Algebro-geometric aspects of Limiting Mixed Hodge Structures

Phillip Griffiths

Talk at the IAS on December 16, 2014.

◆□ → < □ → < Ξ → < Ξ → Ξ < 𝔅 𝔅 1/69</p>

Outline

- I. Guiding questions and notations
- II. Summary of some results about Question A
- III. Summary of some results about Questions B, B', B''
- IV. Brief indication of proofs
- V. Final remarks

References

This talk will be mostly rather expository; the objective is to give an overview that blends the purely Hodge-theoretic and algebro-geometric approaches. It will be largely drawn from the literature and a list of some representative references is given at the end.

I. Discussion of the geometric questions

Question A: What can a smooth, projective variety X_{η} degenerate to?

We imagine $\mathfrak{X} \to \Delta$ with generic fibre X_η and central fibre X a local normal crossing divisor, and are interested in the extreme cases of

- a generic degeneration $X_{\eta} \rightarrow X$;
- the "most singular" degeneration $X_{\eta} \rightarrow X$.

The Hodge theoretic translation is

$$H^n(X_\eta)_{\mathrm{prim}} \to H^n_{\mathrm{lim}}.$$

Here H_{lim}^n is an equivalence class of limiting mixed Hodge structures.

Question A translates to

What are the extremal degenerations of a polarized Hodge structure?

We will define "extremal" below.

Question B: What Hodge-theoretic information about smoothings of X are contained in

- X alone;
- $(X, T_X \text{Def}(X))$?

Here X is a projective variety that is locally a product of normal crossing divisors as arises in a semi-stable reduction in a several parameter family. Among other things we will see that the Hodge-theoretic smoothings (to be defined) may be strictly larger than the algebro-geometric ones. **Question B':** What is the cohomological formulation of the refined differential of the period mapping at infinty?

The "refined differential" will be defined below.

Question B": What are the maximal monodrony cones that can arise algebro-geometrically?

Related to this question is to give algebro-geometric interpretations for the three properties of monodromy cones predicted by Deligne and proved by Cattani-Kaplan-Schmid and discuss a possible fourth such property.

Terms and Notations

From Hodge theory

- $(V, Q, F^{\bullet}) =$ polarized Hodge structure;
- ► D = period domain, or more generally a Mumford-Tate domain with compact dual Ď;
- $D = G_{\mathbb{R}}/H$ and $\check{D} = G_{\mathbb{C}}/P$ is the compact dual;
- $\Phi: S^* \to \Gamma \setminus D$ = variation of Hodge structure, where $S^* = S \setminus Z$ and where the image of $\pi_1(S^*) \to \Gamma$ is abelian with unipotent monodromies having logarithms N_1, \ldots, N_ℓ ;

4 ロ ト 4 回 ト 4 三 ト 4 三 ト 三 の 4 で 6/69

• frequently, but not always, $S^* = \Delta^{*\ell}$;

- ► (V, W, F•) = mixed Hodge structure, throughout assumed to be polarizable;
- $\sigma = \operatorname{span}_{\mathbb{R}>0}\{N_1, \ldots, N_\ell\}$ is a monodromy cone;
- (V, W(N), F[•]) is a limiting mixed Hodge structure, defined for each N ∈ σ with

$$N_k : \operatorname{Gr}_k^{W(N)} V \to \operatorname{Gr}_{-k}^{W(N)} V;$$

◆□▶ ◆□▶ ◆ ■▶ ◆ ■ ▶ ■ ⑦ Q ○ 7/69

$$\left\{ \begin{array}{c} (V, W(N), F^{\bullet}) = \text{limiting mixed Hodge structure} \\ & \uparrow \\ & \\ \text{nilpotent orbit } \begin{cases} N_i F^p \subset F^{p-1} \\ \exp(\Sigma z_i N_i) \cdot F^{\bullet} \in D, \text{Im } z_i \gg 0 \end{cases} \right\};$$

 equivalence class of limiting mixed Hodge structures means F[•] ~ exp(Σλ_iN_i) · F[•], λ_i ∈ C;

< □ > < @ > < E > < E > E のQ 8/69

▶ Y is a grading element for N, [Z, N] = -2N and $W(N)_k = \bigoplus_{j \le k} E(Y)_j;$ From algebraic geometry

• $\mathfrak{X} \to \Delta^{\ell}$ has central fibre X and is given locally by

$$\begin{cases} x_{l_1} = t_{i_1} \\ \vdots \\ x_{l_\ell} = t_{i_\ell} \end{cases}$$

We adopt the convention that if the index set I_i is empty then the equation doesn't appear; e.g., at a smooth point of X there are no equations. It is locally a product of reduced normal crossing divisors times parameters;

 the type of X is the maximum of singular factors that appear in the local descriptions;

- ▶ for $\mathbf{k} = [k_1, \dots, k_\ell]$, $X^{[\mathbf{k}]} = \{x \in X : \text{mult}_x t_i \ge k_i\}$; for $\ell = 1$ we just write $X^{[k]}$ for the usual stratification of X;
- ▶ smoothing of X as above is $X \to S$ where $X = X_{s_0}$ and the fibres over S^* are smooth and the monodromy representation is abelian.

