Monotone Lower Bounds
via Fourier Analysis

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Aaron Potechin — MIT

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What's Known

$$\mathsf{NC}^1 \subseteq \mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{NC}^2 \subseteq \mathsf{NC}^3 \subseteq \cdots \subseteq \mathsf{NC} \subseteq \mathsf{P} \subseteq \mathsf{NP}$$

What's Known

$$m\mathsf{NC}^1 \subsetneq m\mathsf{L} \subsetneq m\mathsf{NL} \subseteq m\mathsf{NC}^2 \subsetneq m\mathsf{NC}^3 \subsetneq \cdots \subsetneq m\mathsf{NC} \subsetneq m\mathsf{P} \subsetneq m\mathsf{NP}$$

k-Clique	m-Circuit Depth	m-SN Size	
[Razborov '85]	$\Omega(\log^2 n)$	$2^{\Omega(\log n)}$	
[Alon–Boppana '87]	$n^{\Omega(1)}$	$2^{n^{\Omega(1)}}$	
[Haken '95]	$n^{\Omega(1)}$	$2^{n^{\Omega(1)}}$	
[Raz–McKenzie '99]	$\Omega(k \log n)$	$2^{\sqrt{\Omega(k \log n)}}$	$k \leq n^{O(1)}$
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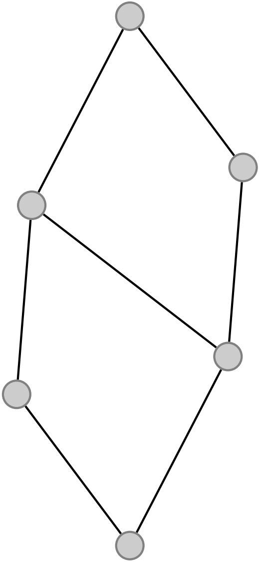
Problem

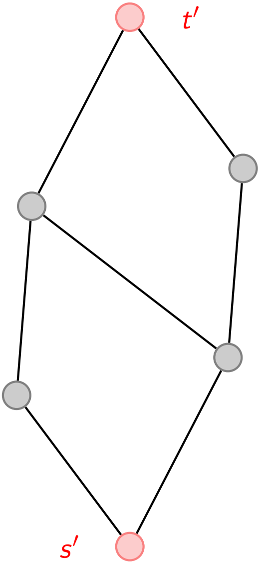
$f\colon \{0,1\}^n \to \{0,1\}$

input size	n	
variables	x_i	$i \in [n]$
literals	$x_i, \bar{x}_i$	$i \in [n]$



SN Model

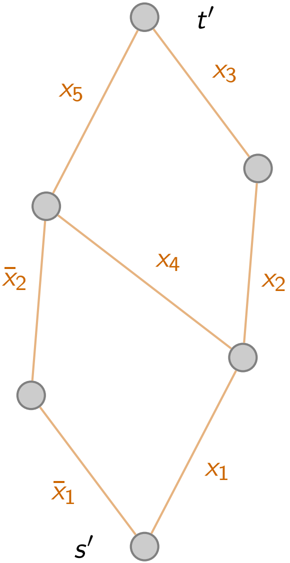
undirected $G' = (V', E')$



SN Model

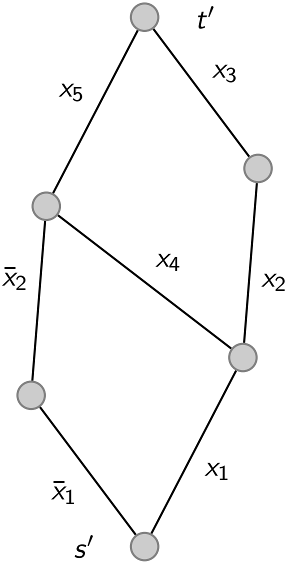
undirected	$G' = (V', E')$
source	$s' \in V'$
sink	$t' \in V'$

The Switching Network Model



SN Model

undirected	$G' = (V', E')$
source	$s' \in V'$
sink	$t' \in V'$
labeling	$\lambda' : E' \to$ literals



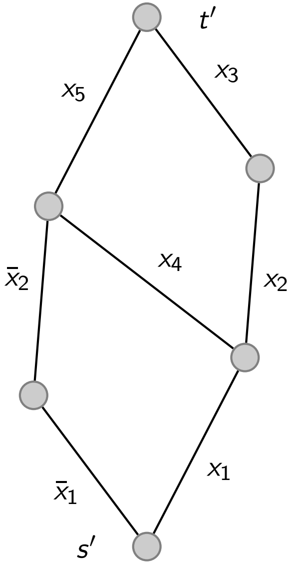
Computation

1) $x \in \{0,1\}^n$ reaches $a' \in V'$

$$\Updownarrow$$

$\exists$ path P' connecting s' and a' labeled with literals in x.

