Sums of three squares and Lattice points on the sphere

(joint work with Z. Rudnick and P. Sarnak)

Sums of 3 squares

Legendre/Gauss $n = x^2 + y^2 + z^2 \Leftrightarrow n \neq 4^a(8b + 7)$

Primitive representation: $gcd(x, y, z) = 1$

n is primitively represented $\iff n \neq 0, 4, 7 \pmod 8$

$$r(n) = \left|\left\{(x, y, z) : x^2 + y^2 + z^2 = n\right\}\right|$$

Gauss Formula $(n \neq 7$ mod 8 and square free)

$$r(n) = C_n \sqrt{n}\, L(1, \mathcal{X}_{d_n})$$

$$d_n = \begin{cases} -4n & -n \equiv 2, 3 \pmod 4 \\ -n & -n \equiv 1 \pmod 4 \end{cases}$$

C_n depends on $n \pmod 8$

Corollary *If n is primitively representable as sum of 3 squares, then*

$$r(n) \approx n^{\frac{1}{2}\pm\varepsilon} \quad \textit{for all } \varepsilon > 0$$

Under GRH

$$\frac{n^{\frac{1}{2}}}{\log\log n} \ll r(n) \ll n^{\frac{1}{2}} \log\log n$$

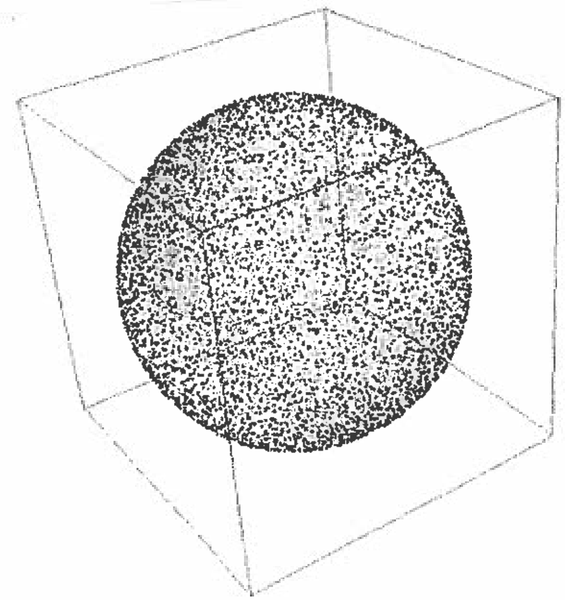
Theorem (Peter) $\frac{r(n)}{\sqrt{n}}$ is Bohr almost periodic function of n

$\big($in fact r-almost periodic for all $r > 0\big)$

Spatial dirstribution of solutions

Project the different representation of n to the unit sphere S^2

$$(x, y, z) \mapsto \frac{1}{\sqrt{n}}(x, y, z) \in S^2$$



We get a set $L(n)$ of $r(n) \approx \sqrt{n}$ points on S^2

- call them "Linnik points"

Uniform distribution on S^2

Definition A collection of subsets $E(n)$ in S^2 becomes uniformly distributed if for any nice set B in S^2

$$\frac{\#(E(n) \cap B)}{\#E(n)} \underset{n\to\infty}{\longrightarrow} \frac{\text{area} B)}{\text{area}(S^2)}$$

Equivalently, for any **continuous** function $f \varepsilon C(S^2)$,

$$\frac{1}{\#E(n)} \sum_{P \in E(n)} f(P) \underset{n\to\infty}{\longrightarrow} \frac{1}{\text{area}(S^2)} \int_{S^2} f(x)dx$$

Theorem *As* $n \to \infty, n \neq 0, 4, 7 \pmod 8$, *the sets* $L(n)$ *become uniformly distributed on* S^2.

