Computations in the Topology of Locally Symmetric Spaces

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Introduction

$\mathbf{G}$ = connected semisimple algebraic group defined over $\mathbb{Q}$.
$G = \mathbf{G}(\mathbb{R})$. Maximal compact $K \subset G$.
$X = G/K$ = symmetric space.
Γ = arithmetic subgroup.

Example. $G = \mathrm{SL}_n(\mathbb{R})$. $K = \mathrm{SO}_n(\mathbb{R})$. $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$ congruence subgroup.

Example. $\mathbf{G}$ is the restriction of scalars of GL_n over a number field k with ring of integers $\mathcal{O}_k$.
Real quadratic k: Hilbert modular forms.
Imaginary quadratic k: Bianchi groups.

Our $\mathbf{G}$ have X contractible. Γ acts properly discontinuously on X.

If Γ is torsion-free,

$$H^*(\Gamma; \mathbb{C}) = H^*(\Gamma \backslash X; \mathbb{C}).$$

$M =$ rational finite-dimensional representation of G over a field $\mathbb{F}$ (typically $\mathbb{C}$ or $\mathbb{F}_p$). Gives a rep'n of Γ, hence a local system $\mathcal{M}$ on $\Gamma \backslash X$, and

$$H^*(\Gamma; M) = H^*(\Gamma \backslash X; \mathcal{M}). \tag{1}$$

If Γ has torsion, (1) is still true as long as the characteristic of $\mathbb{F}$ does not divide the order of any torsion element of Γ.

Theorem.

$$H^*(\Gamma; M) = H^*_{\text{cusp}}(\Gamma; M) \oplus \bigoplus_{\{P\}} H^*_{\{P\}}(\Gamma; M) \tag{2}$$

where the sum is over the set of classes of associate proper $\mathbb{Q}$-parabolic subgroups of G.

Projects We've Done.

- Compute the terms in (2) explicitly.
- Compute the Hecke operators on $H^*(\Gamma; M)$, which will help identify the terms on the right.
- Galois representations.
- Compute both non-torsion and torsion classes.

Case of SL_n: Lattices

$G = \mathrm{SL}_n(\mathbb{R})$ is the space of (det 1) bases of $\mathbb{R}^n$ by row vectors.

$\mathrm{SL}_n(\mathbb{Z})\backslash G$ is the space of lattices in $\mathbb{R}^n$.

$\Gamma\backslash G$ is a space of lattices with extra structure.

Choice of $K \Leftrightarrow$ inner product on lattices.

$X = G/K =$ space of lattice bases, modulo rotations.

$\Gamma\backslash X$ is a space of lattices with extra structure, modulo rotations.

How to Compute Cohomology

For a lattice L, the *arithmetic min* is $\min\{\|x\| : x \in L,\, x \neq 0\}$.
The *minimal vectors* of L are $\{x \in L \mid \|x\| = m(L)\}$.

L is *well-rounded* if its minimal vectors span $\mathbb{R}^n$.

Let $W \subset X$ be the space of bases of well-rounded lattices.

Theorem (Ash, late 1970s).

- There is an $\mathrm{SL}_n(\mathbb{Z})$-equivariant deformation retraction $X \to W$. Call W the *well-rounded retract*.
- $\dim W = \dim X - (n-1)$, the *virtual coh'l dim.*
- W is a locally finite regular cell complex. Cells characterized by coords in $\mathbb{Z}^n$ of their minimal vectors w.r.t. the basis.
- $\Gamma \backslash W$ is a finite cell complex.

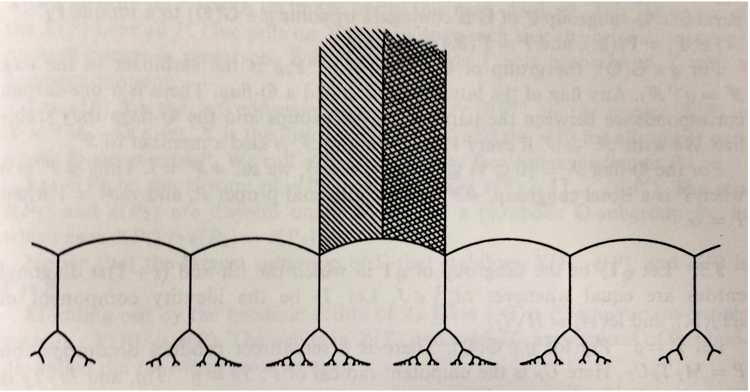
Ash (1984) did this for number fields k, not only $\mathbb{Q}$.