The Switching Network Model



Computation

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$$\Updownarrow$$

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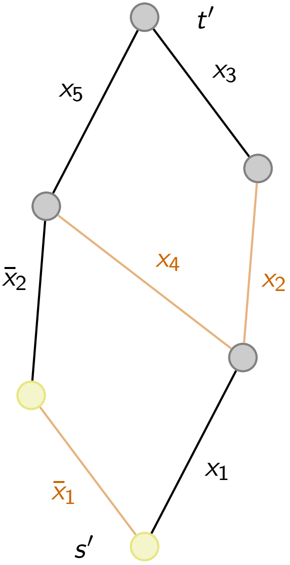
2) $x \in \{0,1\}^n$ is accepted by G'

$$\Updownarrow$$

$x \in \{0,1\}^n$ reaches t'

The Switching Network Model



Computation

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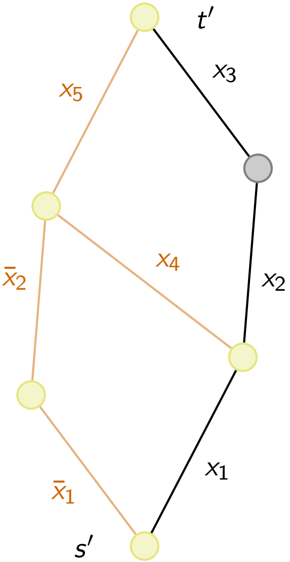
Instance 1

$$x = \begin{array}{ccccc} 0 & 1 & 0 & 1 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

The Switching Network Model



Computation

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$$\Updownarrow$$

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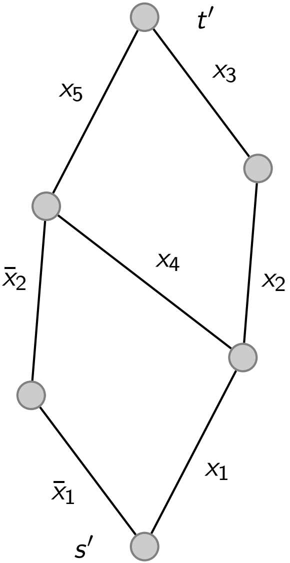
Instance 2

$x =$	0	0	0	1	1
	x_1	x_2	x_3	x_4	x_5



Computation

1) $x \in \{0,1\}^n$ reaches $a' \in V'$

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$\exists$ path P' connecting s' and a' labeled with literals in x.

2) Reachability table $R_{a'}$ for $a' \in V'$

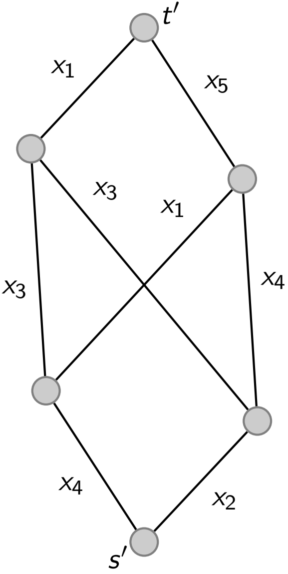
$R_{a'}(x) = \textsc{True}$ if x reaches a'.

$R_{t'}(x) = f(x)$

Monotone SN

A switching network G' labeled only with *positive* literals.
$\lambda' \colon E' \to$ positive literals.

$\Rightarrow$ Computes only *monotone* functions.



k-Clique Problem

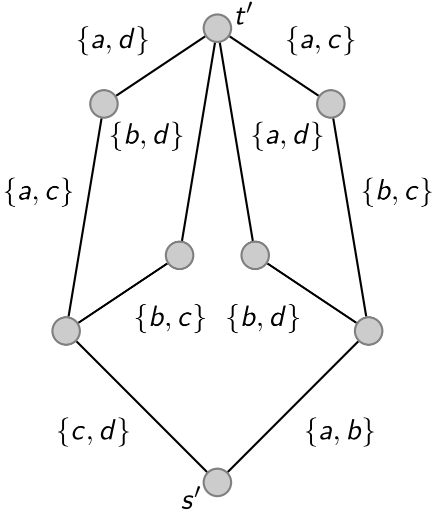
over vertices V

Input a set of edges $E \subseteq \binom{V}{2}$

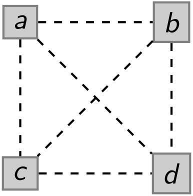
Output if the graph $G = (V, E)$ has a k-clique P

