The $SL(2, \mathbb{R})$ action on Moduli space

Alex Eskin

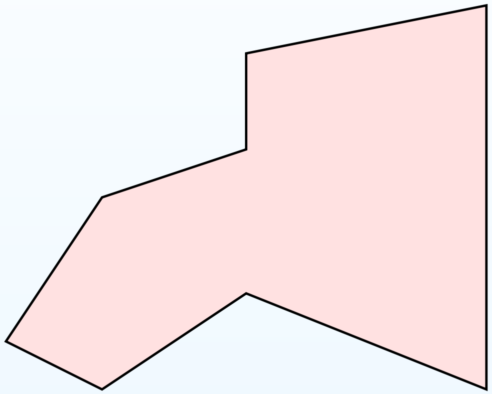
IAS

Nov 16, 2015

Polygons and flat surfaces

Polygons

Let P be a polygon (not necessarily convex).



We consider billiard trajectories inside P.

Problem. Compute the asymptotics as $T \to \infty$ of the number of (cylinders of) periodic trajectories of length at most T.

In general: Seems very difficult. Even for a triangle it is not known if periodic trajectories always exist. Best known upper bound is that the growth rate is subexponential.

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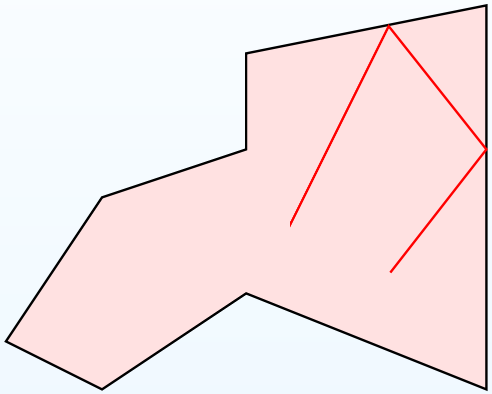

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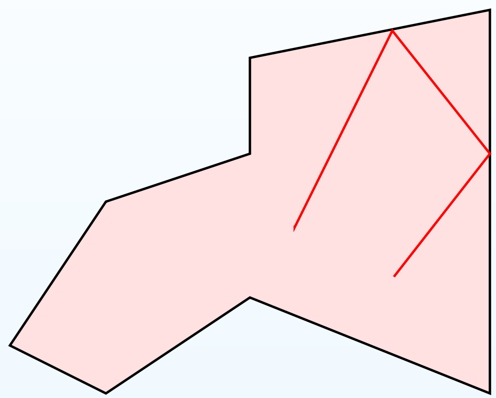

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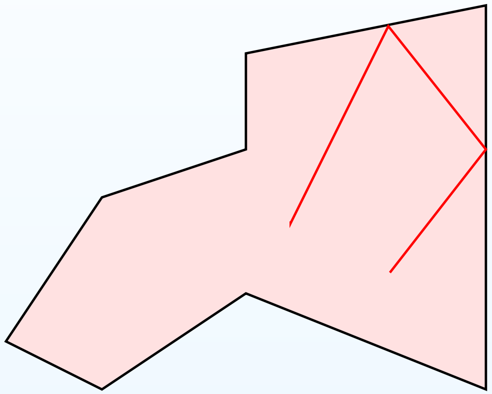

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Rational polygons

Standing Assumption: P is rational (i.e. all angles are rational multiples of π).

Theorem (H. Masur) There exist constants $c_1 > 0$ and $c_2 > 0$ depending on P such that as $T \to \infty$ the number $N(P, T)$ of (cylinders of) periodic trajectories of period at most T satisfies

$$c_1 T^2 < N(P, T) < c_2 T^2.$$

Goal. Convert the upper and lower bounds in this theorem to an asymptotic formula, and compute the constant.

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Construction (Zemlyakov-Katok): Given a rational polygon P construct a surface S such that billiard trajectories on P correspond to straight lines on S.

