# Algebro-geometric aspects of Limiting Mixed Hodge Structures 

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## Outline

I. Guiding questions and notations
II. Summary of some results about Question A
III. Summary of some results about Questions $B, B^{\prime}, B^{\prime \prime}$
IV. Brief indication of proofs
V. Final remarks

References

This talk will be mostly rather expository; the objective is to give an overview that blends the purely Hodge-theoretic and algebro-geometric approaches. It will be largely drawn from the literature and a list of some representative references is given at the end.

## I. Discussion of the geometric questions

Question A: What can a smooth, projective variety $X_{\eta}$ degenerate to?
We imagine $X \rightarrow \Delta$ with generic fibre $X_{\eta}$ and central fibre $X$ a local normal crossing divisor, and are interested in the extreme cases of

- a generic degeneration $X_{\eta} \rightarrow X$;
- the "most singular" degeneration $X_{\eta} \rightarrow X$.

The Hodge theoretic translation is

$$
H^{n}\left(X_{\eta}\right)_{\text {prim }} \rightarrow H_{\text {lim }}^{n} .
$$

Here $H_{\text {lim }}^{n}$ is an equivalence class of limiting mixed Hodge structures.

Question A translates to
What are the extremal degenerations of a polarized Hodge structure?

We will define "extremal" below.
Question B: What Hodge-theoretic information about smoothings of $X$ are contained in

- $X$ alone;
- $\left(X, T_{X} \operatorname{Def}(X)\right)$ ?

Here $X$ is a projective variety that is locally a product of normal crossing divisors as arises in a semi-stable reduction in a several parameter family. Among other things we will see that the Hodge-theoretic smoothings (to be defined) may be strictly larger than the algebro-geometric ones.

Question B': What is the cohomological formulation of the refined differential of the period mapping at infinty?
The "refined differential" will be defined below.
Question B": What are the maximal monodrony cones that can arise algebro-geometrically?

Related to this question is to give algebro-geometric interpretations for the three properties of monodromy cones predicted by Deligne and proved by Cattani-Kaplan-Schmid and discuss a possible fourth such property.

## Terms and Notations

From Hodge theory

- $\left(V, Q, F^{\bullet}\right)=$ polarized Hodge structure;
- $D=$ period domain, or more generally a Mumford-Tate domain with compact dual $\check{D}$;
- $D=G_{\mathbb{R}} / H$ and $\check{D}=G_{\mathbb{C}} / P$ is the compact dual;
- $\Phi: S^{*} \rightarrow \Gamma \backslash D=$ variation of Hodge structure, where $S^{*}=S \backslash Z$ and where the image of $\pi_{1}\left(S^{*}\right) \rightarrow \Gamma$ is abelian with unipotent monodromies having logarithms $N_{1}, \ldots, N_{\ell}$;
- frequently, but not always, $S^{*}=\Delta^{* \ell}$;
- $\left(V, W, F^{\bullet}\right)=$ mixed Hodge structure, throughout assumed to be polarizable;
- $\left(V, W, \widetilde{F}^{\bullet}\right)$ where $\widetilde{F}^{\bullet}=e^{-i \delta} F^{\bullet}$ is the Deligne canonical $\mathbb{R}$-split MHS;
- $\sigma=\operatorname{span}_{\mathbb{R}>0}\left\{N_{1}, \ldots, N_{\ell}\right\}$ is a monodromy cone;
- $\left(V, W(N), F^{\bullet}\right)$ is a limiting mixed Hodge structure, defined for each $N \in \sigma$ with

$$
N_{k}: \operatorname{Gr}_{k}^{W(N)} V \rightarrow \operatorname{Gr}_{-k}^{W(N)} V
$$

$\left(\left(V, W(N), F^{\bullet}\right)=\right.$ limiting mixed Hodge structure $\downarrow$
nilpotent orbit $\left\{\begin{array}{l}N_{i} F^{p} \subset F^{p-1} \\ \exp \left(\sum z_{i} N_{i}\right) \cdot F^{\bullet} \in D, \operatorname{Im} z_{i} \gg 0\end{array}\right\} ;$

- equivalence class of limiting mixed Hodge structures means $F^{\bullet} \sim \exp \left(\Sigma \lambda_{i} N_{i}\right) \cdot F^{\bullet}, \lambda_{i} \in \mathbb{C}$;
- $Y$ is a grading element for $N,[Z, N]=-2 N$ and $W(N)_{k}=\underset{j \leqq k}{\oplus} E(Y)_{j} ;$

From algebraic geometry

- $X \rightarrow \Delta^{\ell}$ has central fibre $X$ and is given locally by

$$
\left\{\begin{array}{c}
x_{l_{1}}=t_{i_{1}} \\
\vdots \\
x_{l_{\ell}}=t_{i_{\ell}}
\end{array}\right.
$$

We adopt the convention that if the index set $l_{i}$ is empty then the equation doesn't appear; e.g., at a smooth point of $X$ there are no equations. It is locally a product of reduced normal crossing divisors times parameters;

- the type of $X$ is the maximum of singular factors that appear in the local descriptions;
- for $\mathbf{k}=\left[k_{1}, \ldots, k_{\ell}\right], X^{[\mathbf{k}]}=\left\{x \in X:\right.$ mult $\left._{x} t_{i} \geqq k_{i}\right\}$; for $\ell=1$ we just write $X^{[k]}$ for the usual stratification of $X$;
- smoothing of $X$ as above is $X \rightarrow S$ where $X=X_{s_{0}}$ and the fibres over $S^{*}$ are smooth and the monodromy representation is abelian.