An example of a several parameter family is

$$\mathfrak{X} \times \mathfrak{X} \to \Delta \times \Delta$$

◆□ ▶ < @ ▶ < E ▶ < E ▶ ○ 2 · ○ Q ○ 10/69</p>

which may be used to study natural classes, such as N, in $\operatorname{Hom}(H^*(X_\eta), H^*(X_\eta))$.

 Def(X) = Kuranishi versal deformation space of X with Zariski tangent space

$$\mathcal{T}_X \mathrm{Def}(X) = \mathbb{E}\mathrm{xt}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X);$$

• $T \subset T_X \text{Def}(X)$ will be a subspace transverse to $T_X \text{Def}^{\text{es}}(X)$ and where a general $\xi \in T$ is smoothing (to 1^{st} order); the corresponding family is

$$\mathfrak{X}_T \to T_{\epsilon}, \quad \mathfrak{O}_{T_{\epsilon}} \cong T^*;$$

for one ξ we write

$$\mathfrak{X}_{\xi} \to \Delta_{\epsilon} = \operatorname{Spec} \mathbb{C}[\epsilon], \quad \epsilon^2 = 0;$$

▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ 三三 - のへで 11/69

II. Statements of results about Question A*

- $D \subset \check{D}$ is an open $G_{\mathbb{R}}$ -orbit and $\partial D = \bigcup G_{\mathbb{R}}$ -orbits \mathfrak{O} with $\operatorname{codim} \mathfrak{O} \geq 1$. In general there are
 - several codimension 1 orbits in ∂D ;
 - a unique closed orbit O_c;

•
$$\mathcal{O}_c$$
 may be
 $\begin{array}{c} \text{totally real} \iff \mathcal{O}_c = G_{\mathbb{R}}/P_{\mathbb{R}} \\ \iff \text{the Levi form } \mathcal{L}_{\mathcal{O}_c} = 0 \\ \text{not totally real (e.g., } G = SO(2a, b)) \end{array}$

Almost simplest example: D = SU(2, 1)/T and $\check{D} = SL(3, \mathbb{C})/B$ is the incidence variety



*Based on joint work with Mark Green and Colleen Robles and uses results from [KP1] and [KP2].

Orbit structure is



<ロト < 母ト < 臣ト < 臣ト 王 のへで 13/69

- D', D'' classical;
- D non-classical and is Mumford-Tate domain for polarized Hodge structures of weight n = 3, Hodge numbers (1, 2, 2, 1) and an action of Q(√-d).

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の Q ℃ 14/69

Reduced limit period mapping (or naïve limit) $\Phi: \Delta^* \to \Gamma \backslash D$ lifts to

$$\widetilde{\Phi}:\mathbb{H} o D\subset \check{D}, \quad \widetilde{\Phi}(z+1)=\exp{\textit{N}\cdot\widetilde{\Phi}(z)}.$$

(i) (Schmid) for $\widetilde{\Psi}(z) = \exp(-zN)\widetilde{\Phi}(z) : \mathbb{H} \to \check{D}$ unwinds Φ and we get

$$\left\{ egin{aligned} \Psi:\Delta^*
ightarrow \check{D},\ \Psi(0) = \mathcal{F}^ullet_{\mathsf{lim}} \end{aligned}
ight.$$

 $\implies \exp(zN)F_{\text{lim}}^{\bullet}$ is a nilpotent orbit with corresponding limiting mixed Hodge structure $(V, W(N), F_{\text{lim}}^{\bullet})$.

(ii) Set

$$F_{\infty}^{\bullet} = \lim_{\mathrm{Im}\, z \to \infty} \widetilde{\Phi}(z) = \lim_{\mathrm{Im}\, z \to \infty} \exp(zN) \cdot F_{\mathrm{lim}}^{\bullet} \in \partial D$$
$$= \lim_{\mathrm{Im}\, z \to \infty} \exp(zN) \cdot \widetilde{F_{\mathrm{lim}}^{\bullet}}.$$

For B(N) = equivalence classes of $(V, W(N), F^{\bullet})$'s and $B(N)_{\mathbb{R}}$ the \mathbb{R} -split ones we have



< □ ▶ < □ ▶ < ≧ ▶ < ≧ ▶ Ξ → ♡ < ♡ 16/69

Then Φ_{∞} is holomorphic, factors as above, and is maximal with respect to these properties.





D classical $\implies \Phi_{\infty} = \operatorname{Gr}^{W}$ (extension data lost). In general, Φ_{∞} retains some but not all of the extension data

Definition: The degeneration $X_{\eta} \rightarrow X$ is

- (i) minimal if Φ_{∞} belongs to a codimension 1 orbit;
- (ii) maximal if Φ_{∞} belongs to the closed orbit.