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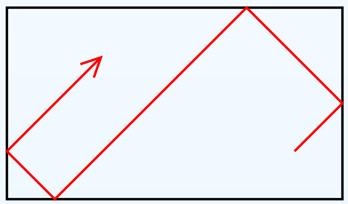

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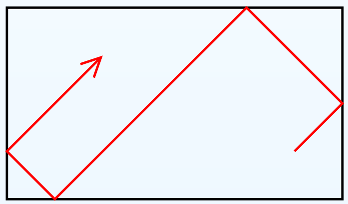

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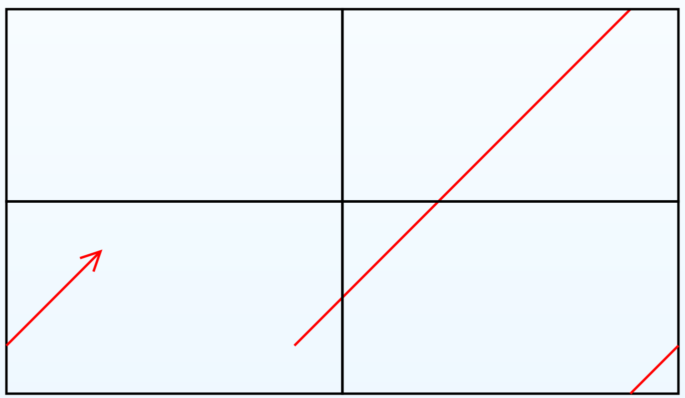

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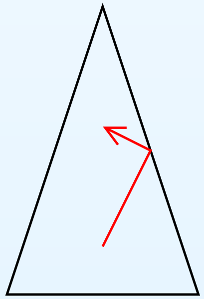

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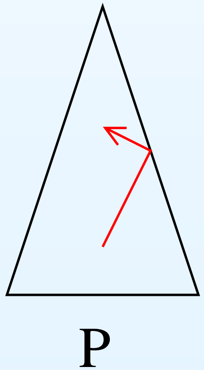

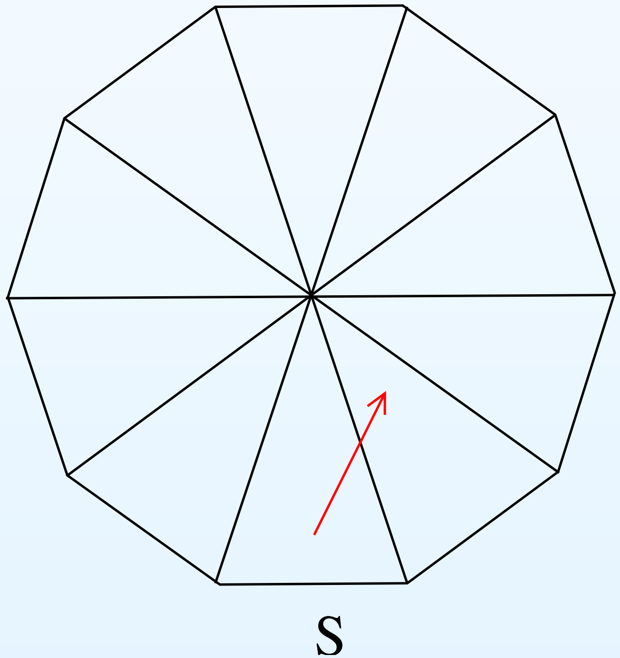

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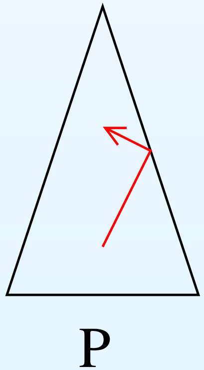

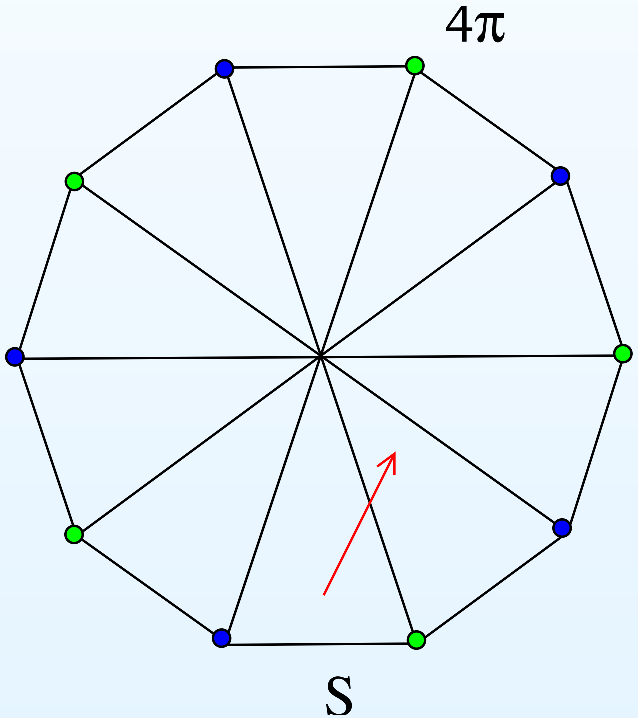

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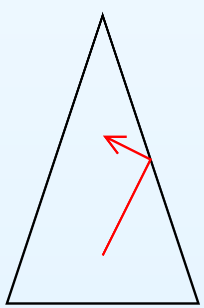

