcal{C}$$

Recall: Bottleneck Argument

$$eta\colon k ext{-cliques} o \mathsf{nodes} \;\mathsf{on} \;\mathsf{SN}$$
 $orall v' \;\mathsf{on} \;\mathsf{SN} \quad |eta^{-1}(v')| \leq \mathsf{few} \; k ext{-cliques}$
 $\# k ext{-cliques} = n^{\Theta(k)}$
 $\# \mathsf{nodes} \;\mathsf{on} \;\mathsf{SN} \geq n^{\Theta(k)} \;\; \square$

1. (non-negligible work)

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in F$
 $\hat{g}_P(U) = 0$ if $suppU \not\subseteq F$

3. ((k-1)-wise ind.) g_{p} is (k-1)-wise independent.

$$\|g_P\| \le k^k$$

$$\forall$$
 k-Clique P, construct $g_P \colon \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) \forall k-Clique P,

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\Big(rac{1}{|G'|}\Big)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
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3. ((k-1)-wise ind.)

$$\|g_P\| \leq k^{l}$$

$$\forall$$
 k-Clique P, construct $g_P \colon \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \ge \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in F$
 $\hat{g}_P(U) = 0$ if $suppU \not\subseteq F$

3. ((k-1)-wise ind.)

$$\langle g_P,f
angle =0$$
 if f is a $(k-1)$ -junta $\hat{g}_P(U)=0$ if $|supp(U)|\leq k-1$

4 (short)

$$\|g_P\| \le k'$$

$$\forall$$
 k-Clique P, construct $g_P \colon \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) gp depends only on coloring on F

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in F$
 $\hat{g}_P(U) = 0$ if $\sup U \not\subseteq F$

3. ((k-1)-wise ind.)

$$||g_P|| \leq k^k$$

$$\forall$$
 k-Clique P, construct $g_P \colon \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in F$
 $\hat{g}_P(U) = 0$ if $\sup_{u \in F} U \not\subseteq F$

3. ((k-1)-wise ind.)

$$\langle g_P,f
angle=0$$
 if f is a $(k-1)$ -junta $\hat{g}_P(U)=0$ if $|supp(U)|\leq k-1$

$$\|g_P\| \le k^k$$

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in P$ (a)

 $\hat{g}_P(U) = 0 \qquad \qquad \text{if } supp U \not\subseteq P$

3. ((k-1)-wise ind.)

$$\langle g_P, f \rangle = 0$$
 if f is a $(k-1)$ -junta $\hat{\sigma}_{-}(I) = 0$ if $|c_{PP}(I)| < k-1$

 $g_P(U) = 0$ If $|supp(U)| \le \kappa - 1$

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in P$ (a)

$$\hat{g}_P(U) = 0$$
 if $supp U \not\subseteq P$ (b)

3. ((k-1)-wise ind.)

$$\langle g_P,f
angle =0$$
 if f is a $(k-1)$ -junta $\hat{g}_P(U)=0$ if $|supp(U)|\leq k-1$

$$||g_P|| \leq k^k$$

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in P$ (a)

$$\hat{g}_P(U) = 0$$
 if $supp U \not\subseteq P$ (b)

3. ((k-1)-wise ind.) g_P is "(k-1)-wise independent"

$$egin{aligned} \langle g_P,f
angle &=0 & ext{if } f ext{ is a } (k-1) ext{-junta} \ \hat{g}_P(U) &=0 & ext{if } |supp(U)| \leq k-1 \end{aligned}$$

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in P$ (a)

$$\hat{g}_P(U) = 0$$
 if $supp U \not\subseteq P$ (b)

3. ((k-1)-wise ind.) g_P is "(k-1)-wise independent"

$$\langle g_P,f\rangle=0$$
 if f is a $(k-1)$ -junta (a)

4 (short)

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

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$$||g_P|| \leq k^k$$

1. (non-negligible work) $\ \ \forall \ \ k ext{-Clique }P ext{,} \quad \exists b_P'\in G' \ \text{s.t.}$

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in P$
 $\hat{g}_P(U) = 0$ if $supp U \not\subset P$

3. ((k-1)-wise ind.)
$$g_P$$
 is " $(k-1)$ -wise independent"