Theorem: For minimal degeneration when D is a period domain, either

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の < ℃ 19/69

▶
$$N^2 \neq 0$$
, $N^3 = 0$ and rank $N = 2$.



For dim $X_{\eta} = 5$,

$$\begin{cases} x_1x_2 + x_3x_4 + x_4x_5 = 0 & \text{double point} \\ x_1x_2 + x_3x_4 = 0 & \text{double surface} \\ x_1x_2 = 0 & \text{double fourfold} \end{cases}$$

Theorem' [KP1] and [GGR]: For maximal deformations and general Mumford-Tate domains

$$\begin{cases} \text{limiting mixed Hodge structure} \\ \text{ is of Hodge-Tate type} \\ \end{cases} \\ & \Longleftrightarrow \{ \text{closed orbit is totally real} \}. \end{cases}$$

It is elementary that a necessary, but in general not sufficient, condition is

$$h^{n,0} \leq h^{n-1,1} \leq \cdots \leq h^{n-[n/2],[n/2]}.$$

Theorem": In the period domain case, if the closed orbit is not totally real, then N = 2m is even and

<□ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

Example: n = 4



Picture a family of fourfolds $X_{\eta} \rightarrow X$ where X_{η} is a product of polarized K3 surfaces each of which has a type III degeneration.

In general, for maximal degenerations, $Gr^{W}(LMHS)$ is rigid.

Remarks:

- The picture for general Mumford-Tate domains is more complex, and in some ways more interesting. For example,
 - G_2 is singled out as an exception in the classification;

<□ ▶ < □ ▶ < ■ ▶ < ■ ▶ < ■ ▶ ■ の へ ? 24/69

▶ if there is a Hodge-Tate degeneration, then for D = G_C/P the Lie algebra p is an even Jacobson-Morosov parabolic.

- ► In general the Hodge-theoretic aspects of the G_R-orbit structure of ∂D may help guide the algebro-geometric properties of boundary strata in a moduli problem
 - curves and abelian varieties, K3 surfaces, cubic threefolds and fourfolds (Laza) (these are all classical D's and only local normal crossings seem to occur), mirror quintics (non-classical D);
 - more non-classical examples? e.g., quintic surfaces and threefolds? May locally products of normal crossing varieties occur as they do Hodge-theoretically?

▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ 三三 - のへで 25/69

 In the classical case Mumford et. al., and in the general case Kato-Usui have constructed maximal completions of Γ/D's



< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の Q ♀ 26/69

Are there minimal completions

 $S^* \longrightarrow \Gamma/D$

 $\bigcap \qquad \cap \\ S \xrightarrow{\Phi_B} \Gamma/D_B$

any
$$S^* = S \setminus Z$$
 and where
 $\mathbb{H}_{\Phi(S)} > 0$ where
 $\mathbb{H} = (\det H^{n,0})^n (\det H^{n-1,1})^{n-1}$
 $\cdots \det H^{1,n-1}$.

In the classical case, as a set
$$D_B = D_{Gr\Sigma}$$
 and a power $\mathbb{H}^a = \omega_D$ gives an ample line bundle on $\Phi_B(S)$.

< □ ▶ < □ ▶ < Ξ ▶ < Ξ ▶ Ξ · ⑦ Q @ 27/69

III. Statements of results about Questions B, B', B''*

Definition: $\operatorname{Ext}^{1}_{\mathcal{O}_{X}}(\Omega^{1}_{X}, \mathcal{O}_{X})$ is the *infinitesimal normal bundle* of X.

• For $\operatorname{Ext}^{1}_{\mathcal{O}_{X}}(\Omega^{1}_{X}, \mathcal{O}_{X}) = \mathcal{E}$ we have a stratification

$$\begin{cases} X_0 \supset X_1 \supset \dots \supset X_\ell, \text{ where if } X_k^0 = X_k \setminus X_{k+1} \\ \mathcal{E}\big|_{X_k^0} \text{ is locally free of rank } k. \end{cases}$$

*Much of this was motivated by [Fr1].

Example: Locally $X = X_1 \times X_2$ and stratification is

$$X_1 imes X_2 \supset (X_{1,\mathrm{sing}} imes X_2) \cup (X_1 imes X_{2,\mathrm{sing}}) \supset X_{1,\mathrm{sing}} imes X_{2,\mathrm{sing}}.$$

Definition: \mathcal{E} is *trivializable* if there are $e_1, \ldots, e_{\ell} \in H^0(X, \mathcal{E})$ such that for each connected component of X_k^0 there are e_{i_1}, \ldots, e_{i_k} that frame \mathcal{E} over that component;

► $X = \text{central fibre in } X \to \Delta^{\ell} \implies \text{Ext}^{1}_{\mathcal{O}_{X}}(\Omega^{1}_{X}, \mathcal{O}_{X}) \text{ is trivializable with } e_{i} = dt_{i}.$

◆□ ▶ < @ ▶ < E ▶ < E ▶ ○ 29/69</p>

Example: X = locally a normal crossing divisor. Then

•
$$X = X_0 \supset X_1 = X_{\text{sing}} = D_X;$$

Ext¹_{O_X} (Ω¹_X, O_X) ≃ O_{D_X}(X) is the infinitesimal normal bundle in [Fr1].