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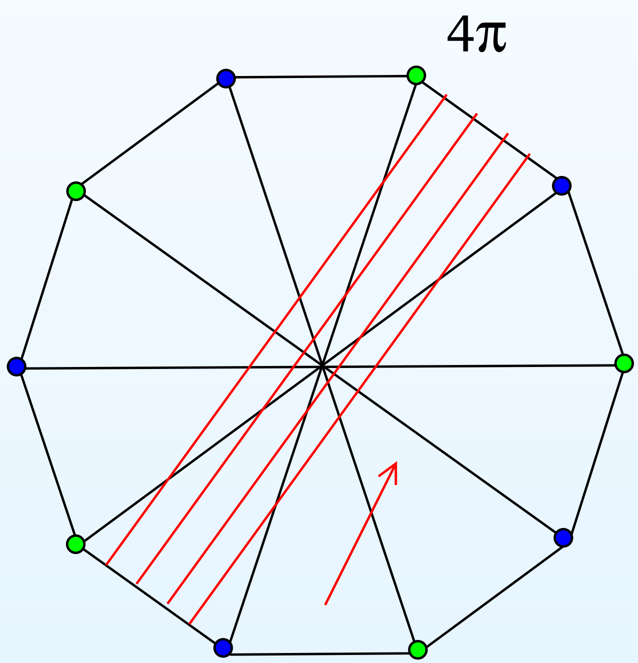

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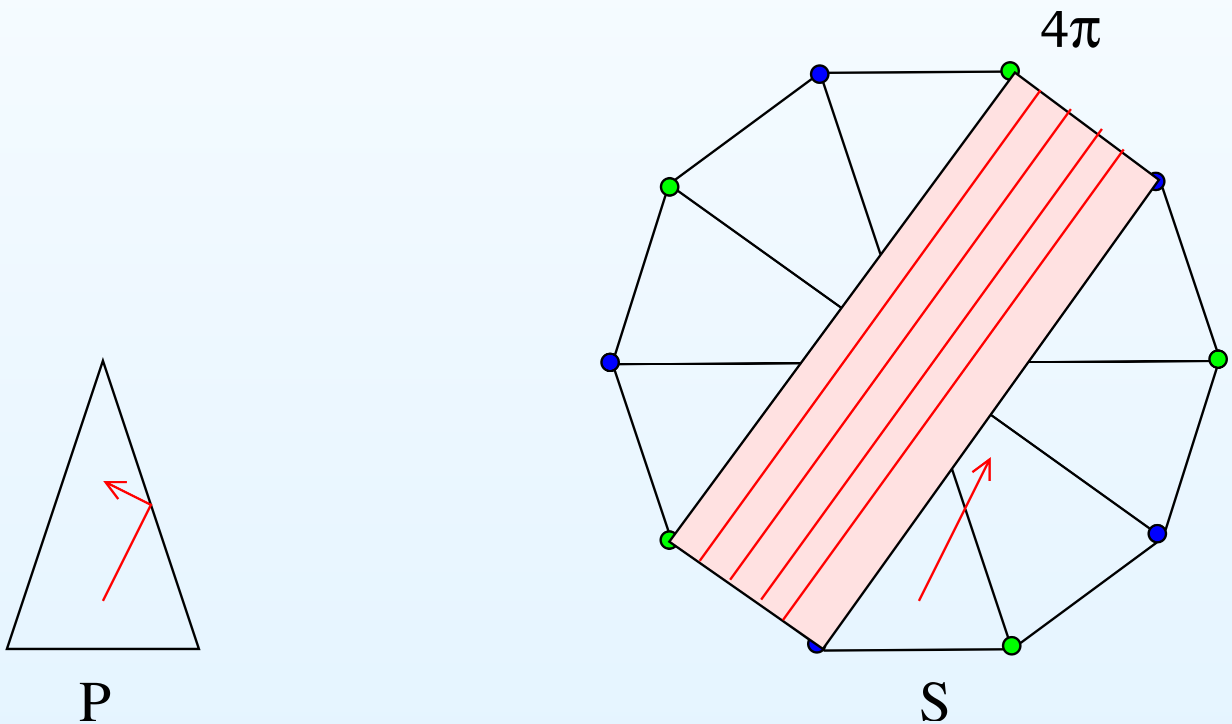

A cylinder of periodic trajectories.