Conclusion. $H^*(\Gamma; M)$ can be computed in finite terms.

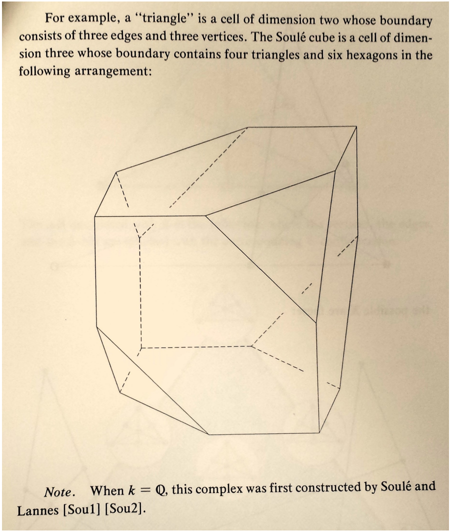
Appendix 1 discusses our improvements in time and memory performance for these difficult computations.

Example. $n = 2$. Then $X = \mathfrak{H}$, the upper half-plane.

Shaded region is fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.
W is the graph.
Vertices of W are bases of the hexagonal lattice $\mathbb{Z}[\zeta_3]$.
Edge-centers of W are bases of the square lattice $\mathbb{Z}[i]$.



Example. $n = 3$. Then $\dim X = 5$ and $\dim W = 3$.
W is glued together from 3-cells like this one, the *Soulé cube*.
Four cells meet at each $\triangle$ face, three at each $\hexagon$ face.
Vertices are bases of the $A_3 = D_3$ lattice (oranges at the market).



For example, a "triangle" is a cell of dimension two whose boundary consists of three edges and three vertices. The Soulé cube is a cell of dimension three whose boundary contains four triangles and six hexagons in the following arrangement:

Note. When $k = \mathbb{Q}$, this complex was first constructed by Soulé and Lannes [Sou1] [Sou2].

Theorem (Ash–M, 1996). The well-rounded retraction extends to the Borel-Serre compactification $\bar{X} \to W$. It is a composition of geodesic flows away from the boundary components.

Hecke Correspondences

Let ℓ be a prime. Take $k \in \{1, \ldots, n\}$.

$\Gamma = \mathrm{SL}_n(\mathbb{Z})$ for simplicity. $\Gamma \backslash X$ is the space of lattices.

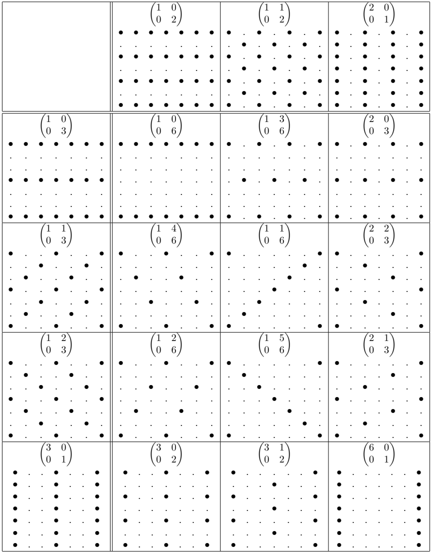
Given a lattice L, there are only finitely many lattices $M \subset L$ with $L/M \cong (\mathbb{Z}/\ell\mathbb{Z})^k$.

Def 1. The *Hecke correspondence* $T(\ell, k)$ is the one-to-many map $\Gamma \backslash X \to \Gamma \backslash X$ given by $L \mapsto M$.

For Γ of level N, need to modify Def 1 when $\ell \mid N$.

Example for $\mathrm{SL}_2(\mathbb{Z})$ on next page. $T(2, 1)$ has 3 sublattices, $T(3, 1)$ has 4 sublattices, and $T(6, 1)$ has the 12 intersections.

Hecke Operators $T(3)$ and $T(2)$ Producing $T(6)$



Alternative def: $t = \mathrm{diag}(1, \dots, 1, \ell, \dots, \ell)$ with k copies of ℓ.

$\Gamma_0(N, k) =$ matrices in $\mathrm{SL}_n(\mathbb{Z})$ congruent to $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ modulo N;
top left block is $(n-k) \times (n-k)$, bottom right $k \times k$.

$$\begin{array}{c} (\Gamma \cap \Gamma_0(\ell, k))\backslash X \\ r \downarrow \quad \downarrow s \\ \Gamma\backslash X \end{array}$$

where $r : (\Gamma \cap \Gamma_0(\ell, k))g \mapsto \Gamma g, \quad s : (\Gamma \cap \Gamma_0(\ell, k))g \mapsto \Gamma t g$.

