Transversality of loop coproduct and cobracket

Dingyu Yang

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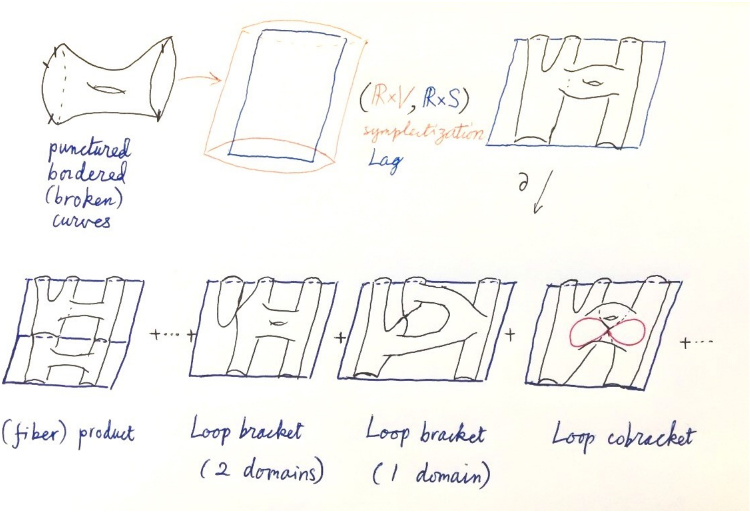
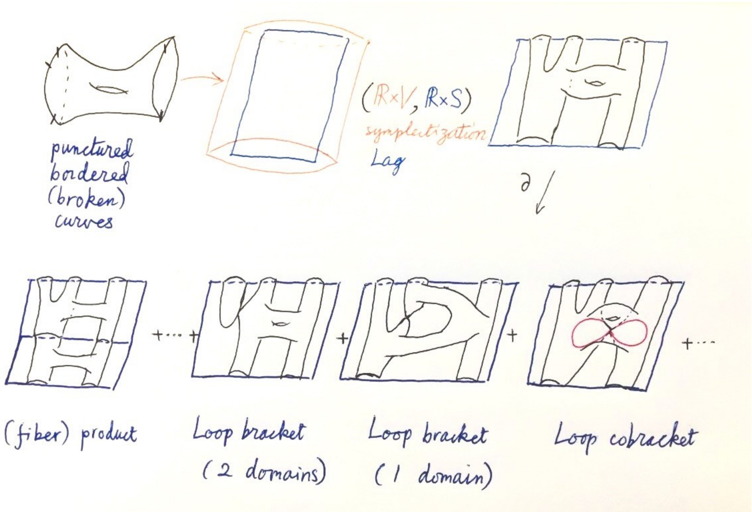
Conjectural picture (Fukaya, C-L-M): codimensional-1 boundary (filtered) $=$ (fiber) product $+$ loop bracket $+$ loop cobracket of evaluation maps from simpler moduli space(s).


punctured
bordered
(broken)
curves
(ℝ×V, ℝ×S)
symplectization
Lag
∂
+ ⋯ +
+
+
+ ⋯
(fiber) product
Loop bracket
(2 domains)
Loop bracket
(1 domain)
Loop cobracket



- transversality of these "intersection"-type operations on maps
- nontrivial transversality of domain moduli space as solution of $\bar{\partial}_J$, Fredholm problem with domain variation, analytic limiting behavior.

Coproduct at chain level is relevant

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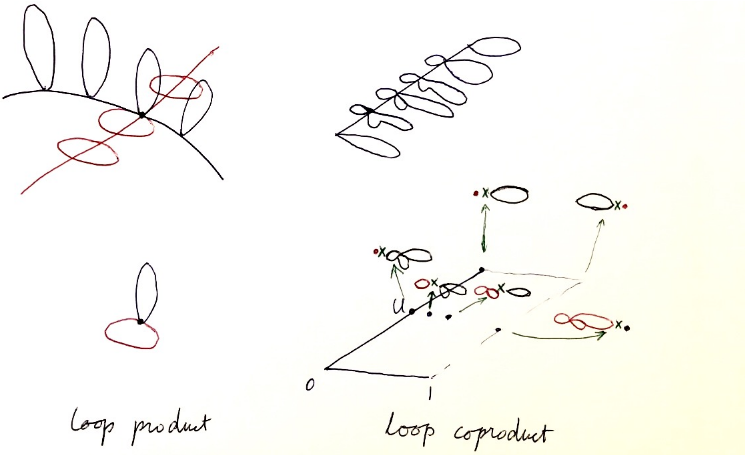