Properties of flat surfaces

- The flat metric is nonsingular outside of a finite number of conical singularities (inherited from the vertices of the polygon).
- The flat metric has trivial holonomy, i.e. parallel transport along any closed path brings a tangent vector to itself.
- In particular, all cone angles are integer multiples of 2π.

- By convention, the choice of the vertical direction ("direction to the North") will be considered as a part of the " flat structure".
  For example, a surface obtained from a rotated polygon is considered as a different flat surface.
- A conical singularity with the cone angle $2\pi \cdot N$ has N outgoing directions to the North.

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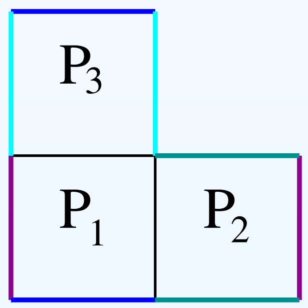
The $SL(2,\mathbb{R})$ action

A flat surface S is a union of polygons, $S = P_1 \cup \ldots P_n$.
We regard each polygon P_i a subset of $\mathbb{R}^2$. The polygons P_i are glued together along parallel sides. Each side is glued to exactly one other.



Suppose $g \in SL(2,\mathbb{R})$, e.g. $g = \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$. Since g acts on $\mathbb{R}^2$, we may define

$$gS = gP_1 \cup \ldots \cup gP_n,$$

with the same identifications of the sides as S.

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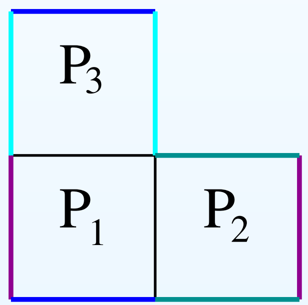

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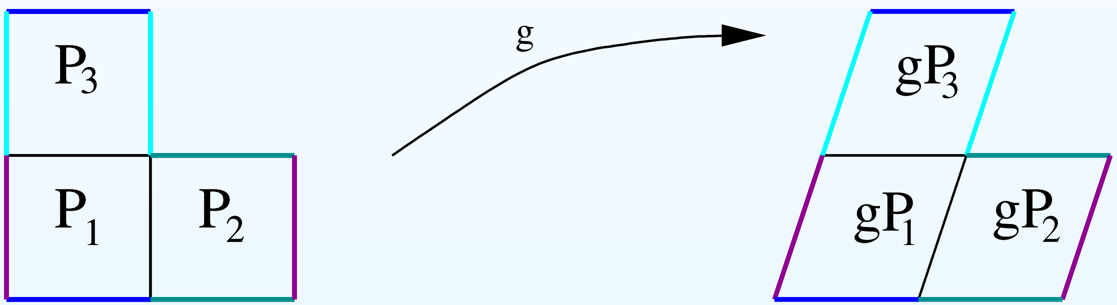

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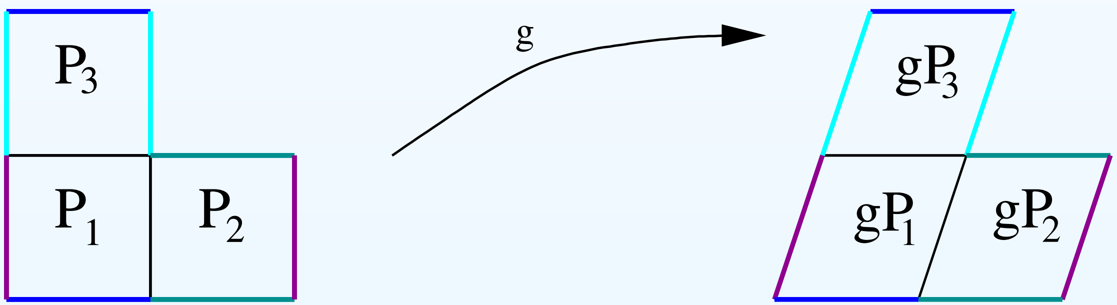

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Holomorphic 1-forms versus flat surfaces

Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points has a natural complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

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Flat structure canonically defined by a holomorphic 1-form

Reciprocally a pair (Riemann surface M, holomorphic 1-form ω) uniquely defines a flat structure.