i.e. $P \subseteq V, |P| = k$

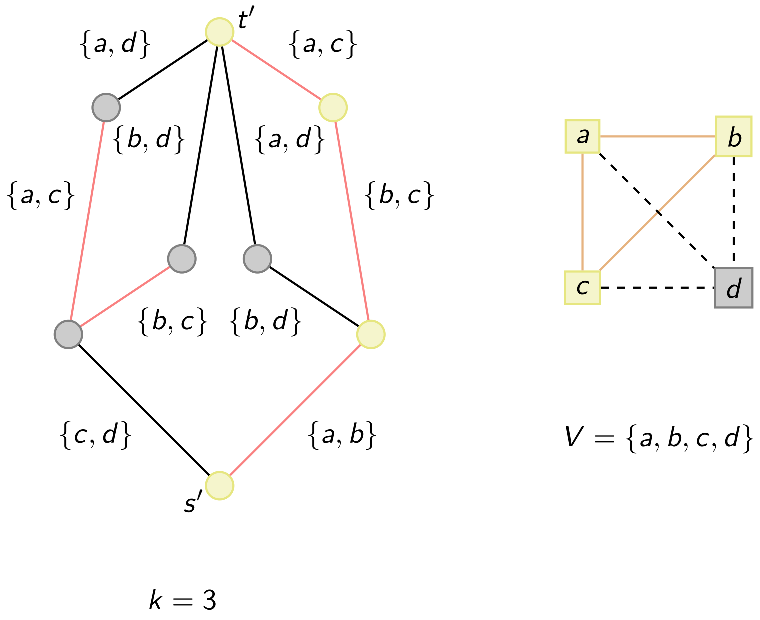
$\forall u \neq v \in P$, $\{u, v\} \in E$.



$k = 3$



$V = \{a, b, c, d\}$



t'
$\{a, d\}$
$\{a, c\}$
$\{b, d\}$
$\{a, d\}$
$\{a, c\}$
$\{b, c\}$
$\{b, c\}$
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s'
$k = 3$
a
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c
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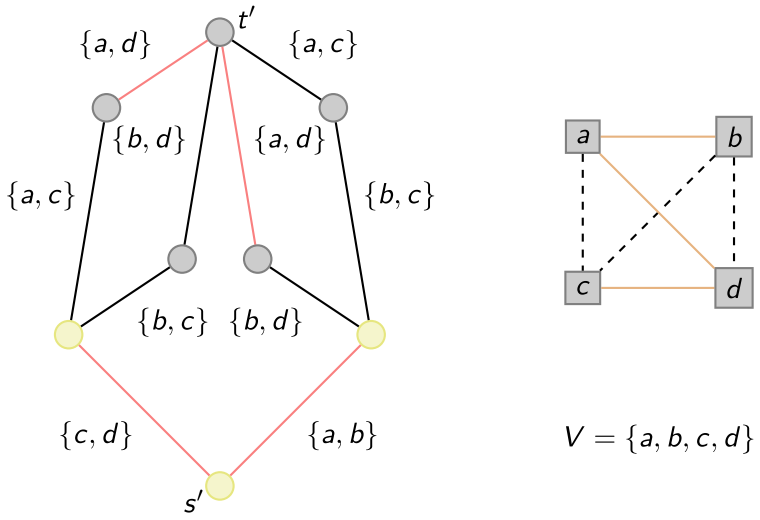

$k = 3$

Monotone SN
for k-CLIQUE

has size $n^{O(k)}$ and $n^{\Omega(k)}$
when $k \leq n^{O(1)}$

Monotone SN

for k-Clique

has size $n^{O(k)}$ and $n^{\Omega(k)}$

when $k \leq n^{O(1)}$

Bottleneck Argument

$$\beta : k\text{-cliques} \to \text{nodes on SN}$$

$$\forall v' \text{ on SN} \quad |\beta^{-1}(v')| \leq \text{few } k\text{-cliques}$$

$$\#k\text{-cliques} = n^{\Theta(k)}$$

$$\#\text{nodes on SN} \geq n^{\Theta(k)}$$

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YES instances	NO instances
min terms	structured
k-clique P	$(k-1)$-colorable C
$P \subseteq V, \lvert P \rvert = k$	$C \colon V \to [k-1]$
$\{u, v\} \in \text{instance}(P)$	$\{u, v\} \in \text{instance}(C)$
$\Updownarrow$	$\Updownarrow$
$u \in P, v \in P$	$C(u) \neq C(v)$

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min terms

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Inner Product Space over $\mathbb{C}^{\mathcal{C}}$

All $(k-1)$-colorings of V ($\cong$ structured No instances)

$$\mathcal{C} \stackrel{\text{def}}{=} \{\, C\colon V \to [k-1] \,\} \cong [k-1]^V$$

