Birational Geometry and Algebraic Dynamics

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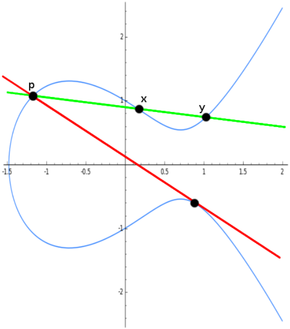
Positive entropy automorphisms of varieties

- $\phi : X \to X$ a self-map of a compact metric space.
- Entropy = "how fast does ϕ separate points of X?"
  - $N(n, \epsilon)$ = maximum size of set of points such that any two have $d(f^k(p_1), f^k(p_2)) \geq \epsilon$ for some $0 \leq k \leq n$.
  - $h_{\text{top}} = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{\log N(n, \epsilon)}{n} \right)$
- Theorem (Gromov–Yomdin). For automorphisms of varieties over $\mathbb{C}$, entropy can be computed in cohomology.
- Non-example: $\phi \in \mathrm{PGL}(3)$ acting on $\mathbb{P}^2$.

An example (J. Blanc)

- Fix an elliptic curve $E \subset \mathbb{P}^2$, $p \in E$.



- Define $\sigma_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to act on L_{pxy} by the involution of $\mathbb{P}^1$ fixing x and y.

- Not defined where the line is tangent to E!

An example, cont.

- To resolve σ_p, blow up p and the four points p_i.
- $\tilde{\sigma}_p : X_p \to X_p$ is an involution, fixing the curve E.
- Same thing with different initial point q gives $\tilde{\sigma}_q : X_q \to X_q$.
- Both σ_p and σ_q lift to the common blow-up $X_{p,q}$.
- $\tilde{\sigma}_p \circ \tilde{\sigma}_q$ is a positive entropy automorphism of $\mathrm{Bl}_{10 \text{ pts}}\, \mathbb{P}^2$.
- Many other constructions: McMullen, Bedford–Kim,...

Algebraic dynamics $\xrightarrow{\text{examples}}$ Birational geometry

- It's sometimes hard to find examples of "pathological" behaviors on high-dimensional varieties.

- Varieties with interesting dynamics provide a good source: rich geometry, tractable computations.

Example example

Question: (Kawamata)

Fix a smooth projective variety X. Is the number of smooth projective varieties Y with $D^b \operatorname{Coh}(X) \cong D^b \operatorname{Coh}(Y)$ finite?

- Yes, for many nice classes: curves, surfaces, abelian, toric, Fano, K_X ample,...

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- Yes, for many nice classes: curves, surfaces, abelian, toric, Fano, K_X ample,...
- No, for X the blow-up of $\mathbb{P}^3$ at ≥ 8 general points.
- The Y's are all $\mathbb{P}^3$ blown up at different configurations of 8 points, arising from dynamics of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$.

Birational geometry $\xrightarrow{\text{classification}}$ Algebraic dynamics

Theorem: (De-Qi Zhang)

Suppose that $\phi : X \dashrightarrow X$ is a birational automorphism of infinite order. Then either:

- ϕ is imprimitive: $0 < \dim B < \dim X$ and

$$\begin{array}{ccc} X & \dashrightarrow & X \\ \downarrow & & \downarrow \\ B & \dashrightarrow & B \end{array}$$

- X is birational to a weak Calabi-Yau or abelian variety;

- X is rationally connected.

Some open questions

Question: (Bedford '11)

Does there exist a blow-up of $\mathbb{P}^3$ that admits an automorphism of positive entropy?

Question: (Kawamata–Matsuda–Matsuki '87)

Does there exist a variety of non-negative Kodaira dimension with infinitely many flipping curves?