If
$$D_X = \prod_{i=1}^{\ell} D_i$$
 then $\mathfrak{O}_{D_X}(X)$ trivializable gives

$$\mathfrak{O}_{D_X}(X)\cong \bigoplus_{i=1}^\ell \mathfrak{O}_{D_i}$$

with sections $1_{D_i} \in H^0(\mathcal{O}_{D_i})$.

We suggest that in this case X should be thought of as potentially the central fibre in a family where over the origin the i^{th} factor in the local product representation of X is singular along D_i and is smoothed along the i^{th} coordinate axis. Thus, even normal crossing divisors should be thought of as occurring in multi-parameter families.

◆□ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ < □ ▶ <

(*) Assumption: $\operatorname{Ext}^{1}_{\mathcal{O}_{X}}(\Omega^{1}_{X}, \mathcal{O}_{X})$ is trivializable.

In the simpler case where X is locally a normal crossing divisor, from the standard local to global spectral sequence for Ext we have



Theorem I: Under the assumption (*) there exists an ℓ -parameter split limiting mixed Hodge structure with $1_{D_i} \leftrightarrow N_i$.

This is defined in terms of X alone. In case $\delta = 0$ and $T \subset T_X \text{Def}(X)$ is unobstructed so that we have a family

$$\mathfrak{X} \to \Delta^\ell$$

where everything is smooth over $\mathfrak{X}^* \to \Delta^{*\ell}$, the limiting mixed Hodge structure is the associated graded to the one given by the family.

We will explain below how the assumption $\mathcal{O}_{D_i}(X) \cong \mathcal{O}_{D_i}$ enters into the construction. Still in the case where X = locally a normal crossing divisor, we have

Theorem II: Let $T_S \subset T$ be a k-dimensional subspace such that $\rho(L)$ does not lie in any coordinate hyperplane in \mathbb{C}^{ℓ} . Then there exists a k-parameter limiting mixed Hodge structure whose commuting monodromies are linear combinations of the 1_{D_i} 's.

If there exists a family $\mathfrak{X} \to S$ whose tangent space is T_S , then we obtain the limiting mixed Hodge structure associated to this family.

In general we do not have $S^* = \Delta^{*\ell}$, as illustrated by the following

Example: X_0 = nodal variety of dimension 2n - 1 with nodes p_1, \ldots, p_ℓ ;

$$X = \widetilde{X}_0 \cup \left(\bigcup_{i=1}^{\ell} X_i\right), \qquad X_i \cong \mathbb{P}^{2n-1} \text{ and } \widetilde{X}_0 \cap X_i = Q_i$$

the normal crossing variety that would be the central fibre in the standard semi-stable reduction of a smoothing of X_0 . Then the limiting mixed Hodge structure in Theorem I is the associated graded to the one that would arise if we could independently smooth the nodes.

In Theorem II, the assumption just below it implies that X_0 may be smoothed. Moreover, we have

$$\mathcal{T}_{X_0}\mathrm{Def}(X_0)\cong\mathcal{T}_X\mathrm{Def}(X)$$

(to 1^{st} order, deforming X_0 is the same as deforming X)

and on $L = \rho^{-1}$ (coordinate hyperplanes in \mathbb{C}^{ℓ}) subsets of the nodes may be smoothed



case $\ell = 3$, k = 2

< □ ▶ < @ ▶ < E ▶ < E ▶ E の < ○ 36/69

Monodromies are $N_1 + N_2$, $N_1 + N_3$, $N_2 + N_3$.

Three properties of nodal degenerations are

- (i) the Koszul group H₁(V; {N₁,..., N_ℓ}) ≅ {group of relations among the nodes};
- (ii) the monodromy cone $\sigma = \{\sigma_{\lambda} = \Sigma \lambda_i N_i, \lambda_i > 0\}$ is strictly smaller than the polarizing cone

 $\sigma_{\text{pol}} =: \{ \sigma_{\lambda} : W(N_{\lambda}) \text{ gives a polarized limiting mixed} Hodge structure} \}$

if, and only if, the nodes are dependent (relations among nodes gives a larger polarizing cone);

(iii) there is an obstruction to extending the N_i to commuting $sl_{2,i}$'s in the split limiting mixed Hodge structure in Theorem I, and this obstruction vanishes if, and only if, the nodes are independent.

Example: Another example where D_X has multiple components is obtained by applying semi-stable reduction to a pencil of surfaces $X_t \subset \mathbb{P}^3$ where the singularities of the base locus are transverse intersections along the double curve of X_0 . These all contribute components to D_X . Taking for example the familiar case when deg $X_t = 4$ all of the possible limiting mixed Hodge structures are extremal in the sense of the theorem and may be pictured as

<□ ▶ < @ ▶ < E ▶ < E ▶ E の < C 38/69



which are familiar from the work of Kulikov, Friedman et al. (cf. [Fr1] and [Fr2]).

