Some recent progress in geometric methods for instability of Hamiltonian systems.

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Report on joint work with M. Capinski, M. Gidea, T.M. Seara and work of other people.

The problem of instability

In the early 60's it was understood that for near-integrable systems, we see two phenomena:

- The effect of small perturbations averages out:
  - Large measure of initial conditions (KAM Theorem)
  - For a long time (Nekhoroshev theorem).
- There are situations where the perturbations do not average out and they grow (Arnol'd example)

The role in applications

- In some applications (e.g. plasma Physics, accelerators) accumulation of perturbations is the main annoyance since it breaks the confinement.
- In some applications (e.g. astrodynamics) accumulation of effects is useful since one can design maneuvers with very small thrusts. (0.05 N in 500Kg spacecraft).
- In some application (motion of asteroids, chemistry), one needs to understand when does it happen since it creates interesting effects: (rapid change of orbital elements; rearrangement of molecules).

How can we understand when the phenomenon of stability/instability takes place?

Some extra remarks

In applications, we are often interested in analyzing very concrete systems. (e.g. the forces have to be Newton/Coulomb).

In chemistry, all atoms of an element have the same mass and exert same forces. This leads to systems that are *degenerate*.

The point of view of this lecture

The phenomenon is ubiquitous anyway.

We want to understand how it happens in concrete systems given to us.

In this lecture we will aim for generality and simplcity. (but in a way that one can add more precision later for concrete systems).

- We will try to identify some geometric structures whose presence implies interesting (or useful) motions.
- We will try that the geometric structures are persistent under small perturbations.
  - Persistent in C^1 open sets.
  - Can be verified in concrete systems by a finite computation.
  - Give quantitative information.

Most of the time, we will not try to tweak the systems to make instability

Some alternative points of view

The problem of stability/instability is a very old problem.

There are many other philosophies and methodologies

- Numerical experiments
- Asymptotic expansions
- Construct the phenomenon by modifying the system, hence showing genericity properties.
- Study stochastic properties

Even with the same philosophy, there are many different techniques:

- 1960s Transition Whiskered Tori
- 1970's Topological methods: correctly aligned windows, Conley index.
- 1980's Local variational methods
- 1990 Global variational methods (several versions)
- Normally hyperbolic manifolds.
  - Separatrix map
  - Scattering map
  - Normally hyperbolic laminations.
  - Normally hyperbolic cylinders.
  - Kissing cylinders.
  - Blenders.
- Mixed methods (geometric/variational)
- Probabilistic methods

It is a very active field (a partial list).
Classical period

V. Arnold, A. Pustilnikov, L. Chierchia, G. Gallavotti, R. Douady, J. P. Marco, J. Cresson, R. Moeckel, E. Fontich, P. Martín.

V. Chirikov, G. Zavslavsky, J. Meiss, I. Percival, M. Kruskal, A. Tennyson. J. Herrera.

Around 2000

- E. Fontich, P. Martín, J. Cresson, J.P. Marco (lambda lemmas)
- U. Bessi, S. Bolotin, R. S. McKay, L. Chierchia, L. Biasco M. Berti, M. Bolle (local variational methods)
- J. Mather, J. Xia (Global variational methods; anouncements)
- J. Bourgain, V. Kaloshin (PDE's)
- A. Delshams, R. L., T. Seara (NHIM/scattering map)
- C.Q. Cheng (global variational/NHIM)
- D. Treshev (NHIM, separatrix map)
- P. Bernard (Lagrangian graphs)
- A. Delshams, R. L., T. Seara (NHIM/scattering map/secondary tori)
- M. Gidea, R. L (NHIM/ correctly aligned windows)
- M. Gidea,C. Robinson, J.P. Marco (Topological methods)
- R. Moeckel, R. L. V. Gelfreich, D. Turaev (Normally Hyperbolic Laminations)
- Nassiri-Pujals (symplectic blenders/NHIL)

- A. Delshams, G. Huguet/ A. Delshams, R. L., T. Seara
- R. L, V. Gelfreich, D. Turaev
- Dolgopyat, De Simoi
- P. Bernard, V. Kaloshin, K. Zhang
- C.Q. Cheng
- J.P. Marco
- V. Kaloshin, M. Levi, M. Saprykina
- V. Kaloshin, J. Fejoz, M. Guardia, P. Rold'an
- J. Xue
- A. Delshams, M. Gidea, P. Roldán
- Capinski, Zglyczinski,
- Arnold, Zarnitsky

