Algebraic Combinatorics in Statistical Mechanics and Complexity Theory

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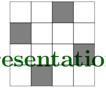
Algebraic Combinatorics

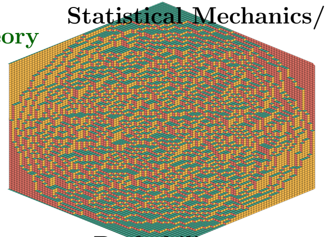
Algebraic Geometry

$[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$

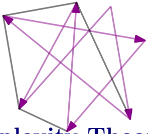


Representation Theory

Statistical Mechanics/

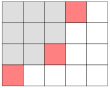


Probability



Complexity Theory

P vs NP



Algebraic Combinatorics

$$s_{(2,2)}(x_1,x_2,x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}$$

Algebraic Combinatorics: Young Tableaux and Schur functions

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the **Specht modules** $\mathbb{S}_\lambda$

Algebraic Combinatorics: Young Tableaux and Schur functions

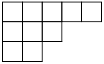
Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

are the **Specht modules** $\mathbb{S}_\lambda$, indexed by

integer partitions $\lambda \vdash n$:

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, $\lambda_1 + \lambda_2 + \cdots = n$

Young diagram of λ: 

Basis for $\mathbb{S}_\lambda$: S*tandard* **Y***oung* **T***ableaux* of shape λ:

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

Young Tableaux and Schur functions

Irreducible representations of the **symmetric group** S_n:

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$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}$

bigskip

Irreducible (polynomial) representations of

the **General Linear group** $GL_N(\mathbb{C})$

Weyl modules V_λ, indexed by highest weights λ, $\ell(\lambda) \leq N$.

Young Tableaux and Schur functions

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bigskip
Irreducible (polynomial) representations of

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Weyl modules V_λ, indexed by highest weights λ, $\ell(\lambda) \leq N$.

Schur functions: $s_\lambda(x_1, \ldots, x_N)$ – characters of V_λ.
Weyl's determinantal formula:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{ij=1}^N}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

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Enumeration and asymptotics

Symmetric group S_n

Irreps $\mathbb{S}_\lambda$, $\lambda \vdash n$

$Tr_{\mathbb{S}_\lambda}[\pi] = \chi^\lambda(\pi)$

General linear group GL_N

V_λ, $\ell(\lambda) \leq N$

$Tr_{V_\lambda}(diag(x_1, .., x_N)) = s_\lambda(x_1, .., x_N)$

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SYTs:

1	3	4	7	9
2	6	10		
5	8			

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Hook-length formula [FRH]:

$$\dim \mathbb{S}_\lambda = f^\lambda = \frac{n!}{\prod_{\square \in \lambda} h_\square}$$

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$$\dim V_\lambda = s_\lambda(1^N) = \prod_{\square \in \lambda} \frac{N + c(\square)}{h_\square}$$

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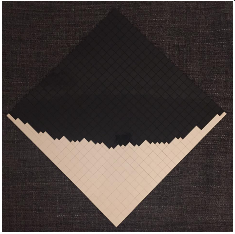

(Lego art by Dan Betea)

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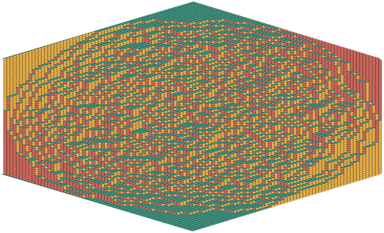

(Computer art by Leonid Petrov)

"Combinatorial interpretations"

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} g(\lambda, \mu, \nu) \mathbb{S}_\nu$$

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$$V_\lambda \otimes V_\mu = \oplus_\nu c^\nu_{\lambda\mu} V_\nu$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

Theorem (Littlewood-Richardson, 1934)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

			1	1	1
	1	2	2		
2	3	3			

			1	1	1
	2	2	2		
1	3	3			

(LR tableaux of shape $(7,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(7,4,3)}_{(3,1)(4,3,2)} = 2$)

"Combinatorial interpretations"

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[Murnaghan]: If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

$$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^\nu_{\lambda\mu}.$$

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. Alternatively, show that KRON is in $\#$P .

"Combinatorial interpretations"

Tensor product decomposition:

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Representation theory via Schur functions:

$$s_\lambda(x) s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)$$

$$s_\lambda(x.y) = \sum_{\mu,\nu} g(\lambda, \mu, \nu) s_\mu(x) s_\nu(y)$$

Schur functions in statistical mechanics

Characters of $U(\infty)$, boundary of the Gelfand-Tsetlin graph

1	1	1	2	2	...
2	2	3	...		
...					