It may be that a similar story about extremal degenerations holds in the work of Laza [La] on the cubic fourfold, but I haven't had a chance to check this.

Two general properties associated to families $\mathfrak{X} \to \Delta^{\ell}$ having as central fibre X are

• dim $\sigma \leq \#$ components of D_X ;

This inequality can be substantially improved by a more precise statement extracted from the mechanics in the proof of Theorem I.

<□ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

For X locally a product of normal crossing divisors

• $N_{i_1}^{a_1} \cdots N_{i_j}^{a_j} = 0$ if some $a_i > |I_i|$, or if j > type of X. This bounds from below the singularities that the central fibre must have in any semi-stable reduction of $\mathcal{X}^* \to \Delta^{*\ell}$. Thus, if $N_1 N_2 \neq 0$ then X must somewhere be locally $X_1 \times X_2 \times$ (parameters) where X_1 and X_2 are singular.

Example: Hodge theory suggests that for some surfaces with $p_g \ge 2$ the moduli space (if such exists) would have to include singular surfaces of type $k \ge 2$.

◆□ ▶ < @ ▶ < E ▶ < E ▶ ○ Q ○ 41/69</p>

One of our original motivations was to have a *refined* and *computationally useful* formulation for the period mapping at infinity.

Example:



For $t = (t_1, t_2)$ and $\ell(t) = (1/2\pi i)(\log t_1 + \log t_2)$ the period matrix is

$$\Omega(t) = egin{pmatrix} 1 & & & \ & 1 & & \ & 1 & & \ & 1 & & \ & 1 & & \ & 1 & & \ & \ell(t) + a_{11}(t) & a_{12}(t) & b_1(t) \ & a_{21}(t) & \ell(t) + a_{22}(t) & b_2(t) \ & b_1(t) & b_2(t) & c(t) \end{pmatrix}, egin{pmatrix} a_{12}(t) = a_{21}(t) & \ & a_{12}(t) &$$

<□▶ < @ ▶ < ≧ ▶ < ≧ ▶ ≧ り < ○ 43/69

The nilpotent orbit is

$$egin{pmatrix} 1 & & & \ & 1 & & \ & & 1 & \ & & 1 & \ \ell(t)+a_{11} & a_{12} & b_1 \ a_{21} & \ell(t)+a_{22} & b_2 \ b_1 & b_2 & c \end{pmatrix}$$

< □ ▶ < @ ▶ < ≧ ▶ < ≧ ▶ E の Q ↔ 44/69

Rescaling gives

$$\begin{cases} \mathbf{a}_{11} \rightarrow \mathbf{a}_{11} + \lambda_1 \\ \mathbf{a}_{22} \rightarrow \mathbf{a}_{22} + \lambda_2. \end{cases}$$

For the correspoinding equivalence class of limiting mixed Hodge structures



▲□▶ ▲□▶ ▲ 三▶ ▲ 三▶ 三三 - のへで 45/69

For $\widetilde{\omega}_1, \widetilde{\omega}_2 = \lim_{t \to 0} \omega_1(t), \omega_2(t)$ where $\omega_1(t), \omega_2(t) =$ normalized differentials of the 3rd kind

•
$$a_{ij} = \int_{q_i}^{p_i} \widetilde{\omega}_j, \quad i \neq j;$$

► may normalize t_i, t₂ to make the logarithmic singularities cancel and have well defined a₁₁, a₂₂.

Refined $d\Omega$ has dc; db_1 , db_2 ; da_{12} and da_{11} , da_{22} . Moduli picture:

$$\partial \mathcal{M}_3 \subset \mathcal{M}_3 - \text{dim} = 6$$

 \cup
 $\mathcal{C} = \text{codimension 2 boundary component}$

- dim C = 4 and c, b_1, b_2, a_{12} give local coordinates;
- a_{11}, a_{22} give normal parameters to $\mathcal{C} \subset \overline{\mathcal{M}}_3$.

Theorem: $\xi \in \mathbb{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ defines a class

$$\xi^{(1)} \in \mathbb{E}\mathrm{xt}^1_{\mathbb{O}_X}\Big(\Omega^1_{\mathfrak{X}_\xi/\Delta\epsilon}(\log X)\otimes \mathbb{O}_X, \mathbb{O}_X\Big),$$

and the refined differential of the period mapping at infinity is expressed by the cup product

$$\begin{split} \mathbb{E} \mathrm{xt}^1_{\mathcal{O}_X} \Big(\Omega^1_{\mathfrak{X}_{\xi}/\Delta\epsilon}(\log X) \otimes \mathcal{O}_X, \mathcal{O}_X \Big) \\ & \to \mathrm{End}_{\mathrm{LMHS}} \, \mathbb{H}^n \Big(\Omega^{\bullet}_{\mathfrak{X}_{\xi}/\Delta\epsilon}(\log X) \otimes \mathcal{O}_X \Big). \end{split}$$