In this lecture, I will just present one method based on:

`https://arxiv.org/abs/1405.0866`

- Based on a very simple geometric structure
- Results hold in C^1 open sets of systems (even non-hamiltonian).
- No non-generic assumptions such as positive definite, twist conditions.
- Can be verified by finite computations in concrete systems. Even in models for celestial mechanics.
- The method relies only on *"soft"* properties of Normally Hyperbolic Invariant Manifolds (NHIM) and their homoclinic orbits
  - Works in ∞ dimensional problems.
  - No need to use Aubry-Mather theory.
  - No need to use averaging theory
  - No need to use KAM theory
  - The big gaps problem gets completely eliminated. (becomes irrelevant).

For the sake of convenience, I will use maps in the exposition.

It all works for flows.

Going to maps allows to explain results in 4-D maps. This requires less cheating than explaining 6-D flows.

The results are true in all higher dimensions; indeed they become easier the higher the dimension.

Main tool used in this lecture:

Normally hyperbolic manifolds (NHIM) with homoclinic intersections.

NHIM: Invariant manifold so that the normal perturbations grow/decrease at exponential rates. The normal rates are larger than the rates of growth decrease of tangent perturbations.

The formal definition of NHIM Fenichel, Hirsch-Pugh-Shub

$\Lambda \subset M$ is a NHIM if it is invariant and

There exists a splitting of the tangent bundle of TM into Df-invariant sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

and there exist a constant $C > 0$ and rates

$$0 < \lambda_+ < \eta_- \leq 1 \leq \eta_+ \leq \mu_-, \tag{1}$$

such that for all $x \in \Lambda$ we have

$$\begin{aligned} v \in E^s_x &\Leftrightarrow \|Df^k_x(v)\| \leq C\lambda^k_+ \|v\| \text{ for a ll } k \geq 0, \\ v \in E^u_x &\Leftrightarrow \|Df^k_x(v)\| \leq C\mu^{-k}_- \|v\| \text{ for al l } k \leq 0, \\ v \in T_x\Lambda &\Leftrightarrow \|Df^k_x(v)\| \leq C\eta^k_+ \|v\|, \quad \|Df^{-k}_x(v)\| \leq C\eta^{-k}_- \|v\|, \text{ for all } k \geq 0. \end{aligned} \tag{2}$$

Assume moreover that the angle between the bundles is bounded from below.

Note: There are versions for locally invariant manifolds. For example, center manifolds.

These manifolds were studied extensively in the 70's. (Jarnich-Kurzweil, Fenichel, Hirsch-Pugh-Shub.

Somewhat later, infinite dimensional versions of the results above: Hale, Chow, Bates, Zeng, Lu, etc.)

Some classical results on NHIMS (stated informally):

- NIHIM's are *smooth* (the order of smoothness depends on the rates and is finite. One cannot expect any better in general).
- They are persistent. When subject to small C^r perturbations, they depend smoothly on the perturbation.
- They posses stable/unstable manifolds.

$$\begin{aligned} W^s_\Lambda &= \{y \,|\, d(f^n(y), \Lambda) \to_{n\to\infty} 0\} \\ &= \{y \,|\, d(f^n(y), \Lambda) \leq C_y \lambda_+^n, n > 0\} \\ &= \cup_{x\in\Lambda} W^s_x \\ W^s_x &= \{y \,|\, d(f^n(y), f^n(x)) \leq C_y \lambda_+^n, n > 0\} \end{aligned}$$

- W^s_Λ are *smooth*, W^s_x are as smooth as the map. They depend *smoothly* on parameters.
- Same results for unstable manifolds.

The results are sophisticated (somewhat subtle fixed point theorems) and very non-trivial, but they are considered *soft* analysis.

