Random band matrices: delocalization and universality

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Random matrices: examples

- Gaussian ensemble (GUE, GOE):
  $\mathrm{M_N}$, $\mathrm{N} = 1, 2, \ldots$ are hermitian (or real symmetric) $\mathrm{N} \times \mathrm{N}$ matrices:

$$\mathrm{M_{ij}} = \mathrm{N}^{-1/2}\mathrm{W_{ij}}, \quad \mathbb{E}\{\mathrm{W_{ij}}\} = 0, \quad \mathbb{E}\{|\mathrm{W_{ij}}|^2\} = 1,$$

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- Wigner ensemble: the elements are i.i.d. but not necessary Gaussian.
- Sample covariance matrices:
  $A_{m,n}$, $m, n = 1, 2, \dots$ are complex $n \times m$ matrices with independent variables $A_{i,j}$

$$M = n^{-1} A_{m,n} A^*_{m,n}, \quad \mathbb{E}\{A_{ij}\} = 0, \quad \mathbb{E}\{|A_{ij}|^2\} = 1.$$

  Both m, n go to $+\infty$ such that $m/n \to c \in (0, +\infty)$.

Global distribution of eigenvalues

Normalized Counting Measure (NCM):

$$\mathcal{N}_N(\triangle) = \sharp\{\lambda_j \in \triangle, j = 1, \ldots, N\}/N, \quad \mathcal{N}_N(\mathbb{R}) = 1,$$

where $\triangle$ is an arbitrary interval of the real axis.

It is shown that for many ensembles of random matrices

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Wigner ensembles (in particular GUE):

$$\rho(\lambda) = (2\pi)^{-1}\sqrt{4-\lambda^2}, \quad \lambda \in (-2, 2).$$

Local statistics, localization and delocalization

One of the key physical parameter of models is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length ℓ is comparable with the matrix size, and it is called localized otherwise. Let Ψ be an eigenvector correspond to some e.v. λ from the bulk of the spectrum.

- Localized eigenvectors: $|\Psi_j|^2 \approx e^{-|j-c|/\ell}$
  lack of transport (insulators), and Poisson local spectral statistics (strong disorder)
- Delocalization: $|\Psi_j|^2 \approx |\Lambda|^{-1}$
  diffusion (electric conductors), and GUE local statistics (weak disorder).

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory.

The main objects of the local regime are k-point correlation functions R_k $(k = 1, 2, \ldots)$, which can be usually defined as

$$R_k(\lambda_1, \ldots, \lambda_k) = \int p_N(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_N) d\lambda_{k+1} \ldots d\lambda_N,$$

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Universality conjecture in the bulk of the spectrum (hermitian case, deloc.regime) (Wigner – Dyson):

$$\frac{1}{(N\rho(\lambda_0))^k} R_k\big(\{\lambda_0 + \xi_j/N\rho(\lambda_0)\}_{j=1}^k\big) \xrightarrow{N\to\infty} \det\left\{\frac{\sin\pi(\xi_i - \xi_j)}{\pi(\xi_i - \xi_j)}\right\}_{i,j=1}^k$$

Anderson model (Random Schrödinger operators)

$$\mathrm{H_{RS}} = -\triangle + \mathrm{V},$$

where $\triangle$ is the discrete Laplacian in lattice box $\Lambda = [1, n]^d \cap \mathbb{Z}^d$, V is a random potential (i.e. a diagonal matrix with i.i.d. entries).

In $d = 1$ it corresponds to a narrow band matrix with i.i.d. diagonal

$$\mathrm{H_{RS}} = \begin{pmatrix} V_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & V_2 & 1 & 0 & \dots & 0 \\ 0 & 1 & V_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & V_{n-1} & 1 \\ 0 & \dots & 0 & 0 & 1 & V_n \end{pmatrix}.$$

Localization, Poisson local spectral statistics (Fröhlich, Spencer, Aizenman, Molchanov)

Random band matrices

Intermediate model that interpolates between random Schrödinger operator and Wigner matrices.