3.
$$((k-1)$$
-wise ind.) gp is $(k-1)$ -wise independent

$$\langle g_P,f
angle = 0$$
 if f is a $(k-1)$ -junta (a) $\hat{g}_P(U) = 0$ if $|supp(U)| \le k-1$ (b)

(a) (b)

$$\langle g_P,g_P
angle^{1/2}\stackrel{\mathsf{def}}{=}\|g_P\|\leq k^k$$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal

 $\Rightarrow \{g_P/\|g_P\|\}$ is orthonormal

$$orall a' \in G'$$
 $1 \geq \langle R_{a'}, R_{a'}
angle \geq \sum_{P} \left| \left\langle R_{a'}, rac{g_P}{\|g_P\|}
ight
angle
ight|^2$

Size of SN
$$= |G'| = \sum_{\sigma' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{P}{2}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq \sum_{P \in \binom{n}{k}} \left| \left\langle R_{b_p}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

for
$$k \leq n^{O(1)}$$

(2) & (3)
$$\Rightarrow$$
 $\{g_P\}$ is orthogonal
$$\Rightarrow \{g_P/\|g_P\|\}$$
 is orthonormal

$$\forall s' \in G' \qquad 1 \geq \langle R_{s'}, R_{s'} \rangle \geq \sum_{P} \left| \left\langle R_{s'}, rac{g_P}{\|g_P\|}
ight
angle
ight|$$

Size of
$$\mathsf{SN} = |G'| = \sum_{a' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|$$

$$\geq \sum_{P \in \binom{r}{k}} \left| \left\langle R_{b_P'}, \frac{g_P}{\|g_P\|} \right\rangle \right|$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

$$|G'|^3 \ge \left(\frac{n}{k^2}\right)^k \ge n^{\Omega(k)}$$
 for $k \le n^{O(1)}$

(2) & (3)
$$\Rightarrow$$
 $\{g_P\}$ is orthogonal
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 is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq \sum_{P \in \binom{n}{k}} \left| \left\langle R_{b'_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal \Rightarrow { $g_P/||g_P||$ } is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of SN =
$$|G'| = \sum_{a' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

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$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{|G'|^2}$$

 $|G'|^3 \ge \left(\frac{n}{\ell^2}\right)^k \ge n^{\Omega(k)}$ for $k \le n^{O(1)}$

(2) & (3)
$$\Rightarrow$$
 $\{g_P\}$ is orthogonal
$$\Rightarrow \{g_P/\|g_P\|\}$$
 is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of
$$SN = |G'|$$

$$\geq \sum_{P \in \binom{r}{k}} \left| \left\langle R_{b_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|$$

$$\geq \sum_{P \in \binom{r}{k}} \left| \left\langle R_{b_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|$$

$$|G'|^3 \ge \left(\frac{n}{\nu^2}\right)^k \ge n^{\Omega(k)}$$
 for $k \le n^{O(1)}$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal
$$\Rightarrow$$
 { $g_P/\|g_P\|$ } is orthonormal

$$\forall a' \in G' \qquad 1 \ge \langle R_{a'}, R_{a'} \rangle \ge \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of SN =
$$|G'| = \sum_{r' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|$$

$$\geq \sum_{P \in \binom{n}{k}} \left| \left\langle R_{b'_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal \Rightarrow { $g_P/\|g_P\|$ } is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of
$$\mathsf{SN} = |G'| = \sum_{\mathsf{a}' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{b}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal
$$\Rightarrow$$
 { $g_P/\|g_P\|$ } is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of SN =
$$|G'| = \sum_{a' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq \sum_{P \in \binom{n}{k}} \left| \left\langle R_{b'_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

 $\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal \Rightarrow { $g_P/||g_P||$ } is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of SN =
$$|G'| = \sum_{a' \in G'} 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq \sum_{P \in \binom{n}{k}} \left| \left\langle R_{b'_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

 $G'|^3 \ge \left(\frac{n}{L^2}\right)^n \ge n^{\Omega(k)}$ for $k \le n^{O(1)}$

(2) & (3)
$$\Rightarrow$$
 { g_P } is orthogonal

$$\Rightarrow \{g_P/\|g_P\|\}$$
 is orthonormal

$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

Size of
$$\mathsf{SN} = |\mathsf{G}'| = \sum 1$$

$$\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{k}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq \sum_{P \in \binom{n}{k}} \left| \left\langle R_{b'_P}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

$$\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$$

$$|G'|^3 \ge \left(\frac{n}{k^2}\right)^k = n^{3/2} \left(\frac{n}{k^2}\right)^k$$