We also note that the decomposition $W^s_\Lambda = \cup_{x\in\Lambda} W^s_x$ is a foliation.

$$W^s_x \cap W^s_y = \emptyset, x \neq y$$

Invariance properties.

$$f(W^s_\Lambda) = W^s_\Lambda$$
$$f(W^s_x) = W^s_{f(x)}$$

If a point converges to Λ in the future, its future orbit is exponentially – with rate λ_+ asymptotic to the orbit of a (unique!!) point in Λ

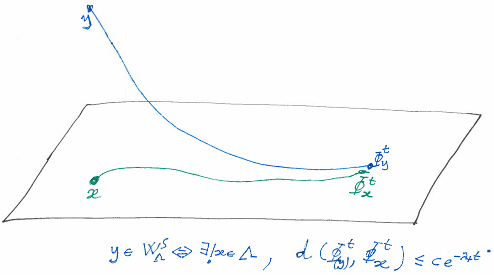

Figure: Trajectories which converge to Λ are asymptotic to the trajectory of a point

An active area of research is to perform effective computations of NHIM's and validate them.

Important fact:

Stable and unstable manifolds of a NHIM can intersect.

This gives origin to very interesting dynamics.

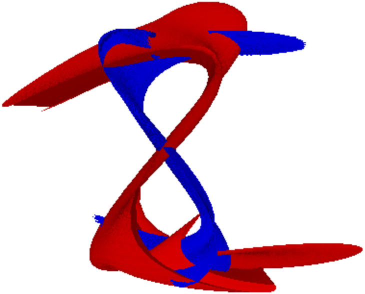

Figure: NHIM's in an oscillating double well: L. Zhang, R. de la Llave Comm. Nonlin. Sci. Num. Anal. (2018)

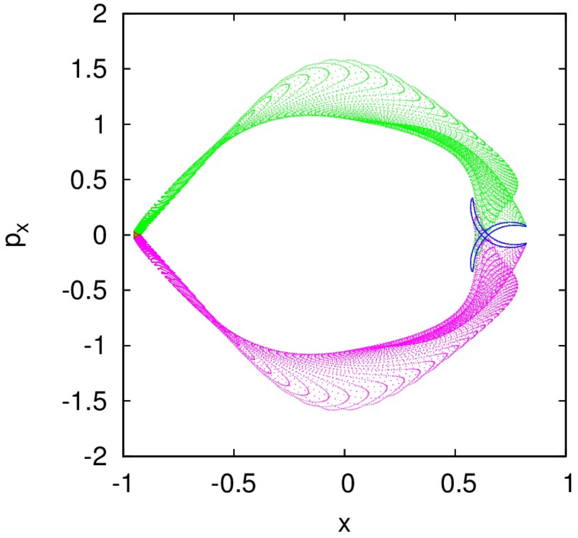

Figure: Stable/unstable manifolds of Lyap orbits near L2 (Sun-Jupiter) M. Capinski, M. Gidea, R. de la Llave Nonlinearity (2017)

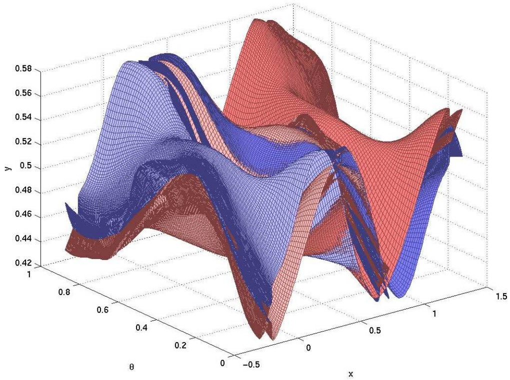

Figure: Stable/unstable manifolds of a quasi-periodic standard map. A. Haro, R. de la Llave SIADS (2006)

Main traditional idea (already used in several contexts)

1. Follow a homoclinic excursion when they are favorable;
2. Stay near the NHIM to recover.

This requires analyzing the dynamics of the jumps and the dynamics in the manifold

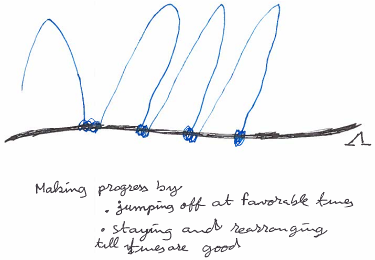

Figure: Heuristic of many instabibliy mechanisms.

Main new development:

We can dispense with analyzing the behavior near the NHIM and use ONLY the dynamics of the excursions

Then, one uses mainly soft methods and dispenses completely with the hard methods (averaging, KAM, Aubry-Mather....). Therefore, many difficult hypothesis (differentiability, twist, low dimensions, positive definite, etc.).

Even Hamiltonian hypothesis can be slightly weakened.