- In the neighborhood of a point where ω is non-zero, there exists a local coordinate z such that $\omega = dz$. This coordinate is unique up to translation $z \to z + c$.
- If we use an atlas of charts using these coordinates in each chart, we get transition functions which are translations.
- In a neighborhood of zero a holomorphic 1-form can be represented as $\zeta^d \, d\zeta$, where d is the **degree** of zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$.
- The moduli space of pairs (complex structure, holomorphic 1-form) is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_m)$ enumerated by unordered partitions $d_1 + \cdots + d_m = 2g - 2$.
- Any holomorphic 1-form corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_m)$ has exactly m zeroes; their degrees are $d_1, \dots, d_m$.

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The (relative) period map and period coordinates

For a path $\gamma \in S = (M, \omega)$ we denote $\mathrm{hol}(\gamma) = \int_\gamma \omega$. Informally, the real and imaginary parts of $\mathrm{hol}(\gamma)$ are how far "east" and "north" one travels along γ.

Coordinates on $\mathcal{H}(\alpha)$. Let Σ denote the set of singularities (aka zeroes). Choose a basis $\{\gamma_1, \dots \gamma_n\}$ for the relative homology group $H_1(S, \Sigma, \mathbb{Z})$. Then the map $\Phi : \mathcal{H}(\alpha) \to (\mathbb{R}^2)^n \approx \mathbb{C}^n$ given by

$$\Phi(S) = (\mathrm{hol}(\gamma_1), \dots, \mathrm{hol}(\gamma_n))$$

is a local coordinate system on $\mathcal{H}(\alpha)$.

Measure. Let λ be the measure on $\mathcal{H}(\alpha)$ which is the pullback of Lebesque measure on $(\mathbb{R}^2)^n$. Then λ is well defined, and is invariant under the $SL(2, \mathbb{R})$ action.

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flat structure (including a choice of the vertical direction)	complex structure and a choice of a holomorphic 1-form ω
conical point with a cone angle $2\pi(d+1)$	zero of degree d of the holomorphic 1-form ω (in local coordinates $\omega = \xi^d \, d\xi$)
side $\vec{v}_j$ of a polygon	relative period $\int_{P_j}^{P_{j+1}} \omega$
family of flat surfaces sharing the same cone angles $2\pi(d_1+1), \dots, 2\pi(d_m+1)$	stratum $\mathcal{H}(d_1, \dots, d_m)$ in the moduli space of holomorphic 1-forms
coordinates in the family: vectors $\vec{v}_i$ corresponding to an independent set of edges in a triangulation	coordinates in $\mathcal{H}(d_1, \dots, d_m)$: cohomology class of ω in $H^1(S, \{P_1, \dots, P_m\}; \mathbb{C})$

Ergodic Theory

The Birkhoff Ergodic Theorem

Let X be a topological space, and let $T : X \to X$ be a map, which preserves a measure ν on X. We assume $\nu(X) = 1$ (so ν is a "probability measure").

Definition *ν is called ergodic if for any T-invariant subset $E \subset X$, either $\nu(E) = 0$ or $\nu(E) = 1$.*

Theorem (Birkhoff) *Suppose ν is ergodic. Then for any $f \in L^1(X, \nu)$ and ν-almost all $x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f\, d\nu$$

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What if we want to know what happens for *all* x?

A big problem

Fact. Let $\mathcal{P} \subset \mathcal{H}_1(\alpha)$ denote the flat surfaces which arise from unfolding a polygon. Then

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Using general ergodic theorems (A. Nevo) one can prove theorems like:

Theorem *(Masur and E., Veech) Let $N(S,T)$ denote the number of cylinders of closed geodesics on S of period at most T. There exists a constant b_α such that for λ-almost all $S \in \mathcal{H}_1(\alpha)$, as $T \to \infty$,*

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$$N(S,T) \sim \pi b_\alpha T^2,$$

but this says nothing about polygons.

Uniquely ergodic systems

Definition *A map $T : X \to X$ is called uniquely ergodic if there exists a unique invariant measure ν.*

Proposition *If $T : X \to X$ is uniquely ergodic and X is compact, then for any $f \in C(X)$ and for all $x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f \, d\nu$$

In a uniquely ergodic system, all points behave the same way.

Unipotent Flows and Ratner's Theorem

Let G be a semisimple Lie group with finite center, and let $\Gamma \subset G$ be a lattice. Let $U \subset G$ be a unipotent one-parameter subgroup. Then U acts on $X = G/\Gamma$ by left multiplication.