Linnik (1940) Dynamical approach (conditional)

Einsiedler, Lindenstrauss, Michel, Venkatesh

Duke, Golubeva-Fomenko (1988) (with input from **Iwaniec**)

Automorphic approach $\to$ quantitative results at scale $n^{-\delta}$

φ non-constant harmonic polynomial of degree r

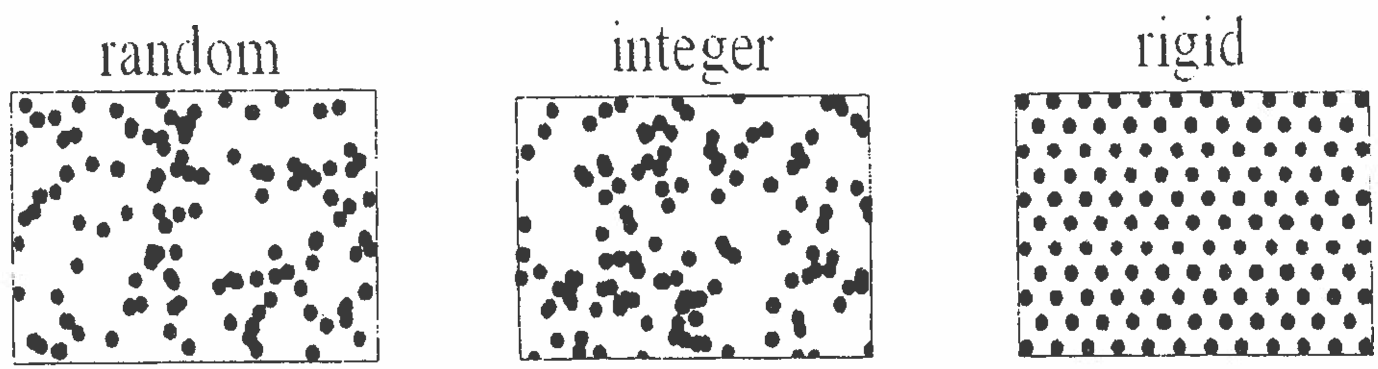
$\Rightarrow$ theta series $$\theta_\varphi(z) = \sum_{|d|\geq 1} \Big(\sum_{a^2+b^2+c^2=|d|} \varphi(a,b,c) \Big) e(|d|z)$$

is a modular form of weight $k = \frac{3}{2} + r$

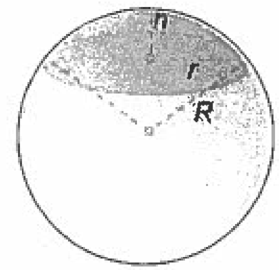
Beyond equidistribution : randomness on smaller scales

Uniform distribution means randomness on scale of O(1) – subsets in S^2 of fixed size.

Question: randomness on smaller scales?



Least spacing for random points
- the birthday paradox



$V(r) = r^2/4$ normalized area of a cap of radius r on S^2:

The probability $P(N, r)$ of placing N random points on the sphere

each at distance $> r$ from the rest is

$$(1 - V(r)) \times (1 - 2V(r)) \times \cdots \times (1 - (N-1)V(r)) \approx \exp\left(-N^2 V(r)/2\right)$$

If $V(r)N^2 > N^{\varepsilon}$ then $p(N, r) \approx 0$ - i.e. if $r > 1/N^{1-\varepsilon}$

If $V(r)N^2 < 1/N^{\varepsilon}$ then $P(N, r) \approx 1$ - i.e. if $r < 1/N^{1+\varepsilon}$

$\Rightarrow$ Minimal distance between N random points is a.s. $1/N^{1 \pm o(1)}$

Compatible with the fact that Linnik points are at least $\frac{1}{\sqrt{n}}$-spaced but not with higher dimensional situation

$d(x) = \min_{y \neq x} |x - y|$ nearest neighbor distance

$$d_{\min}\big(L(n)\big) = \min_{\substack{x \neq y \\ x,y \in L(n)}} |x - y|$$

Theorem *For almost all n*

$$\frac{1}{\sqrt{n}} \leq d_{\min}\big(L(n)\big) \ll \frac{1}{\sqrt{n}} (\log n)^{1+\varepsilon}$$

Theorem (Wooley) *Almost all n is sum of two squares and mini-square*

$$n = x^2 + y^2 + z^2, |z| \ll (\log n)^{1+\varepsilon}$$

Ripley function in statistics

$$\widehat{K}_\delta = \widehat{K}_\delta\big(L(n)\big) = \sum_{\substack{x,y \in L(n) \\ 0<|x-y|<\delta}} 1$$

Conjecture *For* $n^{-\frac{1}{2}+\varepsilon} < \delta < o(1), \widehat{K}_\delta \sim \frac{r(n)^2\delta^2}{2}$

(random model and experimentally verified for $L(n)$)