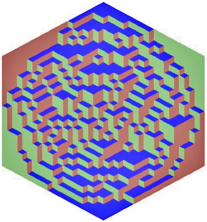
Alternating Sign Matrices (ASM)/ 6-Vertex model:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

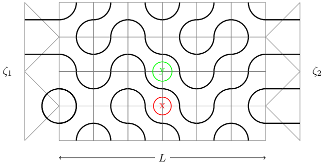
Normalized Schur functions:

$$S_\lambda(x_1,\ldots,x_k;N) = \frac{s_\lambda(x_1,\ldots,x_k,1^{N-k})}{s_\lambda(1^N)}$$

Lozenge tilings:



Dense loop model:



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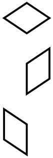
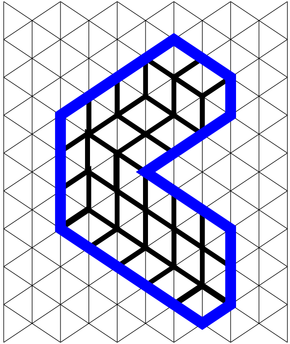
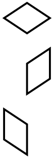


Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



Lozenge tilings



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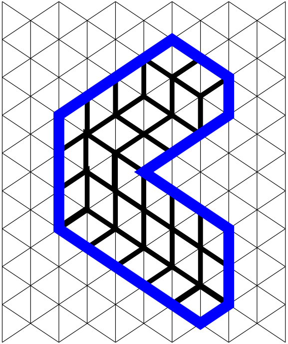
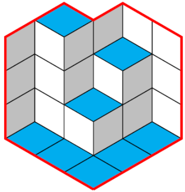
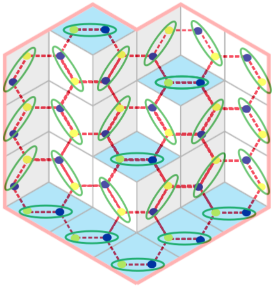


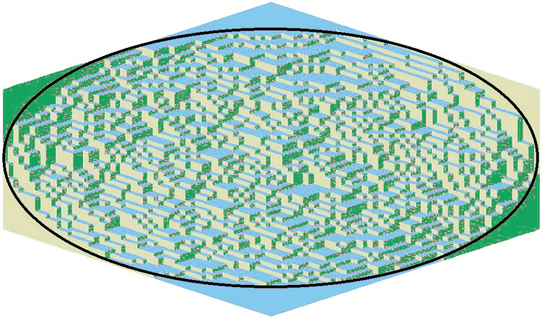


Dimer covers on the hexagonal grid



Classical questions: limit behavior

Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?



Frozen regions (polygonal domains), "limit shapes" of the surface of the height function (plane partition).
([Cohn–Larsen–Propp, 1998], [Kenyon–Okounkov, 2005], [Cohn–Kenyon–Propp, 2001; Kenyon-Okounkov-Sheffield, 2006])

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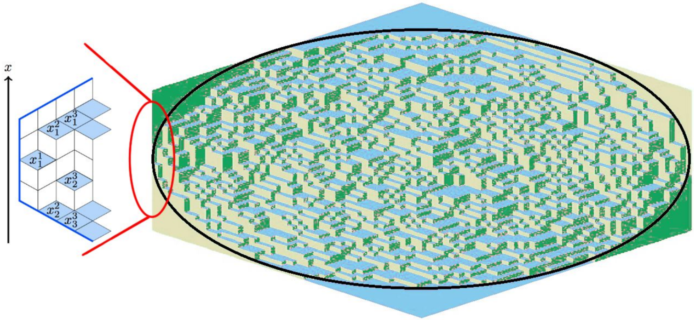
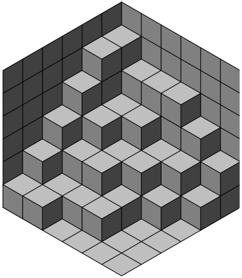


Behavior near boundary: Gaussian Unitary Ensemble eigenvalues,
conjectured by [Okounkov-Reshetikhin, 2006], proofs – hexagon [Johansson-Nordenstam, 2006], [Gorin-Panova, 2013]

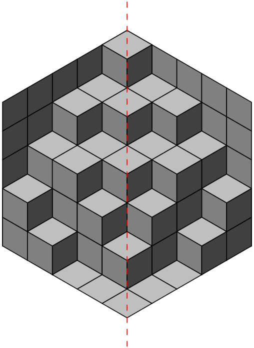
Uniform vs symmetric

Tilings of the hexagon $a \times b \times c \times a \times b \times c$, s.t.