$\mathcal{C}$-vector

$g \in \mathbb{C}^{\mathcal{C}}$

$g : \mathcal{C} \to \mathbb{C}$ a complex vector indexed by $\mathcal{C}$

e.g. $R_{a'}$ is a $\mathcal{C}$-vector for any a' on SN

Inner Product

$f, g \in \mathbb{C}^{\mathcal{C}}$

$$\langle f, g \rangle \stackrel{\text{def}}{=} \mathbb{E}\left[f(C)\overline{g}(C)\right] \stackrel{\text{def}}{=} \underbrace{\frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} f(C)\overline{g}(C)}_{\text{average reachability over No instances}}$$

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Fourier Analysis over $\mathbb{C}^{\mathcal{C}}$

Fourier Basis

$$\chi_U(C) \stackrel{\text{def}}{=} \omega^{\sum_{v \in V} U(v) C(v)}$$

$\chi_U \in \mathbb{C}^{\mathcal{C}}$ for $U \in \mathcal{C}$

Primitive Root of Unity

$$\omega \stackrel{\text{def}}{=} e^{2\pi i/(k-1)}$$

$(k-1)^{\text{st}}$ root

$\{\chi_U\}_{U \in \mathcal{C}}$ is Orthonormal

$$\langle \chi_U, \chi_W \rangle = \begin{cases} 1 & \text{if } U = W \\ 0 & \text{if } U \neq W \end{cases} \quad \text{for } U, W \in \mathcal{C}$$

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Recall: Bottleneck Argument

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$$\forall v' \text{ on SN} \quad |\beta^{-1}(v')| \leq \text{few } k\text{-cliques}$$

$$\#k\text{-cliques} = n^{\Theta(k)}$$

$$\#\text{nodes on SN} \geq n^{\Theta(k)} \quad \square$$

$\forall$ k-Clique P, construct $g_P \colon \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) $\forall$ k-Clique P, $\exists b'_P \in G'$ s.t.

$$\langle g_P, R_{b'_P} \rangle \geq \Omega\left(\frac{1}{|G'|}\right)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2) \qquad \text{if } C_1(u) = C_2(u) \quad \forall u \in P$$

$$\hat{g}_P(U) = 0 \qquad \text{if } suppU \not\subseteq P$$

3. ((k-1)-wise ind.) g_P is "$(k-1)$-wise independent"

$$\langle g_P, f \rangle = 0 \qquad \text{if } f \text{ is a } (k-1)\text{-junta}$$

$$\hat{g}_P(U) = 0 \qquad \text{if } |supp(U)| \leq k-1$$

4. (short)

$$||g_P|| \leq k^k$$

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$$\forall a' \in G' \qquad 1 \geq \langle R_{a'}, R_{a'} \rangle \geq \sum_P \left| \left\langle R_{a'}, \frac{g_P}{\|g_P\|} \right\rangle \right|^2$$

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$$\langle g_P, g_P \rangle^{1/2} \stackrel{\text{def}}{=} \|g_P\| \leq k^k$$

$\forall$ k-Clique P in a NW design, construct $g_P : \mathcal{C} \to \mathbb{C}$ satisfying

1. (non-negligible work) $\forall$ k-Clique P in a NW design, $\exists b'_P \in G'$ s.t.

$$\langle g_P, R_{b'_P} \rangle \geq \Omega\Big(\frac{1}{|G'|}\Big)$$

2. (locality) g_P depends only on coloring on P

$$g_P(C_1) = g_P(C_2) \qquad \text{if } C_1(u) = C_2(u) \quad \forall u \in P \qquad \text{(a)}$$

$$\hat{g}_P(U) = 0 \qquad \text{if } suppU \not\subseteq P \qquad \text{(b)}$$

3. ((k-2)-wise ind.) g_P is "$(k-2)$-wise independent"

$$\langle g_P, f \rangle = 0 \qquad \text{if } f \text{ is a } (k-2)\text{-junta} \qquad \text{(a)}$$

$$\hat{g}_P(U) = 0 \qquad \text{if } |supp(U)| \leq k-2 \qquad \text{(b)}$$

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Invariant Cover

Definition (Invariant)

For a literal ℓ (edge $\{u, v\}$), for a m-k-Clique SN G'

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Scaling factor $\phi(k) \stackrel{\text{def}}{=} (k-1)^{k-1}/(k-2)!(k-2)$

For any vertex v in a k-clique P, define $g_{P,v} : \mathcal{C} \to \mathbb{C}$ by

$$g_{P,v}(C) \stackrel{\text{def}}{=} \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \\ d \end{array}} \right.$$

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A coloring $C : P \to [k-1]$ has a *monopoly*

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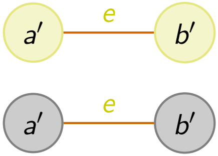
Reachability is Reversible

If a' and b' are connected by an edge e'
whose label $\lambda'(e') = e$ where $e \in C$

$$R_{a'}(C) = R_{b'}(C)$$



For locating invariants $g_P = g_{P,u} - g_{P,v}$ for some $u, v \in P$.