Theorem *There is* $\varepsilon_0 > 0$ *such that for fixed* $0 < \varepsilon < \varepsilon_0$ *and* $\delta = n^{-\frac{1}{2}+\varepsilon}$

$$\widehat{K}_\delta \sim \frac{r(n)^2\delta^2}{2} \quad \textit{for almost all } n$$

Theorem *(in the absence of Siegel zero's)*

For fixed $\varepsilon > 0$ *and* $r(n)^{-1+\varepsilon} < \delta < 2, \widehat{K}_\delta < C_\varepsilon r(n)^2\delta^2$

Theorem *Fix* $\varepsilon < \alpha < \frac{1}{2} - \varepsilon$ *and set* $\delta = n^{-\alpha}$

Then $\widehat{K}_\delta \gg r(n)^2\delta^2$ *for a positive proportion of* n

In particular, for fixed $\lambda > 0$, *for a positive proportion of* n

$$V_\lambda\big(L(n)\big) \equiv \frac{1}{r(n)}\, K_{\sqrt{\frac{\lambda}{r(n)}}}\big(L(n)\big) \asymp \lambda$$

The arithmetical function $A(n,t)$, (I)

$$A(n,t) = \#\{(x,y) \in \mathbb{Z}^3 \times \mathbb{Z}^3;\ |x|^2 = |y|^2 = n \text{ and } x.y = t\}$$

Theorem (**Venkov, Pall** based on **Siegel**)

$A(n,t)$ *is given by an exact formula as product of local densities*

$$A(n,t) = 24\, \alpha_2(n,t). \prod_{\substack{p \mid n^2 - t^2 \\ p \neq 2}} \alpha_p(n,t)$$

Theorem (**Linnik, Pall**)

$$A(n,t) \ll \text{ged}(n,t)^{1/2}\, n^{\varepsilon} \text{ for all } \varepsilon > 0$$

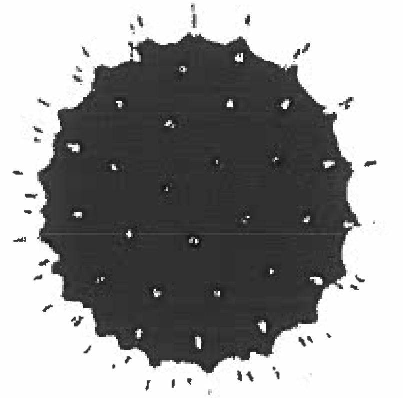
Corollary

$$\#\{(x,y) \in \mathbb{Z}^3 \times \mathbb{Z}^3;\ |x|^2 = |y|^2 = n \text{ and } |x-y| \leq h\} \ll n^{\varepsilon} h^2$$

$$h^2 = 2(n - x.y)$$

Equivalently, $\widehat{K}_\delta \ll n^{1+\varepsilon}\delta^2$

The electrostatic energy

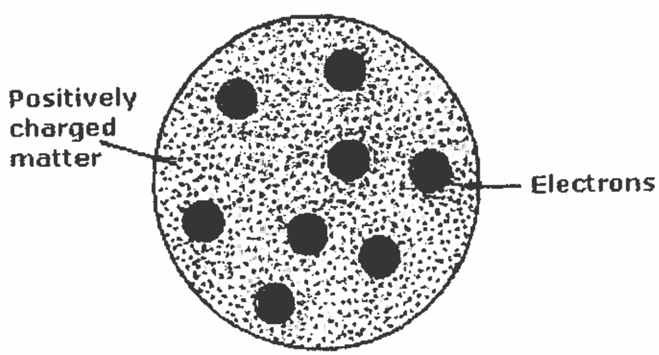


Visualization: Rob Womersley

The electrostatic energy of N points on the sphere S^2 is

$$Energy(P_1,\ldots,P_N) := \sum_{i=1}^{N}\sum_{j\neq i}\frac{1}{|P_i - P_j|}$$

Thomson's question (1904): Find configurations of charges on the sphere which **minimize** energy (stable configurations)