This is meant to give the flavor of the definition of and expression for the refined differential of the period mapping at infinity. **Question A ([GGR]):** Builds on [KP1], [KP2] and the later work [GG]; main points include

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 り < ○ 48/69

►
$$(V, Q, F^{\bullet}) \rightarrow (\mathfrak{g}, B, F^{\bullet}_{\mathfrak{g}}), \quad \mathfrak{g} \subset \operatorname{End}_{Q}(V);$$

►
$$(V, W, F^{\bullet}) \rightarrow (V, W, \widetilde{F^{\bullet}})$$
 with
► $F^{\bullet}_{\infty} = \widetilde{F^{\bullet}_{\infty}},$
► $(\widetilde{F^{\bullet}})_{\mathfrak{g}} = (\widetilde{F^{\bullet}_{\mathfrak{g}}});$

thus we may assume

$$\begin{cases} V_{\mathbb{C}} = \oplus I^{p,q} \\ \mathfrak{g}_{\mathbb{C}} = \oplus \mathfrak{g}^{p,q} \end{cases}, \qquad I^{p,q} = \overline{I^{q,p}} \text{ and } \mathfrak{g}^{p,q} = \overline{\mathfrak{g}^{q,p}}; \end{cases}$$

then one matches the g^{p,q} decomposition with the standard root theoretic description of the orbit structure.

Questions B, B', B'': Recast and extend arguments in the literature, including [Fr1], [Fr2], [St1], [PS],... in the 1-parameter case. Some work in the ℓ -parameter case has been done in [Fu].

Example: In the 1-parameter case, $E_1^{p,q}$ terms are sums of groups $H^a(X^{[b]})(-c)$ where

 d₁ is expressed in terms of various restriction and Gysin maps;

•
$$E_2 = E_\infty = \operatorname{Gr}(\mathsf{LMHS}).$$

Convenient to picture a split limiting mixed Hodge structure in terms of N-strings

$$egin{array}{lll} H^0(-m) o \cdots & \to H^0(-1) o H^0 \ H^1(-1) o \cdots & \to H^1 \ dots \ H^m \end{array}$$

◆□ ▶ < @ ▶ < E ▶ < E ▶ ○ 2 の Q ○ 50/69</p>

where N^k = Hodge structure of weight k.

Theorem: If $\mathcal{O}_D(X) \cong \mathcal{O}_D$, there are complexes

$$egin{aligned} & H^{q-4}(X^{[k+2]})(-2) \ & \oplus & H^{q-2}(X^{[k+1]})(-1) \ &
ightarrow & H^{q-2}(X^{[k]})(-1) &
ightarrow & \oplus &
ightarrow & H^q(X^{[k]}) \ & \oplus & H^q(X^{[k-1]}) \ & H^q(X^{[k-2]}) \end{aligned}$$

such that for $0 \leq j \leq m-i$ the terms in the N-strings are $H^{m-i}(-j) \cong (H^*_{\text{Rest}} + H^*_{\text{Gy}})(H^{m-i}(X^{[i+1]}))(-j).$

Moreover, the monodromy $N = N_1 + \cdots + N_\ell$ is induced by

$$N=\sum_i \mathbb{1}_{D_i}(1)$$

< □ > < @ > < E > < E > E の < C 51/69

where for $b \ge 2$, $X^{[b]} = \coprod_i X_i^{[b]}$ and $H^a(X_i^{[b]})(-c) \xrightarrow{\stackrel{1_{D_i}}{\sim}} H^a(X_i^{[b]})(-(c-1)).$

The important observation is that having trivializations

$$\mathfrak{O}_{D_i}(X) \cong \mathfrak{O}_{D_i}$$

implies that

$$Gy \cdot Rest = -Rest \cdot Gy,$$

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ ミ ∽ Q (~ 52/69)

so that the diagram in the statement of the theorem is actually a complex.

A perhaps new ingredient is the observation that the N decomposes as $\Sigma 1_{D_i}(1)$. As in Theorem II we may then define an ℓ -parameter 1^{st} order family and $N_{\lambda} = \Sigma \lambda_i N_i$, $\lambda_i \neq 0$. Here the 1_{D_i} correspond to a trivialization $\mathcal{O}_{D_i}(X) \cong \mathcal{O}_{D_i}$ and means that to 1^{st} order we have a specified parameter in the smoothing of the component D_i of X_{Sing} . We also have an $\mathrm{sl}_{2,i}$ acting on E_1 where

$$Y_i = (2c - b + 1)1_{D_i}$$

on $H^a(X_i^{[b]})(-c)$, $b \ge 2$. Then for $Y = \Sigma Y_i$,

$$[d_1,Y]=-d_1$$

so that Y acts on $E_2 = E_{\infty} = \operatorname{Gr}^{W(N)} V$.