Theorem (Ratner)

(i) *Any ergodic U-invariant measure on X is homogeneous, i.e. is L-invariant measure supported on a closed orbit of a subgroup L, with $U \subseteq L \subseteq G$.*

(ii) *For any $x \in X$, $\overline{Ux} = Lx$ for some subgroup L, with $U \subseteq L \subseteq G$. In particular, $\overline{Ux}$ is a homogeneous submanifold of X.*

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This theorem makes it possible to control all orbits of U.
Theorems are false if one replaces U by a 1-parameter diagonalizable subgroup (e.g. orbit closures can be Cantor sets).

The $SL(2,\mathbb{R})$ action

The main theorem

Let $P = AU = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset SL(2,\mathbb{R})$.

Definition *An ergodic $SL(2,\mathbb{R})$-invariant probability measure ν on $\mathcal{H}_1(\alpha)$ is called* affine *if in local coordinates it is the restriction to $\mathcal{H}_1(\alpha)$ of the Lebesgue measure on a complex subspace of $H^1(M,\Sigma,\mathbb{C})$.*

Definition *An (immersed) submanifold of $\mathcal{H}_1(\alpha)$ is called* affine *if it is the support of an affine measure. (So in particular, it is closed, $SL(2,\mathbb{R})$-invariant, and in local coordinates it is a linear subspace).*

Theorem *(joint work with Maryam Mirzakhani)*

(i) Any ergodic P-invariant measure on $\mathcal{H}_1(\alpha)$ is $SL(2,\mathbb{R})$-invariant and affine.

(ii) For any x, $\overline{Px} = \overline{SL(2,\mathbb{R})x}$ is an affine submanifold.

(iii) "Any P-orbit is uniformly distributed in its closure".

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Proposition *Any stratum $\mathcal{H}(\alpha)$ contains at most countably many affine submanifolds.*

There is another proof of the Proposition by Alex Wright (showing that any affine manifold is defined over a number field).

Some further results

(joint work with Maryam Mirzakhani and Amir Mohammadi)

Theorem Any closed $SL(2,\mathbb{R})$-invariant (or P-invariant) subset of $\mathcal{H}_1(\alpha)$ is a finite union of affine manifolds.

Theorem The space of ergodic P-invariant measures on $\mathcal{H}_1(\alpha)$ is compact in the weak-* topology.

Theorem Suppose $\mathcal{M}_n$ is a sequence of affine manifolds, and suppose the associated affine measures $\nu_{\mathcal{M}_n}$ converge to ν. Then, $\nu = \nu_{\mathcal{N}}$ is an affine measure supported on an affine manifold $\mathcal{N}$. Furthermore, $\mathcal{M}_n \subset \mathcal{N}$ for all but finitely many n.

(The last theorem essentially says that any infinite sequence of affine manifolds has to equidistribute in a bigger affine manifold).

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Towards a classification of affine manifolds

The basic invariants

We will consider $GL(2,\mathbb{R})$-orbit closures (i.e. drop the restriction on the area). Let $\mathcal{M}$ be an affine manifold (so it is locally a subspace in $\mathbb{C}^n$ cut out by linear equations with real coefficients). Let $T\mathcal{M}$ be the tangent space of $\mathcal{M}$.

- $\dim(\mathcal{M}) = \dim_{\mathbb{C}}(T\mathcal{M})$ means the complex **dimension.**
- Let $p : H^1(M,\Sigma,\mathbb{C}) \to H^1(M,\mathbb{C})$ be the natural map.

  The **rank** of an affine manifold $\mathcal{M}$ is $\frac{1}{2}\dim_{\mathbb{C}} p(T\mathcal{M})$.

  (Directions in $\mathcal{M}$ where one deforms relative periods while keeping the absolute periods fixed contribute to the dimension but not to the rank).
- Let $\kappa(\mathcal{M})$ be the smallest field such that that $\mathcal{M}$ can be defined by linear equations in period coordinates with coefficients in $\kappa(\mathcal{M})$.

  $\kappa(\mathcal{M})$ is called the **affine field of definition** of $\mathcal{M}$.

More on the field of definition and the rank

Theorem (A. Wright)

- $\kappa(\mathcal{M})$ is always a totally real number field. We use the shorthand $\mathrm{degree}(\mathcal{M})$ for the degree of the extension $[\kappa(\mathcal{M}) : \mathbb{Q}]$
- $\mathrm{degree}(\mathcal{M})\,\mathrm{rank}(\mathcal{M}) \leq g$.
- "$\kappa(\mathcal{M})$ knows about the flat structures". E.g.

