Toward a new formalization of real numbers

Catherine Lelay

Institute for Advanced Sciences

September 24, 2015

Toward a new formalization of real numbers in Coq using Univalent Foundations

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Definition of $\mathbb{R}$

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The set of real numbers $\mathbb{R}$ is :

- a non-trivial abelian field
- totally ordered
- archimedean
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- Axiomatic (Coq standard library, PVS)
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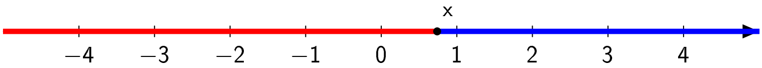
Goal: Build a new library based on Dedekind cuts

Definition of Dedekind cuts



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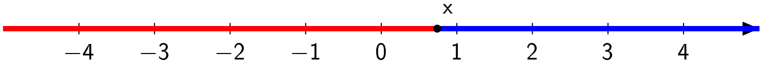


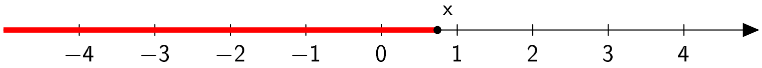
Definition (Dedekind cut)

$(L, U) \subseteq \mathbb{Q} \times \mathbb{Q}$ where :

- $\exists x \in L$, $\exists y \in U$, and $L \cap U = \varnothing$,
- $\forall x \in L, \forall y \in \mathbb{Q}, y < x \Rightarrow y \in L$,
- $\forall x \in L, \exists y \in \mathbb{Q}, x < y \wedge y \in L$,
- $\forall x \in U, \forall y \in \mathbb{Q}, x < y \Rightarrow y \in U$,
- $\forall x \in U, \exists y \in \mathbb{Q}, y < x \wedge y \in U$,
- $\forall (r, x, y) \in \mathbb{Q}^3, x < r < y \Rightarrow x \in L \vee y \in U$.

$$\mathbb{R} = \{\text{Dedekind cuts}\}$$

A second definition of Dedekind cuts



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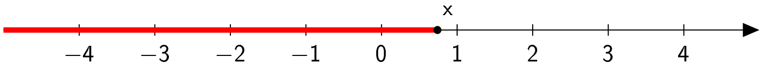

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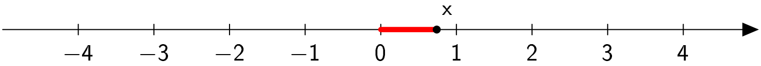
$\ominus$ Difficulties to define multiplication

A better definition of Dedekind cuts



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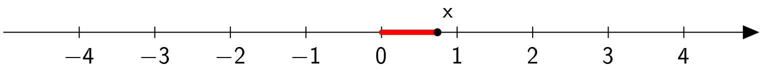

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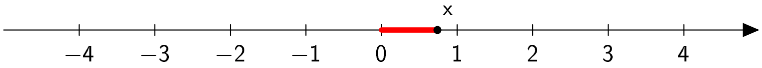

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? Quotients in Coq proof assistant

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For my work:

- New board
- Work with conceptors

Toward a new formalization of real numbers

$$\boxed{\begin{array}{c}\mathbb{R}^+ = [0; +\infty) \\ 0, 1, +, \times, \cdot^{-1}\end{array}}$$

$$\downarrow$$

$$\boxed{\begin{array}{c}\mathbb{R} = (\mathbb{R}^+ \times \mathbb{R}^+)/\approx \\ 0, 1, +, -, \times, \cdot^{-1}\end{array}}$$

To go further: real analysis