$\Lambda = [1, n]^d \cap \mathbb{Z}^d$ is a lattice box, $N = n^d$.

$$H = \{H_{jk}\}_{j,k\in\Lambda}, \quad H = H^*, \quad \mathbb{E}\{H_{jk}\} = 0.$$

Entries are independent (up to the symmetry) but no longer identically distributed. Variance is given by a band profile J (even function with compact support or rapid decay)

$$\mathbb{E}\{|H_{jk}|^2\} = \frac{1}{W^d} J\Big(\frac{|j-k|}{W}\Big)$$

Key parameter: band width $W \in [1; N]$.

It also has nontrivial spatial structure like RS, but technically more accessible.

Anderson transition in random band matrices

$W = O(1)$ [$\sim$ random Schrödinger] $\longleftrightarrow$ $W = N$ [Wigner matrices]

Varying W, we can see the transition between localization and delocalization

Conjecture (in the bulk of the spectrum):

$d = 1:$	$\ell \sim W^2$	$W \gg \sqrt{N}$	Delocalization, GUE statistics
		$W \ll \sqrt{N}$	Localization, Poisson statistics
$d = 2:$	$\ell \sim e^W$	$W \gg \log N$	Delocalization, GUE statistics
		$W \ll \log N$	Localization, Poisson statistics
$d \geq 3:$	$\ell \sim N$	$W \geq W_0$	Delocalization, GUE statistics

At the present time only some upper and lower bounds on the order of localization length are proved rigorously (d = 1).

- Schenker (2009) $\ell \le W^8$ – localization techniques;
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By the developing the Erdős-Yau approach, other results were obtained. In these bounds the localization length is controlled in a rather weak sense, i.e. the estimates hold for "most" eigenfunctions only:

- Erdős, Knowles (2011): $\ell \gg W^{7/6}$;
- Erdős, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$ (not uniform in N).

Main problem: to control of the resolvent $(H - \lambda - i\eta)^{-1}$ for $\eta \sim 1/N$. The techniques allows to obtain the control only for $\eta \sim 1/W$.

Another method, which allows to work with random operators with non-trivial spatial structures, is supersymmetry techniques (SUSY), which based on the representation of the determinant as a integral over the Grassmann variables.

This method is widely (and successively) used in the physics literature and is potentially very powerful but the rigorous control of the integral representations, which can be obtained by this method, is quite difficult.

Part of the formalism is rigorous and can be used. However, good understanding of saddle point approximation on the supermanifold is still a major challenge for mathematicians.

SUSY results, characteristic polynomials

Results for the correlation functions of characteristic polynomials ($d = 1$).

Consider Gaussian case and take special covariance (i.e. take specific J, which is useful for SUSY formalism).

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$$\mathbb{E}\Big\{\det\big(\mathrm{H_n}-\lambda_0-\mathrm{x}/\mathrm{N}\rho(\lambda_0)\big)\det\big(\mathrm{H_n}-\lambda_0-\mathrm{y}/\mathrm{N}\rho(\lambda_0)\big)\Big\} \sim \frac{\sin\pi(\mathrm{x}-\mathrm{y})}{\pi(\mathrm{x}-\mathrm{y})}$$

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- TS, in progress: if $W \ll \sqrt{N}$,

$$\mathbb{E}\Big\{\det\big(H_n - \lambda_0 - x/N\rho(\lambda_0)\big)\det\big(H_n - \lambda_0 - y/N\rho(\lambda_0)\big)\Big\} \sim 1$$

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- Erdős, Bao, 2015: Combining this techniques with Green's function comparison strategy (Erdős-Yau), they proved

$$\ell \geq W^{7/6}$$

  in a strong sense for the block band matrices with more or less general element's distribution (subexponential tails, four Gaussian moments).

Open problems

- In $d = 1$, improvement of existing bounds for localization length up to the crossover point (from both sides), and the proof of universality of the correlation functions for $W \gg \sqrt{N}$.
- Universality of the characteristic polynomials for $d > 1$.
- The order of localization length and universality of the correlation functions for $d > 1$.
- More general covariance, non-Gaussian case...