An example of a several parameter family is

$$
X \times X \rightarrow \Delta \times \Delta
$$

which may be used to study natural classes, such as $N$, in $\operatorname{Hom}\left(H^{*}\left(X_{\eta}\right), H^{*}\left(X_{\eta}\right)\right)$.

- $\operatorname{Def}(X)=$ Kuranishi versal deformation space of $X$ with Zariski tangent space

$$
T_{X} \operatorname{Def}(X)=\mathbb{E x t}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)
$$

- $T \subset T_{X} \operatorname{Def}(X)$ will be a subspace transverse to $T_{X} \operatorname{Def}^{\text {es }}(X)$ and where a general $\xi \in T$ is smoothing (to $1^{\text {st }}$ order); the corresponding family is

$$
X_{T} \rightarrow T_{\epsilon}, \quad \mathcal{O}_{T_{\epsilon}} \cong T^{*}
$$

for one $\xi$ we write

$$
X_{\xi} \rightarrow \Delta_{\epsilon}=\operatorname{Spec} \mathbb{C}[\epsilon], \quad \epsilon^{2}=0
$$

## II. Statements of results about Question A*

- $D \subset \check{D}$ is an open $G_{\mathbb{R}^{-}}$-orbit and $\partial D=\bigcup G_{\mathbb{R}^{-}}$-orbits $\mathcal{O}$ with $\operatorname{codim} \mathcal{O} \geq 1$. In general there are
- several codimension 1 orbits in $\partial D$;
- a unique closed orbit $\mathcal{O}_{c}$;

$$
-\mathcal{O}_{c} \text { may be }<\begin{aligned}
& \text { totally real } \Longleftrightarrow \mathcal{O}_{c}=G_{\mathbb{R}} / P_{\mathbb{R}} \\
& \Longleftrightarrow \text { the Levi form } \mathcal{L}_{\mathcal{O}_{c}}=0 \\
&\text { not totally real (e.g., } G=\operatorname{SO}(2 a, b))
\end{aligned}
$$

Almost simplest example: $D=\operatorname{SU}(2,1) / T$ and $\check{D}=\operatorname{SL}(3, \mathbb{C}) / B$ is the incidence variety

*Based on joint work with Mark Green and Colleen Robles and uses results from [KP1] and [KP2].

Orbit structure is


- $D^{\prime}, D^{\prime \prime}$ classical;
- $D$ non-classical and is Mumford-Tate domain for polarized Hodge structures of weight $n=3$, Hodge numbers $(1,2,2,1)$ and an action of $\mathbb{Q}(\sqrt{-d})$.

Reduced limit period mapping (or naïve limit) $\Phi: \Delta^{*} \rightarrow \Gamma \backslash D$ lifts to

$$
\widetilde{\Phi}: \mathbb{H} \rightarrow D \subset \check{D}, \quad \widetilde{\Phi}(z+1)=\exp N \cdot \widetilde{\Phi}(z) .
$$

(i) (Schmid) for $\widetilde{\Psi}(z)=\exp (-z N) \widetilde{\Phi}(z): \mathbb{H} \rightarrow \check{D}$ unwinds $\Phi$ and we get

$$
\left\{\begin{array}{l}
\Psi: \Delta^{*} \rightarrow \check{D}, \\
\Psi(0)=F_{\lim }^{\bullet}
\end{array}\right.
$$

$\Longrightarrow \exp (z N) F_{\text {lim }}^{\circ}$ is a nilpotent orbit with corresponding limiting mixed Hodge structure $\left(V, W(N), F_{\text {lim }}^{\circ}\right)$.
(ii) Set

$$
\begin{aligned}
F_{\infty}^{\bullet}=\lim _{\operatorname{Im} z \rightarrow \infty} \widetilde{\Phi}(z) & =\lim _{\operatorname{Im} z \rightarrow \infty} \exp (z N) \cdot F_{\lim }^{\bullet} \in \partial D \\
& =\lim _{\operatorname{Im} z \rightarrow \infty} \exp (z N) \cdot \widetilde{F_{\lim }^{\bullet}} .
\end{aligned}
$$

For $B(N)=$ equivalence classes of $\left(V, W(N), F^{\bullet}\right)$ 's and $B(N)_{\mathbb{R}}$ the $\mathbb{R}$-split ones we have


Then $\Phi_{\infty}$ is holomorphic, factors as above, and is maximal with respect to these properties.


Here, $\left\{N, Y, N^{+}\right\}$is an $\mathrm{sl}_{2}$.

$D$ classical $\Longrightarrow \Phi_{\infty}=\mathrm{Gr}^{W}$ (extension data lost). In general, $\Phi_{\infty}$ retains some but not all of the extension data

Definition: The degeneration $X_{\eta} \rightarrow X$ is
(i) minimal if $\Phi_{\infty}$ belongs to a codimension 1 orbit;
(ii) maximal if $\Phi_{\infty}$ belongs to the closed orbit.