  If there exists $S \in \mathcal{M}$ and a direction in which S has exactly one cylinder, then $\kappa(\mathcal{M}) = \mathbb{Q}$.
- "$\mathrm{rank}(\mathcal{M})$ also knows about the flat structures". E.g.

  There exists horizontally periodic $S \in \mathcal{M}$ such that the core curves of horizontal cylinders in S span a subspace of dimension $\mathrm{rank}(\mathcal{M})$.

Strata

Any stratum of genus g and n zeros is an affine manifold. It has rank g, dimension $2g + n - 1$ and is defined over $\mathbb{Q}$.

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Covering constructions. Suppose (Y, ω) is a translation surface, and $f : X \to Y$ is a (branched) covering map. Then $(X, f^*(\omega))$ is also a translation surface. Furthermore f commutes with the $GL(2, \mathbb{R})$ action. This passes to the orbit closures. As a consequence, every point in $\mathcal{M}' \equiv \overline{GL(2, \mathbb{R})(X, f^*(\omega))}$ is a cover of some point in $\mathcal{M} \equiv \overline{GL(2, \mathbb{R})(Y, \omega)}$. We say that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by a *covering constuction*.

We may have $\dim(\mathcal{M}') > \dim(\mathcal{M})$, but always $\text{rank}(\mathcal{M}') = \text{rank}(\mathcal{M})$.

Teichmüller curves

A Teichmüller curve $\mathcal{M}$ is a closed $GL(2,\mathbb{R})$ orbit. Each point of $\mathcal{M}$ is called a “Veech surface”. Veech surfaces are characterized by the property that their stabilizer in $SL(2,\mathbb{R})$ is a lattice (Smillie).

A Teichmuller curve has rank 1 and dimension 2.

Square-tiled surfaces. If a surface is tiled by squares, then it's stabilizer in $SL(2,\mathbb{R})$ is a finite index subgroup of $SL(2,\mathbb{Z})$ (typically non-congruence), and thus it's $SL(2,\mathbb{R})$ orbit is a Teichmüller curve. These curves are defined over $\mathbb{Q}$. Their union is dense in any stratum of the moduli space: in fact any point in moduli space with rational period coordinates parametrizes a square-tiled surface.

Easy fact: $\mathcal{M}$ is an affine manifold of rank 1 which is defined over $\mathbb{Q}$ $\iff$ $\mathcal{M}$ is obtained by a covering constuction from a torus. (The dimension of $\mathcal{M}$ may be large since some branch points may be allowed to move freely).

Towards a classification of Teichmüller curves

First constructions: Thurston and Veech.

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- In $\mathcal{H}(2)$ there is one infinite family of non-square tiled Veech surfaces (construction done independently by Calta and McMullen). These consist of pairs (M, ω) where the curve M has a Jacobian which admits real multiplication by a quadratic number field, and ω is an eigenform. (defined over $\mathbb{Q}(\sqrt{d})$.)
- In $\mathcal{H}(1,1)$ the only non square-tiled Veech surface is the regular 10-gon. (McMullen, using a theorem of Möller). (defined over $\mathbb{Q}(\sqrt{5})$.)

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Genus 3 or higher: Classification problem is open.

- There are infinite families in genus 3 and 4 constructed by McMullen (coming from Prym loci). (defined over $\mathbb{Q}(\sqrt{d})$.)
- There is a family found by Bouw and Möller (with finitely many curves in each genus). (these may have large degrees).
- Two more sporadic examples are known.

Higher dimensional affine submanifolds

Theorem (McMullen) In genus 2: Besides the Teichmüller curves, the only (non-obvious) affine submanifolds are the sets of curves whose Jacobian admits real multiplication by a quadratic number field, with the given holomorphic form as eigenform. (These have rank 1, dimension 3 and are defined over quadratic number fields).

There are similar examples in Prym loci in genus 3, 4 and 5 (McMullen). (These have rank 1, dimension 3 and are defined over quadratic number fields).
Some of these Prym loci contain Teichmüller curves. These are classified by Lanneau and Nguyen.

Mirzakhani's conjectures

Conjectures (Mirzakhani). Let $\mathcal{M}$ be an affine invariant manifold.

- (Arithmeticity Conjecture). If $\kappa(\mathcal{M}) \neq \mathbb{Q}$, then $\mathrm{rank}(\mathcal{M}) = 1$.
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Consistent with all previous examples.