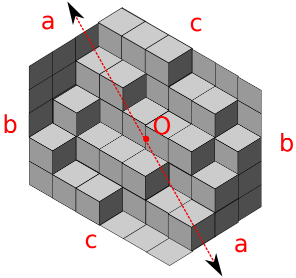
Unrestricted



Vertically symmetric

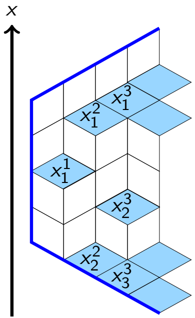


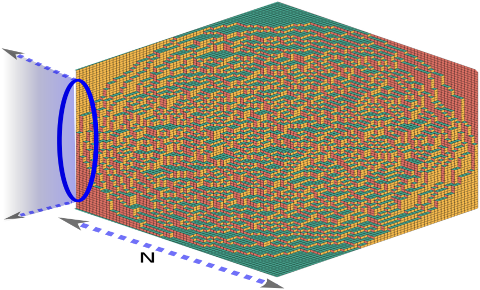
Centrally symmetric



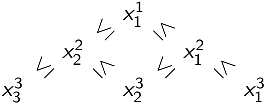
Limit behavior: fluctuations near the boundary, limit surface, CLT?

Behavior near the flat boundary:





Horizontal lozenges near a flat boundary:



Question: Joint distribution of $\{x_j^i\}_{i=1}^k$ as $N \to \infty$ (rescaled)?

Conjecture [Okounkov–Reshetikhin, 2006]:
Fixed boundary: The joint distribution converges to a *GUE*-corners (aka *GUE*-minors) process: eigenvalues of GUE matrices.

Proofs: hexagonal domain [Johansson-Nordenstam, 2006], more general domains [Gorin-P,2012], [Novak, 2014], unbounded [Mkrtchyan, 2013], symmetric tilings [P, 2014, 2015]

Behavior near the flat boundary:GUE

GUE: matrices $A = [A_{ij}]_{i,j}$: $A = \overline{A^T}$

$\mathrm{Re}A_{ij}, \mathrm{Im}A_{ij}$ – i.i.d. $\sim \mathcal{N}(0, 1/2)$, $i \neq j$

A_{ii} – i.i.d. $\sim \mathcal{N}(0, 1)$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$(x_1^k \leq x_2^k \leq \cdots \leq x_k^k)$ – eigenvalues of $[A_{i,j}]_{i,j=1}^k$

Interlacing condition: $x_{i-1}^j \leq x_{i-1}^{j-1} \leq x_i^j$

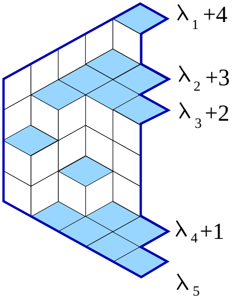
$$\begin{array}{ccccccc} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ \leq & & x_1^2 & & x_2^2 & & \geq \\ & & & x_1^1 & & & \end{array}$$

The *joint distribution* of $\{x_i^j\}_{1 \leq i \leq j \leq k}$ is the
GUE–corners (also, GUE-minors) process, $=: \mathbb{GUE}_k$.

Tilings setup

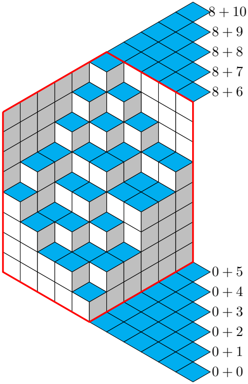
Domain $\Omega_{\lambda(N)}$:
positions of the N horizontal lozenges on right boundary are:

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$



$\lambda(5) = (4, 3, 3, 0, 0)$

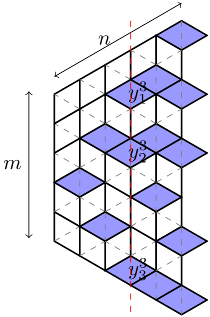
($\frac{1}{N}\Omega_{\lambda(N)}$ is *not necessarily a finite polygon as* $N \to \infty$, e.g. $\lambda(N) = (N, N-1, \ldots, 2, 1)$)



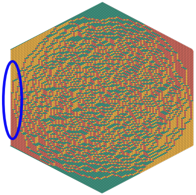
$\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$

$\leftrightarrow a \times b \times c$... hexagon.