In general $[d_1, Y_i] + d_1 \neq 0$ and measures the linking between vanishing cycles and their duals associated to different components of X_{sing} . This gives a geometric description of the obstructions to having commuting $\text{sl}_{2,i}$'s acting on $\text{Gr}^{W(N)} V$.

Example:



< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

In both cases $Gr^{W}(\mathsf{LMHS})$ is the same over \mathbb{Q} .

For general X = locally a product of normal crossing varieties there are a few additional points that arise.

A first step is a fairly straightforward extension from the case $\ell = 1$ of the identification for families $\mathfrak{X} \to \Delta^{\ell}$,

$$H^n(X_\eta)\cong \mathbb{H}^nig(\Omega^ullet_{\mathfrak{X}/\Delta^\ell}(\log X)\otimes \mathbb{O}_Xig).$$

The complex $\Omega^{\bullet}_{\mathcal{X}/\Delta^{\ell}}(\log X) \otimes \mathcal{O}_X$ depends only on $\mathcal{T}_{\{0\}}\Delta^{\ell} \subset \mathcal{T}\mathrm{Def}(X)$ and is identified as the cokernel of the map

$$\Omega^{\bullet}_{\mathfrak{X}}(\log X) \xrightarrow{dt_1/t_1 \wedge \cdots \wedge dt_{\ell}/t_{\ell}} \Omega^{\bullet+\ell}_{\mathfrak{X}}(\log X),$$

◆□ ▶ < @ ▶ < E ▶ < E ▶ ○ 2 の Q ○ 55/69</p>

which can be defined in terms of $T_{\{0\}}\Delta^{\ell}$ alone. Note the ℓ -fold wedge product here.

A second point is that the N_i are defined in terms of the connected components in the highest dimensional strata of the support of $\operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.

One subtlety is that as in the case $\ell = 1$ one cannot define a weight filtration on the complex $\Omega^{\bullet}_{\mathfrak{X}}(\log X) \otimes \mathcal{O}_{X}$. To have a monodromy weight filtration W, whose W_k are expressed on E_1 by linear inequalities among a, b, c, on H^n_{lim} the indices have to have length 2n + 1, and we can only get n + 1 out of the usual filtration on $\Omega^{\bullet}_{\mathfrak{X}}(\log X)$. The correct thing to do is suggested by the map above, which is the first term in a resolution of $\Omega^{\bullet}_{\mathfrak{X}}(\log) \otimes \mathcal{O}_{X}$ from which some numerology suggests what the weight filtration should be. A central point is to use the hard Lefschetz theorem on the normalized strata of X and both Hodge-Riemann bilinear relations to deduce that W, which easily satisfies

$$N_{\lambda}: W_k \to W_{k-2},$$

for all λ in fact satisfies

$$N_{\lambda}^{k}: W_{k} \xrightarrow{\sim} W_{-k}$$

when $k \ge 0$ and $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_i > 0$. Without this from the weight filtration W we get a mixed Hodge structure but *not* a limiting mixed Hodge structure with W = W(N). The proof of the result involves extending the nice argument due to Guillén and Navaro Anzar to the several parameter case.

V. Final remarks

In closing there were three classical properties of monodromy cones for degenerating families of Hodge structures, predicted by Deligne (and established by him in the ℓ -adic setting and proved in the Hodge-theoretic setting in [CKS1], [CKS2]).

- (a) $W(N_{\lambda})$ is independent of λ with $\lambda_i > 0$;
- (b) W(N) is a relative weight filtration for $W(N_i)$;
- (c) the Koszul groups $H_i(V; \{N_1, \ldots, N_\ell\})$ vanish in positive weight (purity).

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の Q ℃ 58/69

The statement (b) means that we easily have that $N: W_k(N_i) \rightarrow W_{k-2}(N_i)$, and then the much more subtle result, whose proof uses Hodge-Riemann I, II,

$$W(N)$$
 induces $W(N | \operatorname{Gr}^{W(N_i)})$ on $\operatorname{Gr}^{W(N_i)}$

is true. It follows from this and the above construction that, e.g., when $\ell=2$ we have the picture



and the limiting mixed Hodge structure "lim_{$t\to0$}(t, t)" along the dotted line may be identified with "lim_{$t_1\to0$} lim_{$t_2\to0$}." This result is true by Deligne's argument in the ℓ -adic case; the geometric argument may be useful in the computation of examples. The result (b) means that the $\mathbb{C}[N_s, \ldots, N_{\lambda}]$ -module V has very special properties. For example, it implies that as a $\mathbb{C}[N_{\lambda}, N_i]$ -module it is a direct sum of $\mathbb{C}[N_{\lambda}, N_i]/(N_{\lambda}^p, N_i^q)$'s.

These statements can be verified using the above, which gives some slight refinements. For instance in the example of a nodal X_0

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の < ℃ 61/69

(iv) the weight filtration $W(N_{\lambda})$ is independent of the $\lambda_i \in \mathbb{C}^*$ if, and only if, the nodes are independent.