Theorem: For minimal degeneration when $D$ is a period domain, either

- $N^{2}=0$ and rank $N=1,2$;
- $N^{2} \neq 0, N^{3}=0$ and rank $N=2$.


double fourfold
double curve

$$
<0>
$$

$n=5$

double point

For $\operatorname{dim} X_{\eta}=5$,

$$
\begin{cases}x_{1} x_{2}+x_{3} x_{4}+x_{4} x_{5}=0 & \text { double point } \\ x_{1} x_{2}+x_{3} x_{4}=0 & \text { double surface } \\ x_{1} x_{2}=0 & \text { double fourfold }\end{cases}
$$

Theorem' ${ }^{\prime}$ [KP1] and [GGR]: For maximal deformations and general Mumford-Tate domains
$\left\{\begin{array}{c}\text { limiting mixed Hodge structure } \\ \text { is of Hodge-Tate type }\end{array}\right\}$
$\Longleftrightarrow\{$ closed orbit is totally real $\}$.

It is elementary that a necessary, but in general not sufficient, condition is

$$
h^{n, 0} \leqq h^{n-1,1} \leqq \cdots \leqq h^{n-[n / 2],[n / 2]}
$$

Theorem": In the period domain case, if the closed orbit is not totally real, then $N=2 m$ is even and

- $k \neq 0 \Longrightarrow \operatorname{Gr}_{n+k, \text { prim }}^{W(N)}$ is Hodge-Tate;
- $\operatorname{Gr}_{n+k, \text { prim }}^{W(N)} \neq 0 \Longrightarrow k \equiv 2 \bmod 4$;
- $\operatorname{Gr}_{n}^{W(N)} \neq 0$ and the non-zero $I_{\text {prim }}^{p, q}$ are $I_{\text {prim }}^{m+1, m-1}$ and $I_{\text {prim }}^{m-1, m+1}$ (no $l_{\text {prim }}^{m, m}$ ).


## Example: $n=4$



Picture a family of fourfolds $X_{\eta} \rightarrow X$ where $X_{\eta}$ is a product of polarized K3 surfaces each of which has a type III degeneration.
In general, for maximal degenerations, $\mathrm{Gr}^{W}(\mathrm{LMHS})$ is rigid.

## Remarks:

- The picture for general Mumford-Tate domains is more complex, and in some ways more interesting. For example,
- $G_{2}$ is singled out as an exception in the classification;
- if there is a Hodge-Tate degeneration, then for $D=G_{\mathbb{C}} / P$ the Lie algebra $\mathfrak{p}$ is an even Jacobson-Morosov parabolic.
- In general the Hodge-theoretic aspects of the $G_{\mathbb{R}^{-}}$-orbit structure of $\partial D$ may help guide the algebro-geometric properties of boundary strata in a moduli problem
- curves and abelian varieties, K3 surfaces, cubic threefolds and fourfolds (Laza) (these are all classical D's and only local normal crossings seem to occur), mirror quintics (non-classical $D$ );
- more non-classical examples? e.g., quintic surfaces and threefolds? May locally products of normal crossing varieties occur as they do Hodge-theoretically?
- In the classical case Mumford et. al., and in the general case Kato-Usui have constructed maximal completions of「/D's

$$
\Delta^{* \ell} \longrightarrow \Gamma / D
$$


$\Delta^{\ell} \longrightarrow \Gamma / D_{\Sigma}$,
$D_{\Sigma}=$ toroidal object

Are there minimal completions
any $S^{*}=S \backslash Z$ and where $\mathbb{H}_{\Phi(S)}>0$ where

$$
S^{*} \longrightarrow \Gamma / D
$$

$$
\begin{array}{cc}
\cap & \cap \\
S \xrightarrow{\Phi_{B}} \Gamma / D_{B}
\end{array}
$$

$$
\begin{aligned}
& \mathbb{H}=\left(\operatorname{det} H^{n, 0}\right)^{n}\left(\operatorname{det} H^{n-1,1}\right)^{n-1} \\
& \cdots \operatorname{det} H^{1, n-1} .
\end{aligned}
$$

In the classical case, as a set $D_{B}=D_{\mathrm{Gr} \Sigma}$ and a power $\mathbb{H}^{a}=\omega_{D}$ gives an ample line bundle on $\Phi_{B}(S)$.

## III. Statements of results about Questions B, B', B"*

Definition: $\operatorname{Ext}_{0_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)$ is the infinitesimal normal bundle of $X$.

- For $\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)=\mathcal{E}$ we have a stratification

$$
\left\{\begin{array}{l}
X_{0} \supset X_{1} \supset \cdots \supset X_{\ell}, \text { where if } X_{k}^{0}=X_{k} \backslash X_{k+1} \\
\left.\mathcal{E}\right|_{X_{k}^{0}} \text { is locally free of rank } k .
\end{array}\right.
$$

*Much of this was motivated by [Fr1].

Example: Locally $X=X_{1} \times X_{2}$ and stratification is

$$
X_{1} \times X_{2} \supset\left(X_{1, \text { sing }} \times X_{2}\right) \cup\left(X_{1} \times X_{2, \text { sing }}\right) \supset X_{1, \text { sing }} \times X_{2, \text { sing }}
$$

Definition: $\mathcal{E}$ is trivializable if there are $e_{1}, \ldots, e_{\ell} \in H^{0}(X, \mathcal{E})$ such that for each connected component of $X_{k}^{0}$ there are $e_{i_{1}}, \ldots, e_{i_{k}}$ that frame $\mathcal{E}$ over that component;

- $X=$ central fibre in $X \rightarrow \Delta^{\ell} \Longrightarrow \operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)$ is trivializable with $e_{i}=d t_{i}$.