We need to quantify the jumps very carefully.

Use language similar to the ones used in Mathematical scattering theory.

- Wave maps
- Scattering map

Using the theory of NHIM's we will discover that these objects have very good smoothness/geometric properties.

It was observed long ago (Wheeler, Heisenberg,etc.) that these objects have extremely nice cancellations and give a very concise description.

Alternative

There is an alternative approach (the separatrix map). Introduced by G. Zavslasky, perfected by D. Treshev.

- Has the dimension of the full space
  - $+$ Gives global information of all orbits.
  - - Requires more work
- The main terms are singular
- Depends on the dynamics of the map on the manifold (hard to deal with resonances in inner map)

Wave maps

To a point y converging to the manifold Λ, we associate the unique point in Λ whose orbit is exponentially asymptotic to the orbit of y.

$$\Omega_+ y = x \iff y \in W^s_x$$
$$\Omega_+ y = x \iff y \in W^u_x$$

The theory of NHIM's guarantees $\Omega_\pm$ are smooth and depend smoothly on parameters.

The scattering map

We consider transversal intersection Γ. of Stable/Unstable manifolds to a manifold Λ.

We also require that the intersection is transversal to the foliation by stable manifolds of points.

$$S^{\Gamma}(x) = \Omega_{+} \circ (\Omega_{-}^{\Gamma})^{-1}$$

Equivalently

$$\begin{aligned} S^{\Gamma}(x_{-}) = x_{+} &\iff \exists z \in \Gamma; z \in W^{s}_{x_{+}} \cap W^{u}_{x_{-}} \\ &\iff \exists z \in \Gamma; d(\phi^{t}(x_{\pm}), \phi^{t}(z) \leq C\lambda_{\pm}^{|t|} \quad t \to \pm\infty \end{aligned}$$

The scattering map gives the asymptotic orbit in the future as a function of the asymptotic orbit in the past.

Math. Physicists will notice the similarity with the S matrix of Wheeler, Heisenberg.

The theory of NHIM's will endow the scattering map with remarkable properties.

Note that each piece of intersection will give us a different scattering map. In general, we expect many scattering maps. (This is a feature, not a bug).

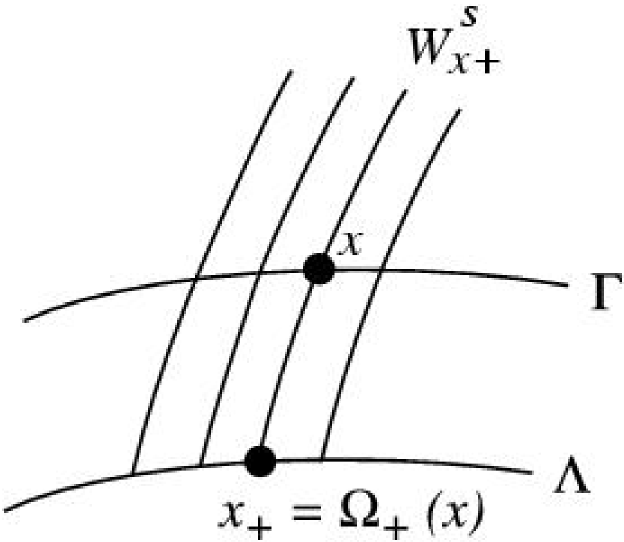

Figure: Illustration of the wave maps and homoclinic channels.

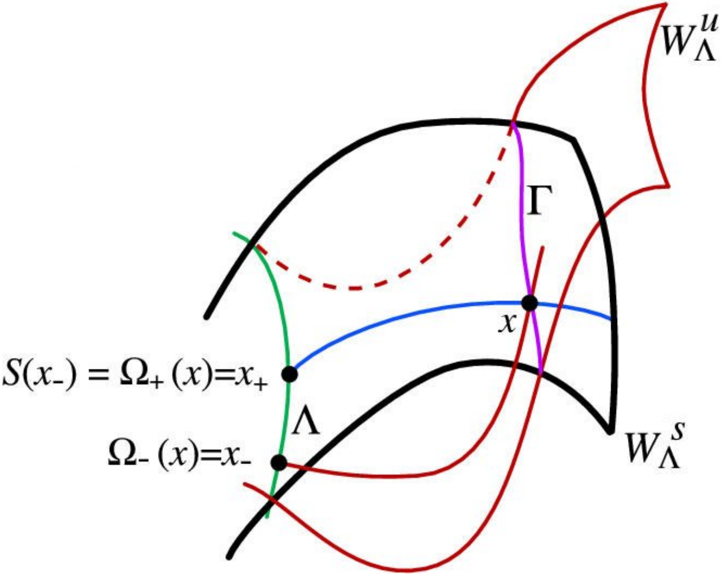

Figure: Illustration of the wave maps and homoclinic channels.