Def 2. The *Hecke correspondence* $T(\ell, k)$ is $s \circ r^{-1}$.

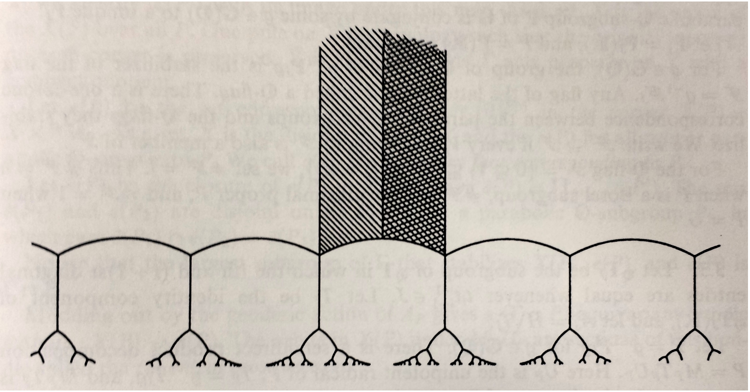
Def. The *Hecke operator* $T(\ell, k)$ on $H^*(\Gamma\backslash X; \mathcal{M})$ is $r_* \circ s^*$.

These $(\forall \ell, k)$ generate a commutative algebra, the *Hecke algebra*.

How to Compute Hecke Operators

Difficulty: Hecke correspondences do not preserve W.

If you retract, cells maps to fractions of cells.



The Sharbly[1] Complex

For $k \geqslant 0$, consider $n \times (n+k)$ matrices A over $\mathbb{Q}$.

Sh_k = formal $\mathbb{Z}$-linear combinations of symbols $[A]$, the *sharblies*.

- Permuting columns of A multiplies $[A]$ by the sign of the permutation.
- Multiplying a column of A by a non-zero scalar does not change $[A]$.
- If $\operatorname{rank} A < n$, then $[A]$ identified with 0.

$$\partial_k : [v_1, \ldots, v_{n+k}] \mapsto \sum_{i=1}^{n+k} (-1)^i [v_1, \ldots, \hat{v}_i, \ldots, v_{n+k}].$$

$(\mathrm{Sh}_*, \partial_*)$ is the *sharbly complex*.

[1] R. Lee, R. H. Szczarba, *On H_* and H^* of Congr. Subgps.*, Invent., 1976.

Tits building T_n: simplicial complex whose vertices are the proper non-zero subspaces of $\mathbb{Q}^n$, with simplices corresponding to flags. Homotopic to a bouquet of spheres S^{n-2}. The *Steinberg module* is $\mathrm{St} = \tilde{H}_{n-2}(T_n)$.

By Borel-Serre duality, if Γ torsion-free, the Steinberg module is the dualizing module.

The *Steinberg homology* of Γ is $H_*(\Gamma; \mathrm{St} \otimes_{\mathbb{Z}} M)$.

Theorem (L-S). $\cdots \to \mathrm{Sh}_1 \to \mathrm{Sh}_0 \to \mathrm{St}$ is an exact sequence of $\mathrm{GL}_n(\mathbb{Q})$-modules. If Γ torsion-free, the sharbly complex is a Γ-free resolution of the Steinberg module.

The *sharbly homology* of Γ is $H_*(\Gamma; \mathrm{Sh}_* \otimes_{\mathbb{Z}} M)$.

If Γ torsion-free, all are the same: $H^*(\Gamma; M)$, $H^*(\Gamma\backslash X; \mathcal{M})$, $H^*(\Gamma\backslash\bar{X}; \mathcal{M})$, $H^*(\Gamma\backslash W; \mathcal{M})$, Steinberg homology, sharbly homology.

Also all the same if M is over $\mathbb{F}$ of characteristic p and p does not divide the order of any torsion element of Γ.

Otherwise, see Appendix 2.

Cells of W are characterized by their minimal vectors $w_1, \ldots, w_{n+k} \in \mathbb{Z}^n$. Cochains for W map into the sharbly complex as $[w_1, \ldots, w_{n+k}]$, the *well-rounded* (or *Voronoi*) sharbly subcomplex.

Only works for a range of dimensions of cells of W. Always works for $n = 2, 3$. For $n = 4$, fortunately, the range contains the range of cuspidal cohomology.