Example: $X=$ locally a normal crossing divisor. Then

- $X=X_{0} \supset X_{1}=X_{\text {sing }}=D_{X}$;
- $\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right) \cong \mathcal{O}_{D_{X}}(X)$ is the infinitesimal normal bundle in [Fr1].
If $D_{X}=\coprod_{i=1}^{\ell} D_{i}$ then $\mathcal{O}_{D_{X}}(X)$ trivializable gives

$$
\mathcal{O}_{D_{X}}(X) \cong \bigoplus_{i=1}^{\ell} \mathcal{O}_{D_{i}}
$$

with sections $1_{D_{i}} \in H^{0}\left(\mathcal{O}_{D_{i}}\right)$.

We suggest that in this case $X$ should be thought of as potentially the central fibre in a family where over the origin the $i^{\text {th }}$ factor in the local product representation of $X$ is singular along $D_{i}$ and is smoothed along the $i^{\text {th }}$ coordinate axis. Thus, even normal crossing divisors should be thought of as occurring in multi-parameter families.
(*) Assumption: $\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)$ is trivializable.

In the simpler case where $X$ is locally a normal crossing divisor, from the standard local to global spectral sequence for Ext we have

$H^{1}\left(\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)\right) \rightarrow \operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right) \rightarrow H^{0}\left(\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)\right) \rightarrow H^{2}\left(\operatorname{Ext}_{\mathcal{O}_{X}}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)\right)$.

Theorem I: Under the assumption (*) there exists an $\ell$-parameter split limiting mixed Hodge structure with $1_{D_{i}} \leftrightarrow N_{i}$.

This is defined in terms of $X$ alone. In case $\delta=0$ and $T \subset T_{X} \operatorname{Def}(X)$ is unobstructed so that we have a family

$$
X \rightarrow \Delta^{\ell}
$$

where everything is smooth over $X^{*} \rightarrow \Delta^{* \ell}$, the limiting mixed Hodge structure is the associated graded to the one given by the family.

We will explain below how the assumption $\mathcal{O}_{D_{i}}(X) \cong \mathcal{O}_{D_{i}}$ enters into the construction.

Still in the case where $X=$ locally a normal crossing divisor, we have

Theorem II: Let $T_{S} \subset T$ be a $k$-dimensional subspace such that $\rho(L)$ does not lie in any coordinate hyperplane in $\mathbb{C}^{\ell}$. Then there exists a $k$-parameter limiting mixed Hodge structure whose commuting monodromies are linear combinations of the $1_{D_{i}}$ 's.
If there exists a family $\mathcal{X} \rightarrow S$ whose tangent space is $T_{S}$, then we obtain the limiting mixed Hodge structure associated to this family.
In general we do not have $S^{*}=\Delta^{* \ell}$, as illustrated by the following

Example: $X_{0}=$ nodal variety of dimension $2 n-1$ with nodes $p_{1}, \ldots, p_{\ell}$;

$$
X=\widetilde{X}_{0} \cup\left(\bigcup_{i=1}^{\ell} X_{i}\right), \quad X_{i} \cong \mathbb{P}^{2 n-1} \text { and } \widetilde{X}_{0} \cap X_{i}=Q_{i}
$$

the normal crossing variety that would be the central fibre in the standard semi-stable reduction of a smoothing of $X_{0}$. Then the limiting mixed Hodge structure in Theorem $I$ is the associated graded to the one that would arise if we could independently smooth the nodes.
In Theorem II, the assumption just below it implies that $X_{0}$ may be smoothed. Moreover, we have

$$
T_{X_{0}} \operatorname{Def}\left(X_{0}\right) \cong T_{X} \operatorname{Def}(X)
$$

(to $1^{\text {st }}$ order, deforming $X_{0}$ is the same as deforming $X$ )
and on $L=\rho^{-1}$ (coordinate hyperplanes in $\mathbb{C}^{\ell}$ ) subsets of the nodes may be smoothed


$$
\text { case } \ell=3, k=2
$$

Monodromies are $N_{1}+N_{2}, N_{1}+N_{3}, N_{2}+N_{3}$.

Three properties of nodal degenerations are
(i) the Koszul group $H_{1}\left(V ;\left\{N_{1}, \ldots, N_{\ell}\right\}\right) \cong\{$ group of relations among the nodes\};
(ii) the monodromy cone $\sigma=\left\{\sigma_{\lambda}=\Sigma \lambda_{i} N_{i}, \lambda_{i}>0\right\}$ is strictly smaller than the polarizing cone
$\sigma_{\mathrm{pol}}=:\left\{\sigma_{\lambda}: W\left(N_{\lambda}\right)\right.$ gives a polarized limiting mixed Hodge structure\}
if, and only if, the nodes are dependent (relations among nodes gives a larger polarizing cone);
(iii) there is an obstruction to extending the $N_{i}$ to commuting $\mathrm{sl}_{2, i}$ 's in the split limiting mixed Hodge structure in Theorem I, and this obstruction vanishes if, and only if, the nodes are independent.