From the general theory of NHIM, we get several remarkable properties of Scattering map.

- Map on Λ (lower dimensional).
- It is smooth.
- It depends smoothly on parameters.
- It can be defined for C^1 open neighborhoods of maps. Even for not symplectic maps.
- If maps are symplectic, Λ is symplectic, then the scattering maps are symplectic
- Smooth dependence on parameters + symplectic properties $\implies$ very effective deformation theory; variational principles, etc.

Even when the intersections are transversal and one can define the map, there are topological reasons why one may have many scattering maps.

Even in the problem of geodesics in $\mathbb{T}^2$ one gets monodromy.

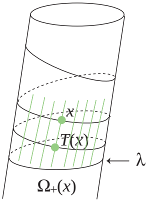

Figure: Illustration of a homoclinic channel with monodromy. Not a bug, a feature.

Main result

Theorem

Consider a Λ with several scattering maps $S^{\Gamma^1}, \dots, S^{\Gamma^n}$ (defined in different domains). $\forall \delta > 0$, given any sequence (possibly infinite) of points

$$y_{n+1} = f^{j_n} \circ S^{\Gamma_{\sigma_n}} \circ f^{l_n} \; y_n$$

j_n , l_n sufficiently large depending on the previous ones

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j_n , l_n sufficiently large depending on the previous ones

$$\implies \exists \text{ an orbit of } f \text{ which passes}$$

$$\text{at a distance } \delta \text{ from all the } y_n$$

(i.e. there exists $z_{n+1} = f^{j_n + l_n}(z_n)$, such that $d(z_n, y_n) \le \delta$)

Note that this construction has many choices.

Starting with y_i, we have many choices for n_i, hence many choices for y_{i+1}. This is consistent with the intution of *diffusion* and allows to reach many points.

For a small perturbation, we can make more elaborate choices the smallest the perturbation to accomplish effects of the same size.

The choices can be optimized for different concrete problems.

Note:

- The only assumption is the existence of transversal intersections. This assumption can be verified explicitly in concrete models (Using e.g. perturbation theory, variational methods, numerics, etc.)
- No assumptions of symplectic or not
- No assumptions on the dynamics on the manifold.
- The hypothesis are true in C^1 open sets of maps.
- It allows to consider infinite orbits.
- Allows to work in infinite dimensional maps.

Corollary

Assume that $f|_\Lambda$ satisfies the non-wandering (in particular, if it satisfies Poincaré recurrence).

Given any finite sequence

$$y_n = S^{\Gamma_{\sigma n}} y_{n+1}$$

can be ϵ shadowed by a true orbit of the map.

Corollary

Assume that $f|_\Lambda$ preserves volume and that there is a finite sequence

$$y_n = S^{\Gamma_{\sigma n}} y_{n+1}$$

that moves more or less the same as the pseudo orbit.

Application to Celestial Mechanics: Capinski, Gidea, R.L.

We consider the *circular restricted thee body problem* CR3BP for the values of the masses Sun, Jupiter.

One studies it in a co-rotating frame (Synodic system).

- There is a conserved quantity (Jacobi constant)
- (Lagrange) There are 5 equilibria.
  For the values of Sun/Jupiter, L_3 has a center directions (hence a center manifold) and hyperbolic directions.
- (Lyapunov) The center manifold is filled with periodic orbits.

If the Sun-Jupiter move not exactly as circles, the system is not autonomous in the rotating frame (energy is not conserved).

Will the effects of the tides accumulate?

Application to Celestial Mechanics: Capinski, Gidea, R.L.

Theorem Nonlinearity (2017)

In the Sun-Jupiter system.

Assume some explicit expression (computed in orbits of the CR3BP) is not identically zero.

Then, we can find $e_0, E^* > 0$. Then, if the primaries (Sun-Jupiter) move in an orbit of eccentricity e, $0 < e \leq e_0$, then there are orbits near L_3 that gain energy E^*.