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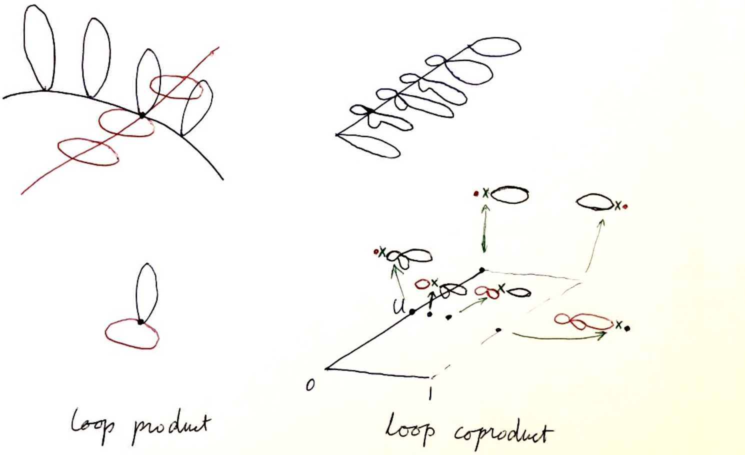

Focus on coproduct, severe transversality issue. Joint w/ Manuel Rivera.

Chas-Sullivan (2002): on "sufficiently transverse" chains.
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$e_\varphi : U \times [0,1] \to M \times M, (x,t) \mapsto (\varphi(x)(0), \varphi(x)(t))$, $\mathring{e}_\varphi := e_\varphi|_{U \times (0,1)}$.

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But $\varphi \Rightarrow$ a chain (Θ_φ, ν) by thickening domain s.t. $P_\varphi := \overline{(\mathring{e}_{\Theta_\varphi})^{-1}(\Delta_M)}$ is a manifold, killing directions by adjoining a Thom form ν.

Achieving transversality of loop coproduct via a thickening

- smooth $\alpha : [0,1]/\sim \to [0,1]/\sim$ is 0 on $[0,\epsilon] \cup [1-\epsilon, 1]$, o/w diffeo.
- $\lambda_1 : [0,1]/\sim \to [0,1]$ smooth $\lambda_1|_{[-\epsilon/2,\epsilon/2]} = ct$ and then cut-off.
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Thickening: $\Theta_\varphi(x, v, w)(t) := \exp_{\varphi(x)(\alpha(t))}(\lambda_1(t) P_t^1(v) + \lambda_2(t) P_t^2(w))$, where $(x, v, w) \in \varphi(\cdot)(0)^* D_\delta(TM) \oplus_U \varphi(\cdot)(1/2)^* D_\delta(TM)$ and P_t^1 and P_t^2 parallel transports along loop $\varphi(x) \circ \alpha$ from 0 to t and 1/2 to t.

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Expect: smooth $\gamma \mapsto (c_\gamma, \epsilon_\gamma)$, Θ_φ can be defined w/o stopping loops via α.

Co-A_∞ for chain map modification, loop cobracket

Thom form ν for domain thickening depending only on M.

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(a loop pair by splitting over $P_\varphi,$ smoothened via $\alpha, \iota^* pr_1^*(\nu \wedge \pi^* \eta)$).

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- Evaluation of varying base point s and time t in the lower 2-simplex, thicken $\Rightarrow$ htpy for commutativity $\overset{\text{symmetrize}}{\Rightarrow}$ loop cobracket.

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Fukaya, Irie (w/o corners) for loop product/bracket.

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Ev map from domain structure $\Rightarrow$ (generalized) dR chains (loop spaces).

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Moduli spaces satisfying codimensional-1 degeneration
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Contains relative symplectic field theory (with genus and multiple boundary components), e.g. splitting a Fukaya algebra along a contact hypersurface.

Thank you!