Example: Another example where $D_{X}$ has multiple components is obtained by applying semi-stable reduction to a pencil of surfaces $X_{t} \subset \mathbb{P}^{3}$ where the singularities of the base locus are transverse intersections along the double curve of $X_{0}$. These all contribute components to $D_{X}$. Taking for example the familiar case when $\operatorname{deg} X_{t}=4$ all of the possible limiting mixed Hodge structures are extremal in the sense of the theorem and may be pictured as

$$
\operatorname{minimal}, N^{2}=0 \quad \text { maximal, } N^{2} \neq 0
$$



$$
X_{0}=Q_{1} \cap Q_{2}
$$


$X_{0}=$ tetrahedron
which are familiar from the work of Kulikov, Friedman et al. (cf. [Fr1] and [Fr2]).
It may be that a similar story about extremal degenerations holds in the work of Laza [La] on the cubic fourfold, but I haven't had a chance to check this.

Two general properties associated to families $X \rightarrow \Delta^{\ell}$ having as central fibre $X$ are

- $\operatorname{dim} \sigma \leq \#$ components of $D_{X}$;

This inequality can be substantially improved by a more precise statement extracted from the mechanics in the proof of Theorem I.

For $X$ locally a product of normal crossing divisors

- $N_{i_{1}}^{a_{1}} \cdots N_{i_{j}}^{a_{j}}=0$ if some $a_{i}>\left|I_{i}\right|$, or if $j>$ type of $X$.

This bounds from below the singularities that the central fibre must have in any semi-stable reduction of $X^{*} \rightarrow \Delta^{* \ell}$. Thus, if $N_{1} N_{2} \neq 0$ then $X$ must somewhere be locally $X_{1} \times X_{2} \times$ (parameters) where $X_{1}$ and $X_{2}$ are singular.

Example: Hodge theory suggests that for some surfaces with $p_{g} \geqq 2$ the moduli space (if such exists) would have to include singular surfaces of type $k \geqq 2$.

One of our original motivations was to have a refined and computationally useful formulation for the period mapping at infinity.

## Example:



For $t=\left(t_{1}, t_{2}\right)$ and $\ell(t)=(1 / 2 \pi i)\left(\log t_{1}+\log t_{2}\right)$ the period matrix is
$\Omega(t)=\left(\begin{array}{ccc}1 & & \\ & 1 & \\ & & 1 \\ \ell(t)+a_{11}(t) & a_{12}(t) & b_{1}(t) \\ a_{21}(t) & \ell(t)+a_{22}(t) & b_{2}(t) \\ b_{1}(t) & b_{2}(t) & c(t)\end{array}\right), \begin{gathered} \\ \left.\begin{array}{c} \\ a_{12}(t)=a_{21}(t) \\ \text { and } \operatorname{Imc}(t)>0 .\end{array}\right]\end{gathered}$

The nilpotent orbit is

$$
\left(\begin{array}{ccc}
1 & & \\
& 1 & \\
\ell(t)+a_{11} & a_{12} & b_{1} \\
a_{21} & \ell(t)+a_{22} & b_{2} \\
b_{1} & b_{2} & c
\end{array}\right)
$$

Rescaling gives

$$
\left\{\begin{array}{l}
a_{11} \rightarrow a_{11}+\lambda_{1} \\
a_{22} \rightarrow a_{22}+\lambda_{2}
\end{array}\right.
$$

For the correspoinding equivalence class of limiting mixed Hodge structures

| $\mathrm{Gr}_{2}$ |  |
| :--- | :--- |
| $\mathrm{Gr}_{1}$ | $\bullet$ |
| $\mathrm{Gr}_{0}$ | $\bullet \bullet$ |
| $\quad \bullet \quad-c$ gives $\mathrm{Gr}_{1}$ |  |
| $-a_{12}=a_{21}$ gives part of the extension upon |  |
| extension data for $\mathrm{Gr}_{2}$ over $\mathrm{Gr}_{0}$. |  |

For $\widetilde{\omega}_{1}, \widetilde{\omega}_{2}=\lim _{t \rightarrow 0} \omega_{1}(t), \omega_{2}(t)$ where $\omega_{1}(t), \omega_{2}(t)=$ normalized differentials of the $3^{\text {rd }}$ kind

- $a_{i j}=\int_{q_{i}}^{p_{i}} \widetilde{\omega}_{j}, \quad i \neq j$;
- may normalize $t_{i}, t_{2}$ to make the logarithmic singularities cancel and have well defined $a_{11}, a_{22}$.
Refined $d \Omega$ has $d c ; d b_{1}, d b_{2} ; d a_{12}$ and $d a_{11}, d a_{22}$. Moduli picture:

$$
\begin{aligned}
& \partial \mathcal{M}_{3} \subset \overline{\mathcal{M}}_{3}-\operatorname{dim}=6 \\
& \cup \\
& \mathcal{C}=\text { codimension } 2 \text { boundary component }
\end{aligned}
$$

- $\operatorname{dim} \mathcal{C}=4$ and $c, b_{1}, b_{2}, a_{12}$ give local coordinates;
- $a_{11}, a_{22}$ give normal parameters to $\mathcal{C} \subset \overline{\mathcal{M}}_{3}$.