Hecke correspondences act on the sharbly complex. They do not carry W to W.

Conclusion. In Ash–Gunnells–M computations for SL_4, we compute sharbly homology, not $H^*(\Gamma\backslash W; \mathcal{M})$.

For char 0 or $p > 5$, all these (co)homologies are the same. For $p = 2, 3, 5$ for SL_4, see Appendix 2.

Computing Hecke Operators in Top Degree

H^{vcd} corresponds to Sh_0, symbols on $n \times n$ matrices. For $n = 2$ and 3, this is in the cuspidal range.

For $n \leqslant 4$, well-rounded 0-sharblies have $|\det| = 1$.

Hecke correspondences carry these to matrices of $|\det| > 1$.

Ash–Rudolph (1979): algorithm to replace $[A]$ with $\sum [A_j]$, homologous in sharbly homology, and where $|\det A_j|$ are decreasing. Recursively, replace any 0-cycle with an equivalent cycle supported on W.

Generalizes *modular symbols* for SL_2 (Birch, Manin, Mazur, Merel, and Cremona). Generalizes continued fractions.

Computing Hecke Operators in Top Degree Minus One

For $n = 4$, top degree is H^6, but cuspidal range is H^5 and H^4.

Gunnells has a Hecke operator algorithm for H^5 in this case. H^5 is Sh_1, using 4×5 matrices. Three classes of well-rounded sharblies up to $\mathrm{SL}_4(\mathbb{Z})$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

All 4×4 subdeterminants are 0 or 1.

Gunnells uses a detailed study of 4×5 matrices and their subdeterminants.

Uses LLL to make subdeterminants smaller. Not proved to converge, but has never failed.

The Well-Tempered Retract

An algorithm for Hecke operators on $H^i(W; M)$ in all degrees i.

M. and Bob MacPherson, 2016–17.

$\mathbf{G}$ = restriction of scalars of GL_n for any number field k. Any n.

Have working code for $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$, $n = 2$ and 3. (Assume these cases in this exposition.)

Fix lattice L. Prime $\ell \nmid N$. $k \in \{1, \ldots, n\}$.
Fix $M \subseteq L$, one of the sublattices so $L/M \cong (\mathbb{Z}/\ell\mathbb{Z})^k$.

$t \in [1, \ell]$ real parameter, the *temperament*.

Definition. $y \in L$ has *tempered length*

$$\begin{cases} t \cdot \|y\| & \text{if } y \notin M \\ \|y\| & \text{if } y \in M. \end{cases}$$

Do well-rounded retraction with this notion, in each t-slice separately. Get $\tilde{W} \subset X \times [1, \ell]$, the *well-tempered retract*. Slice at t is $\tilde{W}_t$. The Γ-action preserves slices.

Continuously interpolates between
$\tilde{W}_1$, making L well-rounded; and
$\tilde{W}_\ell$, making M well-rounded.

Hecke operator $T(\ell, k)$ defined by $\tilde{W}_1$ on left, $\tilde{W}_\ell$ on right.

$$\begin{array}{c}(\Gamma \cap \Gamma_0(\ell, k))\backslash \tilde{W} \\ \downarrow \quad \downarrow \\ \Gamma\backslash W\end{array}$$

X is the space of positive-definite matrices (x_{ij}) modulo homotheties. Open set in $\mathbb{R}^{n(n+1)/2}$. *Linear coordinates.*

Fact. A bounded subset of $\tilde{W}$ can be computed as a big linear programming problem in the variables x_{ij} and $u = 1/t^2$.

Compute a bounded subset of a polyhedron dual to $\tilde{W}$, the *Hecketope*. Uses Sage's class `Polyhedron` over $\mathbb{Q}$.

Depends on n, ℓ, k.

Choose the bounds large enough to get all cells mod Γ.

Hecke Eigenclasses and Galois Representations

$\mathbb{F}$ = finite field of characteristic p. (Not $\mathbb{Q}_p$.)
Representation M is over $\mathbb{F}$.
Let $z \in H^i(\Gamma; M)$ be a Hecke eigenclass.
$a(\ell, k)$ = eigenvalue for $T(\ell, k)$.

$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{F})$ is a *Galois representation*, semisimple and continuous.

Def. ρ is *attached* to z if, $\forall \ell \nmid pN$, the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ is

$$\sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k. \tag{3}$$

Def. ρ *seems to be attached* to z if (3) holds for enough ℓ that you are confident of the result. Hope that some ℓ determine ρ, rest offer check.