Behavior near the flat left boundary



Line $k = 3$



Theorem

Let $Y_n^k = (y_1^k, \ldots, y_k^k)$ – horizontal lozenges on kth line of a uniformly random tiling $T \in \mathcal{T}_n$. As $n \to \infty$ the collection

$$\left\{ \frac{Y_n^j - \mu_n}{\sqrt{n\sigma_n}} \right\}_{j=1}^{k} \to \mathbb{GUE}_k$$

weakly as RVs, where

- *$\mathcal{T}_n$ – all tilings of a hexagon [Johansson-Nordenstam].*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ – $\mu_n = E(f)$, $\sigma_n = S(f)$, "$f(t) = \lim_{n\to\infty} \frac{\lambda(n)_{nt}}{n}$" [Gorin-P, 2013].*
- *$\mathcal{T}_n$ – vertically symmetric lozenge tilings of a $n \times m \times n..$ hexagon, $a = \lim_{n\to\infty} m/n$, $\mu_n = m/2$, $\sigma_n = \frac{a^2+2a}{8}$ [P, 2014].*
- *$\mathcal{T}_n$ – centrally-symmetric tilings of a $a \times b \times c\ldots$ hexagon with $a = 2qn$, $b = 2pn$, $c = 2(1-q)n$: $\mu_n = 2pqn$ and $\sigma_n = 2pq(1-q)(1+p)$ [P, 2015+].*

Limit shape (surface)

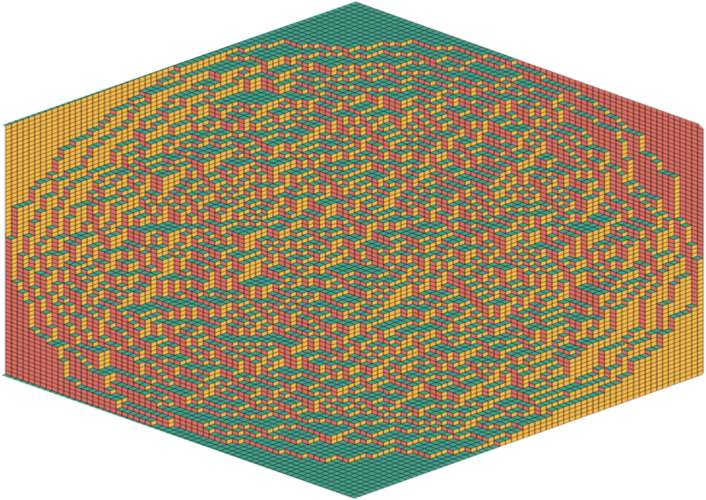


Image: Leonid Petrov

Limit shape (surface)

Theorem (P)

Let $H_n(u, v)$ – height function of a uniformly random tiling from a set $\mathcal{T}_n$, i.e.

$$H_n(u, v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v,$$

where y_i^k is the vertical height of the ith horizontal lozenge on the kth vertical line (left to right). For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u, v)$ converges uniformly in probability to a deterministic function $L(u, v)$ ("the limit shape"), which can be computed explicitly... when $\mathcal{T}_n$ is

- *$\mathcal{T}_n$ – polygonal domain [Cohn, Kenyon, Larsen, Propp, Okounkov]*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ for "nice" family $\lambda(n)$ [Bufetov-Gorin].*
- *$\mathcal{T}_n$ – symmetric tilings [P, 2014].*
- *$\mathcal{T}_n$ – centrally symmetric tilings [P, 2015+].*

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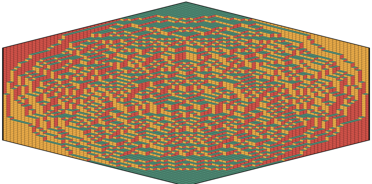
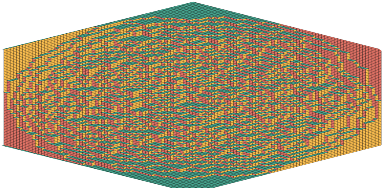
Symmetric:



General:



Main object: Normalized Schur functions

Schur functions: $s_\lambda(x_1,\ldots,x_N)$ – characters of V_λ.

Weyl's determinantal formula:

$$s_\lambda(x_1,\ldots,x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{ij=1}^N}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1,x_2,x_3) = s_{\square\square}(x_1,x_2,x_3) = \underset{\begin{smallmatrix}1&1\\2&2\end{smallmatrix}}{x_1^2x_2^2} + \underset{\begin{smallmatrix}1&1\\3&3\end{smallmatrix}}{x_1^2x_3^2} + \underset{\begin{smallmatrix}2&2\\3&3\end{smallmatrix}}{x_2^2x_3^2} + \underset{\begin{smallmatrix}1&1\\2&3\end{smallmatrix}}{x_1^2x_2x_3} + \underset{\begin{smallmatrix}1&2\\2&3\end{smallmatrix}}{x_1x_2^2x_3} + \underset{\begin{smallmatrix}1&2\\3&3\end{smallmatrix}}{x_1x_2x_3^2}.$$