(2) & (3)
$$\Rightarrow$$
 $\{g_P\}$ is orthogonal \Rightarrow $\{g_P/\|g_P\|\}$ is orthonormal

$$a' \in G'$$
 $1 \geq \langle R_{\mathsf{a}'}, R_{\mathsf{a}'} \rangle \geq \sum_{i} \left| \left\langle R_{\mathsf{a}'}, \frac{\mathsf{g}_P}{\|\varphi_P\|} \right\rangle \right|^2$

$$G'$$
 $1 \geq \langle R_{\mathsf{a}'}, R_{\mathsf{a}'} \rangle \geq \sum_{P} \left| \left\langle R_{\mathsf{a}'}, rac{g_P}{\|g_P\|}
ight
angle$ $N = |G'| = \sum_{P} 1$

Size of
$$SN = |G'| = \sum_{f \in G'} 1$$

$$\mathbb{N} = |G'| = \sum_{a' \in G'} 1$$

 $\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{L}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$

 $\geq \sum_{P \in \binom{n}{\iota}} \left| \left\langle R_{b_P'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$

 $|G'|^3 \ge \left(\frac{n}{k^2}\right)^k \ge n^{\Omega(k)}$ for $k \le n^{O(1)}$

 $\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$

$\forall a' \in G'$	$1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left \left\langle R_{a'}, \cdot \right \right $
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(2) & (3)
$$\Rightarrow$$
 $\{g_P\}$ is orthogonal \Rightarrow $\{g_P/\|g_P\|\}$ is orthonormal

Size of SN = $|G'| = \sum 1$

$$\Rightarrow \{g_P/\|g_P\|\} \text{ is orthonormal}$$

$$\leq G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{|A|} \left| \langle R_{a'}, \frac{g_P}{\|g_P\|} \right|$$

$$\leq G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right| \right|$$

 $\geq \sum_{a' \in G'} \sum_{P \in \binom{n}{L}} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$

 $\geq \sum_{P \in \binom{n}{L}} \left| \left\langle R_{b_P'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$

 $|G'|^3 \ge \left(\frac{n}{L^2}\right)^k \ge n^{\Omega(k)}$ for $k \le n^{O(1)}$

 $\geq n^k \cdot \frac{1}{|G'|^2} \cdot \frac{1}{k^{2k}}$

$$\forall a' \in G' \qquad 1 \ge \langle R_{a'}, R_{a'} \rangle \ge \sum_{P} \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

1. (non-negligible work) \forall k-Clique P, $\exists b_P' \in G'$ s.t.

$$\langle g_P, R_{b_P'} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2)$$
 if $C_1(u) = C_2(u)$ $\forall u \in P$ (a) $\hat{g}_P(U) = 0$ if $suppU \not\subseteq P$ (b)

3. ((k-1)-wise ind.) g_P is "(k-1)-wise independent"

$$egin{aligned} \langle g_P,f
angle &=0 & ext{if f is a $(k-1)$-junta} \end{aligned} \qquad ext{(a)} \ \hat{g}_P(U) &=0 & ext{if $|supp(U)| $\leq k-1$} \end{aligned} \qquad ext{(b)}$$

4 (short)

$$\langle g_P, g_P \rangle^{1/2} \stackrel{\mathsf{def}}{=} \|g_P\| \le k^k$$

 \forall k-Clique P in a NW design, construct $g_P : \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) \forall k-Clique P in a NW design, $\exists b_P' \in G'$ s.t.

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 if $supp U \not\subseteq P$ (b)

3. ((k-2)-wise ind.) g_P is "(k-2)-wise independent"

$$\langle g_P, f \rangle = 0$$
 if f is a $(k-2)$ -junta

$$\hat{g}_P(U) = 0$$
 if $|supp(U)| \le k - 2$ (b)

(a)

$$\langle g_P, g_P \rangle^{1/2} \stackrel{\mathsf{def}}{=} \|g_P\| \le k^k$$

Invariant Cover

Definition (Invariant)

For a literal ℓ (edge $\{u,v\}$), for a m-k-Clique SN G'

$$g$$
 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant on G

Definition (Invariant Cover)

Lemma (Discrepancy in Progress)

Definition (Invariant)