The plum-pudding model

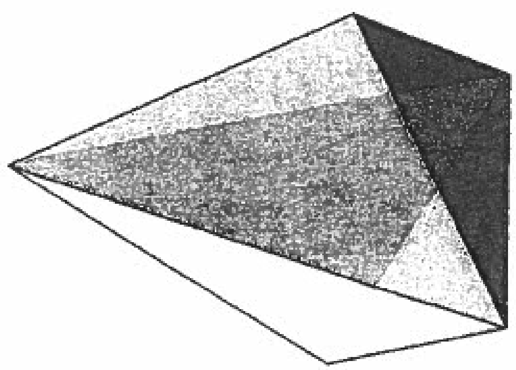
J.J. Thomson, Nobel prize 1903

The known minimal energy configurations

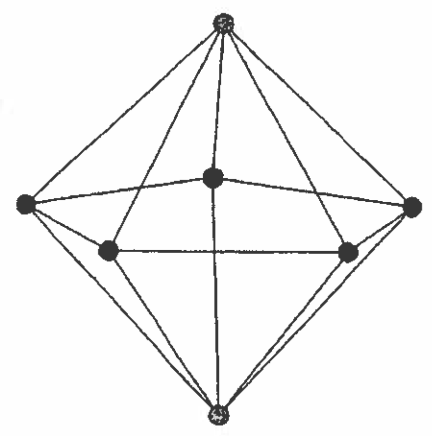
Finding stable configurations is notoriously difficult; known numerically for $N < 112$

Minimum energy configurations have been **rigorously** identified only for $N = 2, 3, 4, 5, 6, 12$

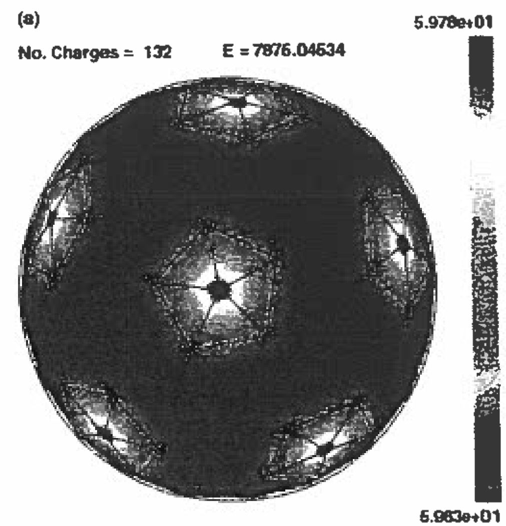
- $N = 2$: antipodal points
- $N = 3$: equilateral triangle about a great circle (Foppl, 1912)
- $N = 4$: regular tetrahedron
- $N = 5$: triangular dipyramid (rigorous proof: E.Schwartz, 2013)
- $N = 6$: regular octahedron (Yudin 1993)
- $N = 12$: regular icosahedron (Andreev 1996)



triangular dipyramid



N=7



Theorem (**G. Wagner, 1992**)

If $S(N) \subset S^2$ *is a stable configuration, then*

$$\text{Energy}\big(S(N)\big) = N^2 \iint_{S^2 \times S^2} \frac{dx\,dy}{|x-y|} - O(N^{3/2}) = N^2 - O(N^{3/2})$$

Theorem (**Dahlberg** , 1978). *For all* $x \in S(N), d(x) \asymp \frac{1}{\sqrt{N}}$

$\Rightarrow S(N)$ is **rigid**

What happens for N random points?

Theorem (**Peled**, 2010) *For* N *random points, Energy* $\sim N^2$ *a.s.*

Energy of Linnik Points

Theorem *The energy of $L(n)$ is close to minimal*

$$\text{Energy}\Big(L(n)\Big) = N^2 + O(N^{2-\delta}) \quad \textit{with} \quad N = r(n)$$

$$\text{Energy}\Big(L(n)\Big) = \sum_{x \in L(n)} \Bigg(\sum_{y \in L(n), y \neq x} \frac{1}{|x-y|} \Bigg)$$

Large distances: use equidistribution

Small distances: use Siegel bound on $\widehat{K}_\delta$.