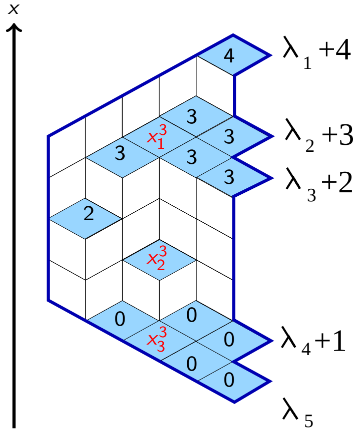
Normalized Schur functions:

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

or other normalized Lie group characters:

$$X_{\gamma(N)}(x_1,\ldots,x_k) := \frac{\chi_{\gamma(N)}(x_1,\ldots,x_k,1^{N-k})}{\chi_{\gamma(N)}(1^N)}$$

Tilings probability I: combinatorics



Tilings of $\Omega_{\lambda(N)}$

$\Leftrightarrow$ Gelfand-Tsetlin triangles, bottom row $\lambda(N)$

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 0 & & 3 & & & \\ & & 0 & & 1 & & 3 & & \\ & 0 & & 0 & & 3 & & 3 & \\ 0 & & 0 & & 3 & & 3 & & 4 \end{array} \quad \Leftrightarrow$$

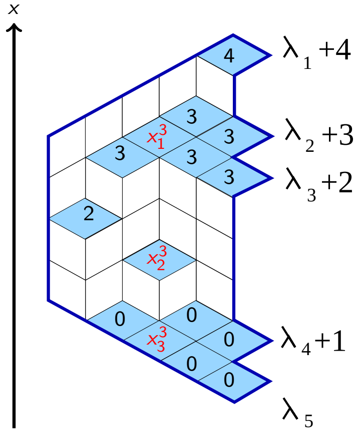
SSYT of shape $\lambda(N)$

$T=$

1	1	2	5
3	4	4	
5	5	5	

Line j:
(x^j) = shape of the subtableaux of T of the entries $1, \ldots, j$.

Tilings probability I: combinatorics



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SSYT of shape $\lambda(N)$

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$x^3 = (3,1,0)$.

Line j:
(x^j) = shape of the subtableaux of T of the entries $1,\ldots,j$.

Tilings probability II: moment generating functions

Proposition

In a uniformly random tiling of Ω_λ

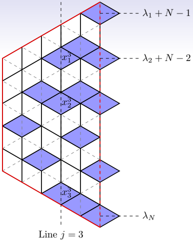
$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

where $s_{\lambda/\eta}$ is the skew Schur polynomial.

Proposition

For any variables $y_1, \ldots, y_k$, the following m.g.f. of x^k (as above) is

$$\mathbb{E}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_\lambda(y_1,\ldots,y_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})} = S_\lambda(y_1,\ldots,y_k).$$



The Schur Generating Functions

$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line j of $T \in \mathcal{T}_n$

MGF: $$\mathbb{E}\left[\frac{s_{x^k(T)}(y_1,\ldots,y_k)}{s_{x^k(T)}(\underbrace{1,\ldots,1}_{k})} \,\middle|\, T \sim Unif(\mathcal{T}_n)\right] = \sum_{\nu} \frac{s_\nu(y_1,\ldots,y_k)}{s_\nu(1^k)} \Pr(x^k(T)=\nu) = \ldots$$

- $= S_{\lambda(n)}(y_1,\ldots,y_k) = \frac{s_{\lambda(n)}(y_1,\ldots,y_k,1^{n-k})}{s_{\lambda(n)}(1^n)}$ for $\mathcal{T}_n = \Omega_{\lambda(n)}$.
- $= \prod_i y_i^{m/2} \cdot \frac{\mathfrak{so}_{(\frac{m}{2})^n}(y_1,\ldots,y_k,1^{n-k})}{\mathfrak{so}_{(\frac{m}{2})^n}(1^n)}$ for $\mathcal{T}_n$ – symmetric tilings of $n \times m \times n$....
- $= S_{(\frac{b}{2})^{a/2}}(y_1,\ldots,y_k)^2$ for $\mathcal{T}_n$ – centrally symmetric tilings of $a \times b \times c$... hexagon.