Numerical Computation

The assumption of the above theorem is true
Computer assisted proof Capinski-Gidea, in progress

The assumption of the above theorem is true.

Application to Celestial Mechanics: Capinski, Gidea, R.L.

Work in progress (Anderson, Gidea, R.L)

Supplement the above mechanism with small thrusts to make the gain of energy happen in reasonable time.

Idea: the thrusts allow you to clip the time spent in the NHIM. A small thrust allows you to jump from stable to unstable when you are close to NHIIM.

Application to the a-priori unstable model

Theorem (Gidea, Seara, R. L)

Consider the models described by the Hamiltonian

$$H(p, q, I, \phi, t) = \sum_j \pm(p_i^2/2 + V(q_i)) + h_0(I) + \varepsilon H_1(p, q, I, \phi, t; \varepsilon)$$

Assume

- $V_i'(0) = 0$, $V_i'' \neq 0$, no other critical point same energy.
- Some explicit integral on H_1 do not vanish identically.
- Some mild regularity assumptions.

Then, you get instability of order 1.

If there are several non-zero expressions, and they satisfy some non-degeneracy condition.

Note:

- No assumptions on the integrable part h_0.
- Arbitrary dimensions (even ∞, work in progress).
- The assumptions are C^2 open, analytic dense

Similar results were obtained in this model with different assumptions:

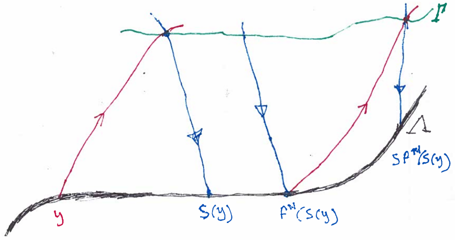
Treschev (2012)

- h_0 is non-degenerate
- The variables p, q are 1-D
- To some orders in perturbation, the system remains a product system.
- No resonances in the first order perturbation theory. '
- Obtained estimates on time.
  - An example by D. Turaev, shows that in the non-product case, the estimates on time have to be more complicated than in the product case.
  - An interesting question for applications to astrodynamics is how to shorten times by adding thrusts (small thrusts).

Delshams, Seara R.L. (2016)

- h_0 is non-degenerate

Some idea of a topological proof



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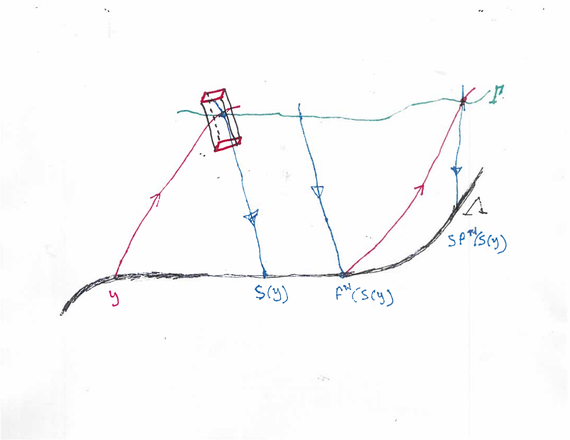

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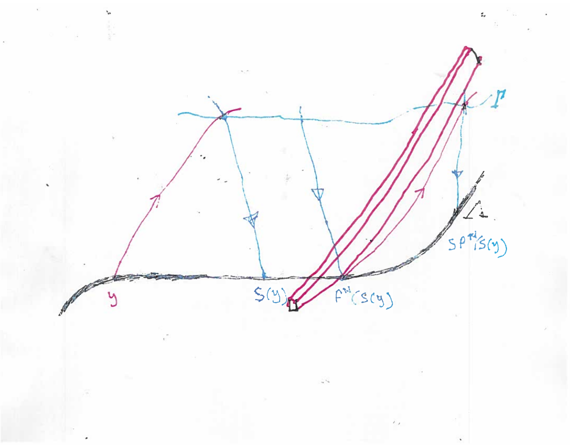