The arithmetical function $A(n,t)$, II

Recall that

$$A(n,t) = 24\alpha_2(n,t) \prod_{\substack{p \mid n^2-t^2 \\ p \neq 2}} \alpha_p(n,t)$$

Analysis of local factors show that

$$A(n,t) \leq 24.F_n(n^2-t^2)$$

where F_n is a multiplicative function essentially given by

$$F_n(a) \approx \sum_{\substack{d \mid a \\ d \text{ odd}}} \left(\frac{-n}{d}\right)$$

Problem Evaluate

$$\sum_{n-\frac{h^2}{2}<t<n} F_n(n^2-t^2)$$

Nair's theorem (1992)

- F-multiplicative function: $F(1)=1, F(ab)=F(a)F(b)$ if a,b coprime
- F non-negative: $F \geq 0$, slowly growing: $F(n) \ll n^{\varepsilon}$
- $P(t)\varepsilon\mathbb{Z}[t]$ polynomial

Then for $x^a < y < x$

$$\sum_{x-y<m<x} F\Big(|P(m)|\Big) \ll_{F,P} y \times \prod_{p\leq x}\Big(1-\frac{\rho(p)}{p}\Big) \times \exp\Big(\sum_{p\leq x}\frac{\rho(p)F(p)}{p}\Big)$$

$$\rho(m) = \#\{x \in \mathbb{Z}/m\mathbb{Z} : P(x) = 0 \mod m\}$$

In our case

$$P(t)=t(2n-t), F(a)=F_n(a) \approx \sum_{\substack{d|a \\ d \text{ odd}}} \Big(\frac{-n}{d}\Big), x=n, y=\frac{h^2}{2}$$

This gives bound

$$\sum_{n-\frac{h^2}{2}<t<n} A(n,t) \ll h^2 \exp\left[2 \sum_{p<n} \frac{1}{p}\, \mathcal{X}_{-n}(p)\right]$$

Recall $\widehat{K_\delta} = \sum_{n-\frac{h^2}{2}<t<n} A(n,t)$ with $h = \delta\sqrt{n}$ and $r(n) \sim \sqrt{n}L(1,\mathcal{X}_{-n})$

$$\Rightarrow \frac{\widehat{K_\delta}}{r(n)^2} \ll \delta^2 \left\{\frac{1}{L(1,\mathcal{X}_{-n})} \exp\left[\sum_{p<n} \frac{1}{p}\, \mathcal{X}_{-n}(p)\right]\right\}^2$$

$\ll \delta^2$ (assuming no Siegel zero)

Estimating the variance

$$\mathcal{E}(n) = \{x \in \mathbb{Z}^3, |x|^2 = n\} \qquad r(n) = \#\mathcal{E}(n)$$

For $\xi \in S^2, \delta > 0$, denote $\mathrm{Cap}(\xi, \delta)$ the cap centered at ξ of normalized area δ^2

Denote

$$Z(n, \delta, \xi) = \#(\sqrt{n}\mathrm{Cap}(\xi, \delta) \cap \mathbb{Z}^3)$$

Conjecture As $n \to \infty, n \neq 0, 4, 7 \pmod 8$

$$\int_{S^2} |Z(n, \delta, \xi) - r(n)\delta^2|^2 d\sigma(\xi) \sim r(n)\delta^2$$

Theorem *A slightly weaker form holds assuming the Lindelöf hypothesis for automorphic L-functions on $GL(2)$.*

The Conjecture holds on average, more precisely

Theorem *Let $\delta < X^{-\varepsilon}$. Then*

$$\frac{1}{X} \sum_{\frac{X}{2}<n<X} \int_{S^2} \left|Z(n,\delta;\xi) - r(n)\delta^2\right|^2 d\sigma(\xi) =$$

$$\frac{1}{X} \sum_{\frac{X}{2}<n<X} r(n)\delta^2 + O(\delta^3 X^{\frac{1}{2}})$$

Corollary 1 *For most n, the covering radius of $L(n)$ is $\ll n^{-\frac{1}{8}}$*

(conjectured to be $\ll n^{-\frac{1}{4}+\varepsilon}$)

Corollary 2 *For most n and most $\xi \in S^2$, $Z(n, n^{-\frac{1}{4}+\varepsilon}; \xi) \neq \phi$*

Corollary 3 *For fixed $X^{-\frac{1}{2}+\varepsilon} < \delta < X^{-\varepsilon}$, $\widehat{K_\delta} \gg r(n)^2\delta^2$ for a positive proportion of $\frac{X}{2} < n < X$.*

Proof

$$Av_{\frac{X}{2}<n<X} \int_{[Z(n,\delta;\xi)\geq 2]} Z(n,\delta;\xi)\Big(Z(n,\delta;\xi) - 1\Big)\sigma(d\xi) \gg X\delta^4$$

and

$$\int Z(n,\delta;\xi)^2\sigma(d\xi) \ll X\delta^4$$