Theorem *(Aulicino, Nguyen, Wright) Mirzakhani's conjectures are true in $\mathcal{H}(4)$, $\mathcal{H}(3,1)$, $\mathcal{H}(2,2)$ (genus 3).*

Theorem *(Apisa) Mirzakhani's conjectures are true in hyperelliptic components of strata.*

Proofs use following theorem:

Cylinder deformations

Let $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $a_s = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \subset GL(2, \mathbb{R})$.

Let x be a translation surface. Let $\mathcal{C}$ be the (possibly empty) set of horizontal cylinders in x.

Let $u_t^{\mathcal{C}}(x)$ be the translation surface obtained by applying the horocycle flow u_t to the cylinders in $\mathcal{C}$, but not to the rest of x.

Let $a_t^{\mathcal{C}}(x)$ be the translation surface obtained by applying the vertical stretch a_t to the cylinders in $\mathcal{C}$, but not to the rest of x.

Theorem (A. Wright) For any x,t, the “Cylinder Shear” $u_t^{\mathcal{C}}(x)$ and the “Cylinder Stretch” $a_t^{\mathcal{C}}(x)$ are both contained in the orbit closure $\overline{GL(2, \mathbb{R})x}$.

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This has many important consequences.

A sharp finiteness result.

(Previous results: Bainbridge-Habegger-Möller, Matheus-Wright, Lanneau-Nguyen-Wright.)

Theorem *(Filip-Wright-E) In any fixed stratum $\mathcal{H}(\alpha)$ there are only finitely many affine manifolds except for the following situations:*

(i) If there exists $\mathcal{M} \subset \mathcal{H}(\alpha)$ with $\mathrm{rank}(\mathcal{M}) = 2$, $\kappa(\mathcal{M}) = \mathbb{Q}$ with $\dim(\mathcal{M}) = k$, then $\mathcal{M} = \overline{\bigcup \mathcal{M}_n}$ where $\forall n, \mathrm{rank}(\mathcal{M}_n) = 1$ $\dim(\mathcal{M}_n) = k - 2$, and degree $\kappa(\mathcal{M}_n) = 2$.

(ii) For any $\mathcal{M} \subset \mathcal{H}(\alpha)$ with $\kappa(\mathcal{M}) = \mathbb{Q}$, we have $\mathcal{M} = \overline{\bigcup \mathcal{M}_n}$ where $\forall n, \mathrm{rank}(\mathcal{M}_n) = 1$ and $\kappa(\mathcal{M}_n) = \mathbb{Q}$.

Cor 1: In any genus there exist only finitely many $\mathcal{M}$ with degree $\kappa(\mathcal{M}) \geq 3$.

Cor 2: $\mathcal{H}(\alpha)$ contains infinitely many Teichmuller curves not defined over $\mathbb{Q}$ if and only if $\mathcal{H}(\alpha)$ contains an affine manifold of rank 2 dimension 4 defined over $\mathbb{Q}$. (e.g $\mathcal{H}(2)$, Prym loci).

Sketch of the proof

Suppose there are infinitely many $\mathcal{M}_n$ in a fixed stratum. Then $\mathcal{M} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}_n}$ is (a finite union of) affine manifolds.

For every affine manifold $\mathcal{M}$ we can consider the monodromy group. (Parts of) the Zariski closure of this group are well understood.

The "algebraic hull" is the analogous object but when you only consider monodromy along the $SL(2,\mathbb{R})$ orbits ("dynamical monodromy"). This is simple to define but usually uncomputable.

The main part of our proof is the computation of the algebraic hull for any affine manifold (in terms of the Zariski closure of monodromy). We also prove that if $\mathcal{M}_n \to \mathcal{M}$ then eventually the algebraic hulls of $\mathcal{M}_n$ coincide with the algebraic hull of $\mathcal{M}$. This (with some more work) implies the theorem.

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Theorem (S. Filip) Any affine manifold is an quasi-projective subvariety of moduli space.

The proof combines ideas from dynamics and Hodge theory. The main technique is constructing (in various clever ways) subharmonic functions on affine manifolds, and then using the dynamics to show they are constant. Some intermediate results (of independent interest):

- The Jacobians of the family of curves comprising the affine manifold all admit real multiplication.
- The difference between zeroes is always a (twisted) torsion point in the Jacobian.

These two items in fact define the manifold: these are the algebraic equations.

For Teichmüller curves, these results are due to Matrin Möller. Filip's proof is substantially different.