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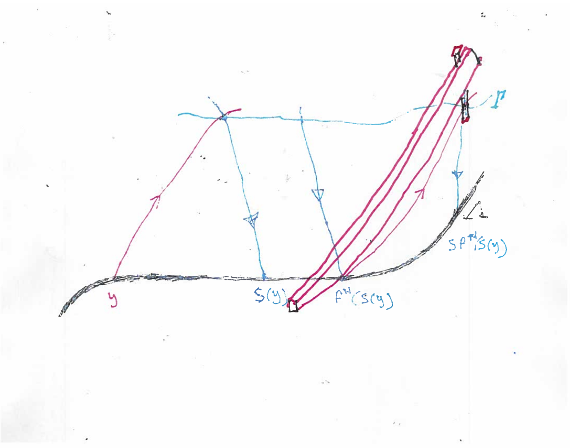

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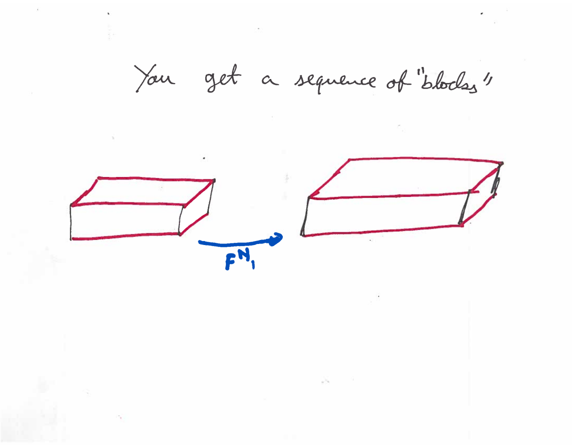

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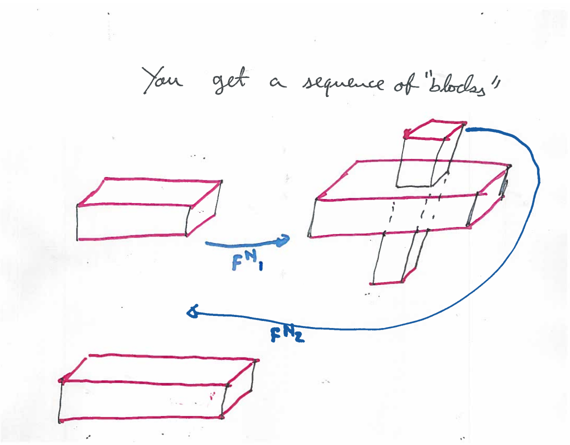

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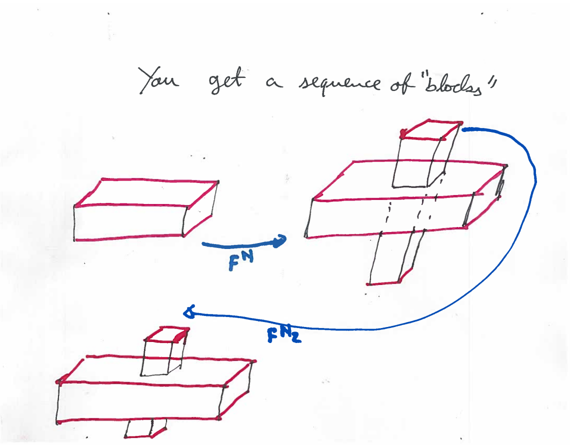

Work in Progress

Quasi dense orbits under Chow's condition

Consider the a-priori unstable model.

Assume that f satisfies the Poincare recurrence. '

Assume that there are scattering maps $S_1, \cdots, S_K$,

$$S_j = \mathrm{Id} + \epsilon \mathcal{S}_j$$

and that they satisfy in a neighborhood:

$$\mathrm{Span}\left(\mathcal{S}_i, [\mathcal{S}_{i_1}, \mathcal{S}_{i_2}], \cdots [\mathcal{S}_{i_1}, [\mathcal{S}_{i_2}, [\cdots \mathcal{S}_{i_L}] \cdots]\right) = T\Lambda$$

Then, for every path in action space, we can get orbits at distance $\epsilon^{1/2^L}$.

Work in Progress

Normally hyperbolic Laminations

If there is a transversal intersection, one gets a normally hyperbolic lamination. Hence infinitely many scattering maps.

NOTE: Had been done assuming non-resonance in the inner map. We want to do it more generally.

IDEA: Use Horseshoe theorem in spaces of embedding.

Work in Progress

Applications to infinite dimensional systems (concatenating breathers).

Thank you for your attention