Tilings probability III: MGF asymptotics

Proposition

$$\mathbb{E}\left[\frac{s_{\nu-\delta_k}(y_1,\ldots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\ldots,1}_{k})} \quad \nu \sim \mathbb{GUE}_k \right] = \exp\left(\frac{1}{2}(y_1^2 + \cdots + y_k^2) \right),$$

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Compare:

$$S_\lambda(y_1,\ldots,y_k) = \mathbb{E}_{tiling}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right)$$

Proposition (Gorin-P)

For any k real numbers $h_1,\ldots,h_k$ and $\lambda(N)/N \to f$ (as earlier) we have:

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right) e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} \quad = \quad \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

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Theorem. Let $\Upsilon^k_{\lambda(N)} = \{x^k, x^{k-1},\ldots\}$ –collection of positions of the horizontal lozenges on lines $k, k-1,\ldots,1$ of tiling from $\Omega_{\lambda(N)}$, then

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$ (GUE-corners process of rank k).

The limit surface

Counting measure for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_L)$

$$m[\mu] := \frac{1}{L} \sum_{i=1}^{L} \delta\left(\frac{\mu_i + L - i}{L}\right),$$

Random measure on partitions $\rho^n(\mu)$ (e.g. $= \mathrm{Prob}\{x^k(T) = \mu\}$ in tilings of size n), $m[\rho]$ – pushforward.

$$S_\rho(u_1, \ldots, u_k) := \sum_{\mu} \rho(\mu) \frac{s_\mu(u_1, \ldots, u_k)}{s_\mu(1^k)} = \mathbb{E}\left[\frac{s_{x^k(T)}(y_1, \ldots, y_k)}{s_{x^k(T)}(\underbrace{1, \ldots, 1}_{k})} \,\middle|\, T \sim Unif(T_n) \right]$$

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$k = \alpha n$ – not fixed!

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Theorem (Bufetov-Gorin)

Suppose that ρ^N is a sequence of measures on partitions, s.t. for every r

$$\lim_{N\to\infty} \frac{1}{N} \ln\left(S_{\rho^N}(u_1, \ldots, u_r, 1^{N-r})\right) = Q(u_1) + \cdots + Q(u_r),$$

uniformly in a $\mathbb{C}$ nbhd of (1^r), Q – analytic. Then the random measures $m[\rho^N]$ converge, as $N \to \infty$, in probability to a deterministic *measure M on $\mathbb{R}$, whose moments are*

$$\int_{\mathbb{R}} t^p M(dt) = \sum_{\ell=0}^{p} \binom{p}{\ell} \frac{1}{(\ell+1)!} \frac{\partial^\ell}{\partial u^\ell} u^p Q'(u)^{p-\ell} \Big|_{u=1}$$

The limit shape of the partitions μ under ρ^N (as histograms) is then M.
"Deterministic = Concentration".

The limit surface

Counting measure for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_L)$

$$m[\mu] := \frac{1}{L} \sum_{i=1}^{L} \delta\left(\frac{\mu_i + L - i}{L}\right),$$

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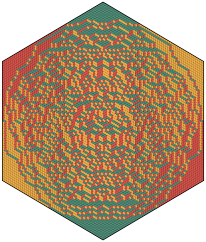
Our cases: MGF = normalized Schur $S_{\lambda(n)}$, SO characters, etc.
Asymptotics using [Gorin-P, 2013] for fixed r:

$$\lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_1, \ldots, u_r) = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_i) = \sum_{i=1}^{r} \Phi(u_i)$$

Limit surface



Theorem (P, 2014)

Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u, v)$ – height function of a symmetric tiling of $n \times m \times n$... hexagon,i.e.

$$H_n(u, v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v.$$

For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u, v)$ converges uniformly in probability to a deterministic function $L(u, v)$ ("the limit surface").

For any fixed $u \in (0, 1)$, $L(u, v)$ is the distribution function of the measure **m**, given by its moments:

$$\int_{\mathbb{R}} t^r \mathbf{m}(dt) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} u^{-r+\ell} \frac{\partial^\ell}{\partial z^\ell} z^p \Phi_a'(z)^{p-\ell} \Big|_{z=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y; a) - 2$ and...

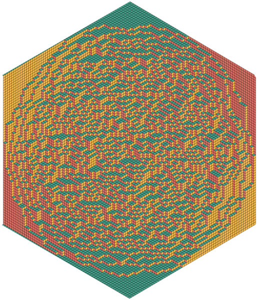
$$h(y) = \frac{1}{4}\left((e^y+1) + \sqrt{(e^y+1)^2 + 4(a^2+a)(e^y-1)^2}\right)$$

$$\phi(y;a) = (\frac{a}{2}+1)\ln\left(h(y) - (\frac{a}{2}+1)(e^y-1)\right) - (\frac{a}{2}+\frac{1}{2})\ln\left(h(y) - (\frac{a}{2}+\frac{1}{2})(e^y-1)\right)$$
$$+ \frac{a}{2}\ln\left(h(y) + \frac{a}{2}(e^y-1)\right) - (\frac{a}{2}-\frac{1}{2})\ln\left(h(y) + (\frac{a}{2}-\frac{1}{2})(e^y-1)\right)$$

Limit surface

Theorem (P, 2015+)

The scaled height function $H_n(u, v)$ of a centrally symmetric tiling of an $a \times b \times c \ldots$ hexagon converges uniformly in probability to a deterministic function $L(u, v)$ – the limit surface, as $n \to \infty$, where $n = \frac{a+c}{2}$ and $a/n, b/n$ – approx constant.
The limit surface coincides with the limit surface for the uniformly random tilings of the hexagon (without symmetry constraints).