Another motivation for the above is this: Given a nilpotent $N \in \operatorname{End}(V)$, Robles has formulated and proved a precise result which has as a corollary that N is the monodromy of a variation of Hodge structure over Δ^* constructed in an explicit way from (N, V). One may ask

Given commuting $N_i \in End(V)$, is $\sigma = \operatorname{span}_{\mathbb{R}>0}\{N_1, \ldots, N_\ell\}$ that satisfies (a), (b), (c) above a monodromy cone?

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の Q ℃ 62/69

She has shown by example that when $\ell > 1$ this may not be the case. This of course raises the question: can one find additional conditions that σ be a monodromy cone? By having an explicit description of σ in the geometric case one might hope to identify additional conditions, but so far this has not been carried out.

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ E の < ⊙ 63/69

We conclude with a final speculative comment. On $\operatorname{Gr}^{W} V$ there is an induced action of the N_{λ} and N_{1}, \ldots, N_{ℓ} . For each λ with the $\lambda_{i} > 0$ we have $\{N_{\lambda}, Y, N_{\lambda}^{+}\}$ giving an $\operatorname{sl}_{2,\lambda} \subset \operatorname{End}(\operatorname{Gr}^{W} V)$, and by a result of Looijenga-Lunts (whose proof uses Hodge-Riemann I, II) these $\operatorname{sl}_{2,\lambda}$'s generate a semi-simple Lie algebra $\mathfrak{g}_{\sigma} \subset \operatorname{End}(\operatorname{Gr}^{W} V)$.

We have noted above that in the geometric case the Y_i for N_i do not in general pass to $E_2 = \operatorname{Gr}^W V$. However, a linear algebra construction of Deligne gives canonically a set Y'_i of grading elements for the N_i , and from this we obtain a set of $\operatorname{sl}_{2,i}$'s. These generate a Lie algebra $\mathfrak{g}'_{\sigma} \subset \operatorname{End}(\operatorname{Gr}^W V)$, one that in the geometric case measures the obstruction to having the commuting $\operatorname{sl}_{2,i}$'s on E_1 survive to $\operatorname{Gr}^W V$. In some examples, using the polarization conditions, one may show that

$$\mathfrak{g}_{\sigma}=\mathfrak{g}_{\sigma}^{\prime}.$$

If true in general the condition

(d) \mathfrak{g}'_{σ} is semi-simple

should then be added to the properties of monodromy cones.

—Thank you—

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の < で 65/69

References

The main references for Question A are

- [KP1] M. Kerr and G. Pearlstein, Boundary components of Mumford-Tate domains. arXiv:1210.5301.
- [KP2] M. Kerr and G. Pearlstein, Naïve boundary strata and nilpotent orbits, preprint, 2013.

4 ロ ト 4 日 ト 4 王 ト 4 王 ト 王 の 4 で 66/69

[GGR] M. Green, P. Griffiths, and C. Robles, Extremal degenerations of polarized Hodge structures, arxiv:1403.0646.

- [GG] M. Green and P. Griffiths, Reduced limit period mappings and orbits in Mumford-Tate varieties, to appear in the Herb Clemens volume.
 - So far as Question B is concerned, it mainly consists of proof analysis and extension of the works
- [Fr1] R. Friedman, Global smoothings of varieties with normal crossings, Ann. of Math. 118 (1983), 75–114.
- [Fr2] R. Friedman, The period map at the boundary of moduli, in *Topics in Transcendental Algebraic Geometry*, 183–208, (P. Griffiths, ed.), Princeton Univ. Press, Princeton, NJ, 1984.
- [La] R. Laza, The moduli space of cubic fourfold via the period map, *Ann. of Math* **172** (2010), 673–711.

- [Fu] T. Fujisawa, Mixed Hodge structures on log smooth degenerations, *Tohoku Math.* **60** (2008), 71–100.
- [St1] J. Steenbrink, Limits of Hodge structures, Invent. Math. 31 (1979), 229–257.
- [SZ] J. Steenbrink and S. Zucker, Variation of mixed Hodge structure. I, Invent. Math. 80 (1985), 489–542.

<ロト < 母 ト < 臣 ト < 臣 ト 王 の へ で 68/69

The Hodge-theoretic aspects of the discussion are based on

- [CK] E. Cattani and A. Kaplan, Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure, *Invent. Math.* 67 (1982),
- CKS1] E. Cattani, A. Kaplan, and W. Schmid, Degenerations of Hodge structures, Ann. of Math. 123 (1986), 457–535.
- [CKS2] E. Cattani, A. Kaplan, and W. Schmid, L² and intersection cohomologies for a polarizable variation of Hodge structure, *Invent. Math.* 87 (1987), 217–252.

There are of course many related works that are not listed above.

< □ ▶ < □ ▶ < 三 ▶ < 三 ▶ 三 の < で 69/69