For a literal ℓ (edge $\{u,v\}$),

```
g is \ell-invariant on G' if \forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle g is \ell-invariant on G if \forall G', g is \ell-invariant on G
```

Definition (Invariant Cover)

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

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g is \ell-invariant on G' if \forall a' \stackrel{\iota}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle g is \ell-invariant on G
```

Definition (Invariant Cover)

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Definition (Invariant Cover)

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For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

g is
$$\ell$$
-invariant on G' if $\forall a' \stackrel{\ell}{-} b'$,

g is ℓ -invariant $\qquad \qquad \qquad ext{if } orall G', \quad g$ is ℓ -invariant on G

Definition (Invariant Cover)

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For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

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Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$ is an invariant cover if

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

$$g$$
 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant if $\forall G', \quad g$ is ℓ -invariant on G'

Definition (Invariant Cover)

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Definition (Invariant)

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Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

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 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant if $\forall G', \quad g$ is ℓ -invariant on G'

Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$ is an invariant cover if

- **I.** $\forall \{u,v\} \in G(P) \quad \exists v \in P \quad g_{P,v} \text{ is } \{u,v\}\text{-invariant}$
- **II.** $\forall v \in P$ $\langle 1, g_{P,v} \rangle = \langle \chi_{\emptyset}, g_{P,v} \rangle = \hat{g}_{P,v}(\emptyset) = 1$

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

$$g$$
 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant if $\forall G', \quad g$ is ℓ -invariant on G'

Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,\nu}\}_{\nu\in P}$ is an invariant cover if

- 1. $\forall \{u,v\} \in G(P)$
- II. $\forall v \in P$ $\langle 1, g_{P,v} \rangle = \langle \chi_{\emptyset}, g_{P,v} \rangle = \hat{g}_{P,v}(\emptyset) = 1$

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

$$g$$
 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant if $\forall G', \quad g$ is ℓ -invariant on G'

Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$ is an invariant cover if

- I. $\forall \{u,v\} \in G(P)$ $\exists v \in P$
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Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

$$\begin{array}{ll} \textit{g} \ \text{is} \ \ell\text{-invariant on} \ \textit{G'} & \quad \text{if} \ \forall \textit{a'} \stackrel{\ell}{-} \textit{b'}, \quad \langle \textit{R}_{\textit{a'}}, \textit{g} \rangle = \langle \textit{R}_{\textit{b'}}, \textit{g} \rangle \\ \textit{g} \ \text{is} \ \ell\text{-invariant} & \quad \text{if} \ \forall \textit{G'}, \quad \textit{g} \ \text{is} \ \ell\text{-invariant} \ \text{on} \ \textit{G'} \end{array}$$

Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$ is an invariant cover if

I.
$$\forall \{u, v\} \in G(P)$$
 $\exists v \in P$ $g_{P,v}$ is $\{u, v\}$ -invariant

II. $\forall v \in P$ $\langle 1, g_{P,v} \rangle = \langle \chi_{\emptyset}, g_{P,v} \rangle = \hat{g}_{P,v}(\emptyset) = 1$

Definition (Invariant)

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 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant if $\forall G', \quad g$ is ℓ -invariant on G'

Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$ is an invariant cover if

- I. $\forall \{u, v\} \in G(P)$ $\exists v \in P$ $g_{P,v}$ is $\{u, v\}$ -invariant
- II. $\forall v \in P$ (1. so P = P) (1. so P = P)

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

$$g$$
 is ℓ -invariant on G' if $\forall a' \stackrel{\ell}{-} b', \quad \langle R_{a'}, g \rangle = \langle R_{b'}, g \rangle$ g is ℓ -invariant if $\forall G', \quad g$ is ℓ -invariant on G'

Definition (Invariant Cover)

For a k-Clique P, say a collection $\{g_{P,v}\}_{v\in P}$ is an invariant cover if

1.
$$\forall \{u,v\} \in G(P)$$
 $\exists v \in P$ $g_{P,v}$ is $\{u,v\}$ -invariant

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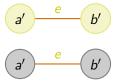
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Reachability is Reversible

If a' and b' are connected by an edge e' whose label $\lambda'(e') = e$ where $e \in C$

$$R_{a'}(C) = R_{b'}(C)$$



For locating invariants $g_P = g_{P,u} - g_{P,v}$ for some $u, v \in P$.