Behind the scene: asymptotics of symmetric functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

Theorem (Gorin-P)

For any partition λ and any $x \in \mathbb{C} \setminus \{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))} dz,$$

where the contour C includes all the poles of the integrand. Similar formulas hold for the other normalized Lie group characters.

Theorem (Gorin-P)

If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ [under certain convergence conditions], for all fixed $y \neq 0$:

$$\lim_{N\to\infty} \frac{1}{N} \ln S_{\lambda(N)}(e^y;N,1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y-1),$$

where $\mathcal{F}(w;f) = \int_0^1 \ln(w-f(t)-1+t)dt$, w_0 – root of $\frac{\partial}{\partial w}\mathcal{F}(w;f) = y$. If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ ["other" conv. cond.], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

$$\text{where } E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t)-t+1/2)^2 dt - 1/6 - E(f)^2.$$

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Theorem (Gorin-P)

Let $D_{i,1} = x_i \frac{\partial}{\partial x_i}$, $\Delta-$ Vandermonde det. Then $\forall$ λ, $k \leq N$, we have

$$S_\lambda(x_1,\ldots,x_k;N) = \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\ldots,x_k)} \prod_{j=1}^{k} S_\lambda(x_j;N,1)(x_j-1)^{N-1}.$$

Corollary (Gorin-P)

Suppose that the sequence $\lambda(N)$ is such that, as $N \to \infty$,

$$\frac{\ln\left(S_{\lambda(N)}(x;N,1)\right)}{N} \to \Psi(x) \quad \textit{uniformly on a compact } M \subset \mathbb{C}. \textit{ Then for any } k$$

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1,\ldots,x_k;N,1)\right)}{N} = \Psi(x_1) + \cdots + \Psi(x_k)$$

uniformly on M^k. More informally, under various regimes of convergence for $\lambda(N)$ we have $S_{\lambda(N)}(x_1,\ldots,x_k) \sim S_{\lambda(N)}(x_1)\cdots S_{\lambda(N)}(x_k)$.

(Boolean) Complexity

Input: string of n bits, i.e. size($input$) = n.

Decision problems:

Is there an object, s.t.... ?

P = solution can be found in time $\mathrm{Poly}(n)$

NP = solution can be *verified* in $\mathrm{Poly}(n)$ (polynomial witness)

NP **–Complete** = in NP , and every NP problem can be reduced to it poly time;

Counting problems:

Compute $F(input) = ?$

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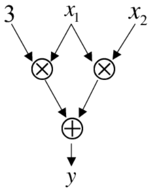
An approach [Mulmuley, Sohoni]: **Geometric Complexity Theory**

VP vs VNP: determinant vs permanent

Arithmetic Circuits:

$y = 3x_1 + x_1x_2$



Polynomials $f_n \in \mathbb{F}[X_1, \ldots, X_n]$. Circuit – nodes are $+, \times$ gates, input – $X_1, \ldots, X_n$ and constants from $\mathbb{F}$.

Class VP (Valliant's P):
polynomials that can be computed with $poly(n)$ large circuit (size of the associated graph).

Class VNP:
the class of polynomials f_n, s.t. $\exists g_n \in \mathrm{VP}$ with

$$f_n = \sum_{b \in \{0,1\}^n} g_n(X_1, \ldots, X_n, b_1, \ldots, b_n).$$

VP vs VNP: determinant vs permanent

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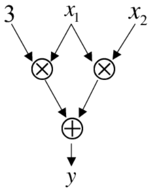

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Theorem[Bürgisser]: If VP = VNP, then P = NP if $\mathbb{F}$ - finite or the Generalized Riemann Hypothesis holds.

VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
For every polynomial $p(X)$ there exists some n s.t.

$$p(X) = \det(A),$$

where $A = [\ell_{i,j}(X)]_{i,j=1}^n$ with $\ell_{i,j}(X) \in \{a_0 + a_1 X_1 + \cdots + a_k X_k | a_i \in \mathbb{F}\}$.
The smallest n possible is the *determinantal complexity* $\mathrm{dc}(p)$.