Results

Ash and collaborators have many papers on SL_3.
Use $\Gamma_0(N) := \Gamma_0(N, 1)$ for a range of N.

Various M: constant coefficients, Dirichlet characters, $\mathrm{Sym}^r(x, y, z)$ for a range of r.

Give Hecke eigenvalues for a range of ℓ, and ρ that seem to be attached.

Ash–Grayson–Green (1984) found cuspidal cohomology in $H^3(\Gamma_0(N); \mathbb{C})$ for $N = 53, 61, 79, 89$. (More found since.)

Report on Ash–Gunnells–M's papers on $H^5(\Gamma_0(N); M)$ for SL_4.

Coefficients M:

- Constant coefficients:
  - Characteristic 0 (pretend $\mathbb{F}_{12379} = \mathbb{C}$). Did all $N \leqslant 56$, prime $N \leqslant 211$. Largest sparse matrix was 1M by 4M.
  - $\mathbb{F}_p$ for a few p not dividing the order of torsion elements of Γ (coefficients in $\mathbb{Z}$).
  - $\mathbb{F}_3$, $\mathbb{F}_5$, and $\mathbb{F}_2$.
- (being written, 2017) All nebentypes, i.e., all Dirichlet characters on the bottom-right entry of $\Gamma_0(N)$, taking values in $M = \mathbb{F}_p$ (generic p). Did all $N \leqslant 28$, prime $N \leqslant 41$.

Recall

$$H^*(\Gamma; M) = H^*_{\text{cusp}}(\Gamma; M) \oplus \bigoplus_{\{P\}} H^*_{\{P\}}(\Gamma; M) \tag{2}$$

We split left side $H^5(\Gamma_0(N); M)$ into Hecke eigenspaces for the ℓ we compute.

Each eigenspace always seems to be attached to a Galois representation we recognize. In fact, *uniquely*. We partly understand the summands for each $\{P\}$.

We have not yet seen any *autochthonous* cuspidal cohomology, i.e., not a functorial lifting from a lower-rank group. ☹

What Galois Reps do we Search For?

Let $\mathbb{F}'$ be a large enough finite extension of $\mathbb{F}_p$.

Let χ be any Dirichlet character $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}'^\times$.
$\varepsilon =$ cyclotomic character for p.
$\mathcal{L}_1 = \{\chi \otimes \varepsilon^i \mid \forall \chi, \forall i = 0, 1, 2, 3\}$.

Let $N_1 \mid N$. Let ψ be any nebentype character $(\mathbb{Z}/N_1\mathbb{Z})^\times \to \mathbb{C}^\times$.
Let f be a classical newform of weight 2, 3, 4 for $\Gamma_1(N_1)$ with nebentype character ψ.
Gives a Galois rep'n φ_f in characteristic 0 defined over a cyclotomic field K_f. Let $\mathfrak{P}$ be a prime of K_f over p. If $\mathbb{F}'$ is large enough, φ_f factors through to a rep'n over $\mathbb{F}'$.
$\mathcal{L}_2 =$ set of all these φ_f.
$\mathcal{L}_3 =$ symmetric squares of rep'ns in $\mathcal{L}_2$.

Tensor together repn's from $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$. Take direct sums of the tensors so total dim $= 4$.

The cuspidal SL_3 classes from AGG appear for $N = 53, 61, \ldots$.

For $N = 41$ and quartic nebentype, a cuspidal SL_3 class for that nebentype appears.

We get some classes in $H^*_{\mathrm{cusp}}(\Gamma; M)$. They are functorial liftings from holomorphic Siegel modular forms of weight 3 on $\mathrm{GSp}_4(\mathbb{Q})$. Ibukiyama: dims of weight 3 cuspidal Siegel modular forms on the paramodular groups of prime level. Gritsenko constructed a lift from Jacobi forms to Siegel modular forms on the paramodular group; ours are not Gritsenko lifts.

For cusp forms of weight 4 and prime N, we conjecture that they lift to cohomology if the central special value $\Lambda(2, f)$ vanishes.

We always observe the "epsilon powers" of the rep'ns are $[0, 1, 2, 3]$. The "epsilon power" of ε^i is i, of χ is 0, and of φ_f is $[0, \mathrm{weight} - 1]$.

Converses

Ash conjectured (1992) that any eigenclass z has an attached ρ.
$n = 2$: Eichler-Shimura, and Deligne.
Proved by Scholze (2014). The ρ will be odd.