Theorem: $\quad \xi \in \mathbb{E x t}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)$ defines a class

$$
\xi^{(1)} \in \operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{x_{\xi} / \Delta_{\epsilon}}^{1}(\log X) \otimes \mathcal{O}_{X}, \mathcal{O}_{X}\right)
$$

and the refined differential of the period mapping at infinity is expressed by the cup product
$\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X_{\xi} / \Delta \epsilon}^{1}(\log X) \otimes \mathcal{O}_{X}, \mathcal{O}_{x}\right)$
$\rightarrow \operatorname{End}_{\mathrm{LMHS}} \mathbb{H}^{n}\left(\Omega_{x_{\xi / \Delta \epsilon}^{\bullet}}(\log X) \otimes \mathcal{O}_{X}\right)$.
This is meant to give the flavor of the definition of and expression for the refined differential of the period mapping at infinity.

## IV. Discussion of proofs

Question A ([GGR]): Builds on [KP1], [KP2] and the later work [GG]; main points include

- $\left(V, Q, F^{\bullet}\right) \rightarrow\left(\mathfrak{g}, B, F_{\mathfrak{g}}^{\bullet}\right), \quad \mathfrak{g} \subset \operatorname{End}_{Q}(V)$;
- $\left(V, W, F^{\bullet}\right) \rightarrow\left(V, W, \widetilde{F^{\bullet}}\right)$ with
- $F_{\infty}^{\circ}=\widetilde{F_{\infty}^{-}}$,
- $(\widetilde{F} \bullet)_{\mathfrak{g}}=\left(\widetilde{F_{\mathfrak{g}}^{\bullet \bullet}}\right) ;$
- thus we may assume

$$
\left\{\begin{array}{l}
V_{\mathbb{C}}=\oplus I^{p, q} \\
\mathfrak{g}_{\mathbb{C}}=\oplus \mathfrak{g}^{p, q}
\end{array} \quad, \quad I^{p, q}=\overline{I^{q, p}} \text { and } \mathfrak{g}^{p, q}=\overline{\mathfrak{g}^{q, p}}\right.
$$

- then one matches the $\mathfrak{g}^{p, q}$ decomposition with the standard root theoretic description of the orbit structure.

Questions B, $\mathbf{B}^{\prime}, \mathbf{B}^{\prime \prime}$ : Recast and extend arguments in the literature, including [Fr1], [Fr2], [St1], [PS],... in the 1 -parameter case. Some work in the $\ell$-parameter case has been done in [Fu].

Example: In the 1-parameter case, $E_{1}^{p, q}$ terms are sums of groups $H^{a}\left(X^{[b]}\right)(-c)$ where

- $d_{1}$ is expressed in terms of various restriction and Gysin maps;
- $E_{2}=E_{\infty}=\operatorname{Gr}(\mathrm{LMHS})$.

Convenient to picture a split limiting mixed Hodge structure in terms of N -strings

$$
\begin{gathered}
H^{0}(-m) \rightarrow \cdots \cdots \cdots \cdots \rightarrow H^{0}(-1) \rightarrow H^{0} \\
H^{1}(-1) \rightarrow \cdots \rightarrow H^{1} \\
\vdots \\
H^{m}
\end{gathered}
$$

where $N^{k}=$ Hodge structure of weight $k$.

Theorem: If $\mathcal{O}_{D}(X) \cong \mathcal{O}_{D}$, there are complexes

$$
\left.\begin{array}{cccc} 
& H^{q-4}\left(X^{[k+2]}\right)(-2) & & \\
& H^{q-2}\left(X^{[k+1]}\right)(-1) & & \\
\rightarrow & H^{q-2}\left(X^{[k]}\right)(-1) & \rightarrow & \oplus \\
\oplus & & H^{q}\left(X^{[k-1]}\right) & \\
& H^{q}\left(X^{[k-2]}\right) & &
\end{array} X^{[k]}\right)
$$

such that for $0 \leqq j \leqq m-i$ the terms in the $N$-strings are

$$
H^{m-i}(-j) \cong\left(H_{\text {Rest }}^{*}{ }^{*}+" H_{\text {Gy }}^{*}\right)\left(H^{m-i}\left(X^{[i+1]}\right)\right)(-j) .
$$

Moreover, the monodromy $N=N_{1}+\cdots+N_{\ell}$ is induced by

$$
N=\sum_{i} 1_{D_{i}}(1)
$$

where for $b \geqq 2, X^{[b]}=\coprod_{i} X_{i}^{[b]}$ and

$$
H^{a}\left(X_{i}^{[b]}\right)(-c) \xrightarrow{{\stackrel{1}{D_{i}}}_{\rightarrow}^{\rightarrow}} H^{a}\left(X_{i}^{[b]}\right)(-(c-1)) .
$$

The important observation is that having trivializations

$$
\mathcal{O}_{D_{i}}(X) \cong \mathcal{O}_{D_{i}}
$$

implies that

$$
\text { Gy } \cdot \text { Rest }=- \text { Rest } \cdot \mathrm{Gy},
$$

so that the diagram in the statement of the theorem is actually a complex.