Example: $p = x_1^2 + x_1 x_2 + x_2 x_3 + 2x_1$, then

$$p = \det \begin{bmatrix} x_1 + 2 & x_2 \\ -x_3 + 2 & x_1 + x_2 \end{bmatrix}, \qquad \mathrm{dc}(p) = 2$$

VP vs VNP: determinant vs permanent

Universality of the determinant [Cohn, Valiant]:
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The permanent:

$$\mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} X_{i,\sigma(i)}.$$

Theorem:[Valiant] per_m is VNP-complete.

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Known: $\mathrm{dc}(\mathrm{per}_m) \leq 2^m - 1$ (Grenet 2011), $\mathrm{dc}(\mathrm{per}_m) \geq \frac{m^2}{2}$ (Mignon, Ressayre, 2004).

Ryser's formula: $\mathrm{per}_m(X) = (-1)^m \sum_{S \subset [1..m]} (-1)^{|S|} \prod_{i=1}^{m} (\sum_{j \in S} X_{i,j})$

Geometric Complexity Theory

GL_N action on polynomials:
$A \in GL_N(\mathbb{C})$, $v := (X_1, \ldots, X_N)$, $f \in \mathbb{C}[X_1, \ldots, X_N]$,
then $A.f = f(A^{-1}v)$
(replaces variables with linear forms)

$GL_{n^2} \det_n := \{g \cdot \det_n \mid g \in GL_{n^2}\}$ – **determinant orbit**.

$\Omega_n := \overline{GL_{n^2} \det_n}$ - **determinant orbit closure**.

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$$\mathrm{per}_m^n \in \overline{GL_{n^2}\,\det_n} \iff \underbrace{\overline{GL_{n^2}\mathrm{per}_m^n}}_{=:\Gamma_m^n} \subseteq \underbrace{\overline{GL_{n^2}\det_n}}_{\Omega_n}.$$

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Exploit the symmetry! Coordinate rings as GL_{n^2} representations:

$$\mathbb{C}[\overline{GL_{n^2}\det_n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}}, \qquad \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

Definition (Representation theoretic obstruction)

If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction**. Its existence shows $\overline{GL_{n^2}\mathrm{per}_m^n} \not\subseteq \overline{GL_{n^2}\,\det_n}$ and so $\mathrm{dc}(\mathrm{per}_m) > n$!

(Non)existence of obstructions

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If $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$, then λ is a **representation theoretic obstruction** and $\mathrm{dc}(\mathrm{per}_m) > n$. If $n > poly(m) \implies \mathsf{VP} \neq \mathsf{VNP}$.

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There exist representation theoretic obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

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There exist representation theoretic obstructions that show superpolynomial lower bounds on $\mathrm{dc}(\mathrm{per}_m)$.

If also $\delta_{\lambda,d,n} = 0$, then λ is an **occurrence obstruction**.

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Theorem (Bürgisser-Ikenmeyer-P(FOCS 2016))

This Conjecture is false. There are no occurrence obstructions.

(Non)existence of obstructions

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Question: What are these $\delta_{\lambda,d,n}$ and $\gamma_{\lambda,d,n,m}$???

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Question: What are these $\delta_{\lambda,d,n}$ and $\gamma_{\lambda,d,n,m}$???
Kronecker coefficients of the Symmetric Group:

$$\delta_{\lambda,d,n} \leq sk(\lambda, n^d) \leq g(\lambda, n^d, n^d)$$

(Symmetric Kronecker: $sk(\lambda,\mu) := \dim \mathrm{Hom}_{S_{|\lambda|}}(\mathbb{S}^\lambda, S^2(\mathbb{S}^\mu)) = mult_\lambda \mathbb{C}[GL_{n^2}\det_n]_d$)
Plethysm coefficients: of GL.

$$a_\lambda(d[n]) := mult_\lambda Sym^d(Sym^n(V)) \geq \gamma_{\lambda,d,n,m}.$$

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Conjecture (Mulmuley and Sohoni 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$ but not in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$, where $n = m^c$.

Theorem (Ikenmeyer-P (2015, FOCS'16))

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[GL_{n^2}\det_n] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}(X_{1,1})^{n-m} per_m}]) = 0$.

Theorem (Bürgisser-Ikenmeyer-P (FOCS'16))

Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[\overline{GL_{n^2} X_{11}^{n-m} per_m}]$, then λ also occurs in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]$. In particular, the Conjecture is false, there are no "occurrence obstructions".

Proofs: Positivity of Kronecker and plethysm coefficients via semigroup properties and building blocks. Nonvanishing of relevant highest weight vectors on sums of powers of linear forms.

Thank you