Conversely,
Conjecture: For any odd ρ, $\exists \Gamma \; \exists M \; \exists z$ to which ρ is attached.
Conjectured by Ash-Sinnott (2000).
Ash-Doud-Pollack-Sinnott (ADPS): refined to predict which Γ and M will arise.
Refined further by Florian Herzig (for generic rep'ns).
When $n = 2$, this was Serre's Conjecture. Proved by Khare and Wintenberger (2008).

Next project (Ash–Gunnells–M–Pollack, 2018?) Test the ADPS conjecture.

Appendix 1: Computational Issues

In our (co)homology calculations, the boundary maps are sparse.

Computing $H^*(\Gamma; \mathcal{M})$ when M is a $\mathbb{Z}$-module needs *Smith normal form* of the boundary operators A. If A is $m \times n$ over $\mathbb{Z}$ of rank r, then SNF is

$$A = PDQ, \qquad P \in \mathrm{GL}_m(\mathbb{Z}), \quad Q \in \mathrm{GL}_n(\mathbb{Z}),$$

and D is diagonal with entries $d_1, \ldots, d_r$, the *elementary divisors*, with $d_i \mid d_{i+1}$. (Possibly $d_{r+1} = \cdots = 0$.)

Two approaches to find elementary divisors.

(•) Find elementary divisors A mod $p_i^{n_i}$ for many primes p_i in parallel, and reconstruct D by Chinese remainder theorem.

Dumas–Saunders–Villard 2000
Eberly–Giesbrecht–Giorgi–Storjohann–Villard 2006: sub-cubic complexity on sparse matrices.

(•) Parallel methods don't give you P and Q. Need P, Q, P^{-1}, Q^{-1} to compute cohomology and Hecke operators. Much slower than parallel methods.

Use a Markowitz pivoting strategy to reduce fill-in of the sparse matrix.

Two tricks I found for computing H^i at large level (Ash–Gunnells–M 2009):

$$\cdots \leftarrow C^{i+1} \xleftarrow[A_i]{P_i D_i Q_i} C^i \xleftarrow[A_{i-1}]{P_{i-1} D_{i-1} Q_{i-1}} \leftarrow C^{i-1} \cdots$$

1. Store P_{i-1} and Q_i^{-1} on disk as a product of elementary matrices. Get their inverses by reading the elementary matrices in reverse order and inverting them.

2. Once you know Q_i, compute SNF of $\eta = Q_i A_{i-1}$, not A_{i-1}.

The topmost $\text{rank}(D_i)$ rows of $Q_i A_{i-1}$ are zero. This compression lets Markowitz be more intelligent at limiting fill-in for η.

Improvement on a 13614×52766 matrix is shown by dotted blue line in the figure [A–G–M 2009, p. 10].

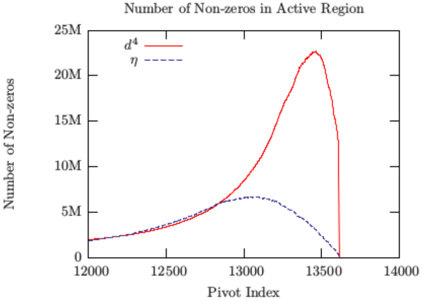


FIGURE 2. Size of the active region during an SNF computation for d^4 and η. Here M denotes million.

I have two main bodies of code.

- SHEAFHOM, for linear algebra and SNF for large sparse matrices over $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{Z}$, or other PIDs. In Common Lisp. http://www.bluzeandmuse.com/oldMarkGeocities/math.html
- Sage code.
  - Find W for $\mathrm{SL}_n(\mathbb{Z})$ for any n. In practice, $n \leqslant 4$.
  - Finite-dim rep'ns of Γ over $\mathbb{Q}$ or $\mathbb{F}_q$. Rep'n-theory operators $\oplus$, Res, Ind, Coind, $\otimes$.
  - Hecke operators: Ash-Rudolph for H^i at $i =$ vcd.
  - Hecke algorithm with MacPherson for H^i for all i.

Gunnells and Yasaki have code for W for SL_n for a range of n for $k = \mathbb{Q}$, real and imaginary quadratic fields, and some cubic fields. Also rank-one symmetric spaces like $SU(2,1)$. Hecke algorithms.

Appendix 2: SL_4 Sharbly Homology at $p = 2, 3, 5$

Theorem (A–G–M 2012) If p odd divides the order of a torsion element, then the sharbly homology, Steinberg homology, and well-rounded homology are all the same for SL_4 in the cuspidal range. At $p = 2$, the Steinberg and well-rounded homologies are the same in this range.