A perhaps new ingredient is the observation that the $N$ decomposes as $\Sigma 1_{D_{i}}(1)$. As in Theorem II we may then define an $\ell$-parameter $1^{\text {st }}$ order family and $N_{\lambda}=\Sigma \lambda_{i} N_{i}, \lambda_{i} \neq 0$. Here the $1_{D_{i}}$ correspond to a trivialization $\mathcal{O}_{D_{i}}(X) \cong \mathcal{O}_{D_{i}}$ and means that to $1^{\text {st }}$ order we have a specified parameter in the smoothing of the component $D_{i}$ of $X_{\text {sing }}$. We also have an $\mathrm{sl}_{2, i}$ acting on $E_{1}$ where

$$
Y_{i}=(2 c-b+1) 1_{D_{i}}
$$

on $H^{a}\left(X_{i}^{[b]}\right)(-c), b \geqq 2$. Then for $Y=\Sigma Y_{i}$,

$$
\left[d_{1}, Y\right]=-d_{1}
$$

so that $Y$ acts on $E_{2}=E_{\infty}=\operatorname{Gr}^{W(N)} V$.

In general $\left[d_{1}, Y_{i}\right]+d_{1} \neq 0$ and measures the linking between vanishing cycles and their duals associated to different components of $X_{\text {sing. }}$. This gives a geometric description of the obstructions to having commuting $\mathrm{sl}_{2, \text {, }}$ 's acting on $\mathrm{Gr}^{W(N)} V$.

## Example:


commuting sl2 ${ }^{\prime}$ s
none commuting $\mathrm{sl}_{2}$ 's

In both cases $\mathrm{Gr}^{W}(\mathrm{LMHS})$ is the same over $\mathbb{Q}$.

For general $X=$ locally a product of normal crossing varieties there are a few additional points that arise.

A first step is a fairly straightforward extension from the case $\ell=1$ of the identification for families $X \rightarrow \Delta^{\ell}$,

$$
H^{n}\left(X_{\eta}\right) \cong \mathbb{H}^{n}\left(\Omega_{x / \Delta^{e}}(\log X) \otimes \mathcal{O}_{X}\right)
$$

The complex $\Omega_{x / \Delta^{\ell}}^{\bullet}(\log X) \otimes \mathcal{O}_{X}$ depends only on $T_{\{0\}} \Delta^{\ell} \subset T \operatorname{Def}(X)$ and is identified as the cokernel of the map

$$
\Omega_{x}^{\bullet}(\log X) \xrightarrow{d t_{1} / t_{1} \wedge \cdots \wedge d t_{\ell} / t_{\ell}} \Omega_{X}^{\bullet+\ell}(\log X),
$$

which can be defined in terms of $T_{\{0\}} \Delta^{\ell}$ alone. Note the $\ell$-fold wedge product here.

A second point is that the $N_{i}$ are defined in terms of the connected components in the highest dimensional strata of the support of $\operatorname{Ext}_{\mathcal{O}_{X}}^{1}\left(\Omega_{X}^{1}, \mathcal{O}_{X}\right)$.
One subtlety is that as in the case $\ell=1$ one cannot define a weight filtration on the complex $\Omega_{x}^{\bullet}(\log X) \otimes \mathcal{O}_{X}$. To have a monodromy weight filtration $W$, whose $W_{k}$ are expressed on $E_{1}$ by linear inequalities among $a, b, c$, on $H_{\text {lim }}^{n}$ the indices have to have length $2 n+1$, and we can only get $n+1$ out of the usual filtration on $\Omega_{x}^{\bullet}(\log X)$. The correct thing to do is suggested by the map above, which is the first term in a resolution of $\Omega_{x}^{\bullet}(\log ) \otimes \mathcal{O}_{x}$ from which some numerology suggests what the weight filtration should be.

A central point is to use the hard Lefschetz theorem on the normalized strata of $X$ and both Hodge-Riemann bilinear relations to deduce that $W$, which easily satisfies

$$
N_{\lambda}: W_{k} \rightarrow W_{k-2}
$$

for all $\lambda$ in fact satisfies

$$
N_{\lambda}^{k}: W_{k} \xrightarrow{\sim} W_{-k}
$$

when $k \geqq 0$ and $\lambda=\left(\lambda_{1}, \ldots \lambda_{\ell}\right), \lambda_{i}>0$. Without this from the weight filtration $W$ we get a mixed Hodge structure but not a limiting mixed Hodge structure with $W=W(N)$. The proof of the result involves extending the nice argument due to Guillén and Navaro Anzar to the several parameter case.

## V. Final remarks

In closing there were three classical properties of monodromy cones for degenerating families of Hodge structures, predicted by Deligne (and established by him in the $\ell$-adic setting and proved in the Hodge-theoretic setting in [CKS1], [CKS2]).
(a) $W\left(N_{\lambda}\right)$ is independent of $\lambda$ with $\lambda_{i}>0$;
(b) $W(N)$ is a relative weight filtration for $W\left(N_{i}\right)$;
(c) the Koszul groups $H_{i}\left(V ;\left\{N_{1}, \ldots, N_{\ell}\right\}\right)$ vanish in positive weight (purity).

The statement (b) means that we easily have that $N: W_{k}\left(N_{i}\right) \rightarrow W_{k-2}\left(N_{i}\right)$, and then the much more subtle result, whose proof uses Hodge-Riemann I, II,

$$
W(N) \text { induces } W\left(N \mid \operatorname{Gr}^{W\left(N_{i}\right)}\right) \text { on } \mathrm{Gr}^{W\left(N_{i}\right)}
$$

is true. It follows from this and the above construction that, e.g., when $\ell=2$ we have the picture

variation of mixed Hodge structure induced by $X_{1}^{*} \rightarrow \Delta_{1}^{*} \times\{0\}$
and the limiting mixed Hodge structure " $\lim _{t \rightarrow 0}(t, t)$ " along the dotted line may be identified with " $\lim _{t_{1} \rightarrow 0} \lim _{t_{2} \rightarrow 0}$." This result is true by Deligne's argument in the $\ell$-adic case; the geometric argument may be useful in the computation of examples.

The result (b) means that the $\mathbb{C}\left[N_{s}, \ldots, N_{\lambda}\right]$-module $V$ has very special properties. For example, it implies that as a $\mathbb{C}\left[N_{\lambda}, N_{i}\right]$-module it is a direct sum of $\mathbb{C}\left[N_{\lambda}, N_{i}\right] /\left(N_{\lambda}^{p}, N_{i}^{q}\right)$ 's.

These statements can be verified using the above, which gives some slight refinements. For instance in the example of a nodal $X_{0}$
(iv) the weight filtration $W\left(N_{\lambda}\right)$ is independent of the $\lambda_{i} \in \mathbb{C}^{*}$ if, and only if, the nodes are independent.

Another motivation for the above is this: Given a nilpotent $N \in \operatorname{End}(V)$, Robles has formulated and proved a precise result which has as a corollary that $N$ is the monodromy of a variation of Hodge structure over $\Delta^{*}$ constructed in an explicit way from ( $N, V$ ). One may ask

Given commuting $N_{i} \in \operatorname{End}(V)$, is
$\sigma=\operatorname{span}_{\mathbb{R}>0}\left\{N_{1}, \ldots, N_{\ell}\right\}$ that satisfies (a), (b), (c)
above a monodromy cone?

She has shown by example that when $\ell>1$ this may not be the case. This of course raises the question: can one find additional conditions that $\sigma$ be a monodromy cone? By having an explicit description of $\sigma$ in the geometric case one might hope to identify additional conditions, but so far this has not been carried out.

We conclude with a final speculative comment. On $\mathrm{Gr}^{w} V$ there is an induced action of the $N_{\lambda}$ and $N_{1}, \ldots, N_{\ell}$. For each $\lambda$ with the $\lambda_{i}>0$ we have $\left\{N_{\lambda}, Y, N_{\lambda}^{+}\right\}$giving an $\mathrm{sl}_{2, \lambda} \subset \operatorname{End}\left(\mathrm{Gr}^{W} V\right)$, and by a result of Looijenga-Lunts (whose proof uses Hodge-Riemann I, II) these $\mathrm{sl}_{2, \lambda}$ 's generate a semi-simple Lie algebra $\mathfrak{g}_{\sigma} \subset \operatorname{End}\left(\operatorname{Gr}^{W} V\right)$.
We have noted above that in the geometric case the $Y_{i}$ for $N_{i}$ do not in general pass to $E_{2}=\mathrm{Gr}^{W} V$. However, a linear algebra construction of Deligne gives canonically a set $Y_{i}^{\prime}$ of grading elements for the $N_{i}$, and from this we obtain a set of $\mathrm{sl}_{2, i}$ 's. These generate a Lie algebra $\mathfrak{g}_{\sigma}^{\prime} \subset \operatorname{End}\left(\mathrm{Gr}^{W} V\right)$, one that in the geometric case measures the obstruction to having the commuting $\mathrm{sl}_{2, i}$ 's on $E_{1}$ survive to $\mathrm{Gr}^{W} V$.

In some examples, using the polarization conditions, one may show that

$$
\mathfrak{g}_{\sigma}=\mathfrak{g}_{\sigma}^{\prime}
$$

If true in general the condition
(d) $\mathfrak{g}_{\sigma}^{\prime}$ is semi-simple
should then be added to the properties of monodromy cones.
—Thank you-

## References

The main references for Question A are
[KP1] M. Kerr and G. Pearlstein, Boundary components of Mumford-Tate domains. arXiv:1210.5301.
[KP2] M. Kerr and G. Pearlstein, Naïve boundary strata and nilpotent orbits, preprint, 2013.
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[GG] M. Green and P. Griffiths, Reduced limit period mappings and orbits in Mumford-Tate varieties, to appear in the Herb Clemens volume.

So far as Question B is concerned, it mainly consists of proof analysis and extension of the works
[Fr1] R. Friedman, Global smoothings of varieties with normal crossings, Ann. of Math. 118 (1983), 75-114.
[Fr2] R. Friedman, The period map at the boundary of moduli, in Topics in Transcendental Algebraic Geometry, 183-208, (P. Griffiths, ed.), Princeton Univ. Press, Princeton, NJ, 1984.
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The Hodge-theoretic aspects of the discussion are based on
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CKS1] E. Cattani, A. Kaplan, and W. Schmid, Degenerations of Hodge structures, Ann. of Math. 123 (1986), 457-535.
[CKS2] E. Cattani, A. Kaplan, and W. Schmid, $L^{2}$ and intersection cohomologies for a polarizable variation of Hodge structure, Invent. Math. 87 (1987), 217-252.
There are of course many related works that are not listed above.

