# From the Fukaya category to curve counts via Hodge theory

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# Calabi-Yau Kähler manifolds

A **Kähler manifold** $M$ comes equipped with two compatible structures:

- A **complex structure** $J \in \mathrm{End}(TM)$, $J^2 = -I$, which is 'integrable';
- A **symplectic structure**, which is a closed, non-degenerate two-form $\omega$.

We require these structures be **compatible**, in the sense that

$$\omega(J\cdot, \cdot)$$

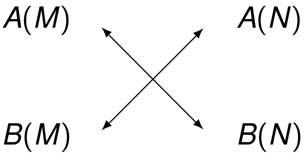
is a Riemannian metric on $M$.
$M$ is said to be **Calabi-Yau** if it admits a nowhere-vanishing holomorphic $n$-form $\Omega$ (a 'holomorphic volume form').

# 'Meta' mirror symmetry

- On a Calabi-Yau Kähler manifold $M$, one has

$$
\begin{aligned}
A(M) &= \text{some invariants of the symplectic structure of } M \\
B(M) &= \text{some invariants of the complex structure of } M
\end{aligned}
$$

- $M$ and $N$ are said to be **mirror** if there are equivalences



# Closed-string invariants: B-model

- Let $N_t$ be a smooth family of $n$-dimensional Calabi-Yau varieties parametrized by $t \in \Delta^*$ (a punctured disk), with maximally unipotent monodromy around $t = 0$.
- The B-model (complex invariant) is the corresponding variation of Hodge structures on $H^n(X_t)$.
- There exists a unique family of 'normalized' holomorphic volume forms $\Omega_t$; we define the **Yukawa coupling**

$$Y_N^B(t) := \int_{N_t} \Omega_t \wedge \nabla^n_{t\frac{\partial}{\partial t}} \Omega_t$$

where $\nabla$ is the Gauss-Manin connection.

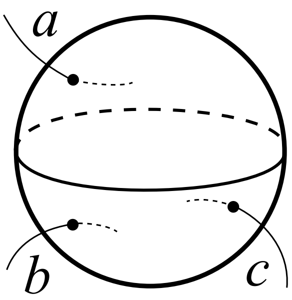
# Closed-string invariants: A-model

- Let $M$ be a Calabi-Yau variety.
- The $A$-model (symplectic invariant) is the **Gromov-Witten invariants** of $M$.
- These are invariants which count holomorphic maps of a Riemann surface into $M$. For example:
  - Number of degree-1 curves (lines) $u : \mathbb{CP}^1 \to \mathbb{CP}^n$, passing through two generic points: 1.
  - Number of degree-2 curves (conics) $u : \mathbb{CP}^1 \to \mathbb{CP}^2$, passing through five generic points: 1.
  - Number of lines on a cubic surface: 27.
- These can be assembled into a variation of Hodge structures on $H^*(M)$ (!), parametrized by a formal variable $T \in \Delta^*$ (Dubrovin, Morrison).

# Quantum cohomology

One part of the variation of Hodge structures is the **quantum cohomology** ring, $QH^*(M)$, an associative deformation of $H^*(M)$ over $\mathbb{C}[\![T]\!]$:

$$\langle \alpha * \beta, \gamma \rangle = \# \left( \quad \right) T^{u \cdot D}$$




where $a, b, c, D$ are cycles Poincaré dual to $\alpha, \beta, \gamma, [\omega]$. We define the **Yukawa coupling**

$$Y_M^A(T) := \langle [\omega]^{*n}, 1 \rangle.$$

# A mirror symmetry prediction (Candelas-de la Ossa-Green-Parkes, 1991)

- $M^5 :=$ quintic hypersurface in $\mathbb{CP}^4$, $N^5 :=$ its mirror.
- Mirror symmetry predicts that $Y_N^B(t) = Y_M^A(T)$, for an explicit change of variables relating $t$ and $T$ (the **mirror map**).
- $\rightsquigarrow$ Prediction: if $n_d := \#$(degree-$d$ curves on $M^5$), then $n_1 = 2875, n_2 = 609250, \ldots$
- Verified mathematically by Givental, Lian-Liu-Yau (1996).

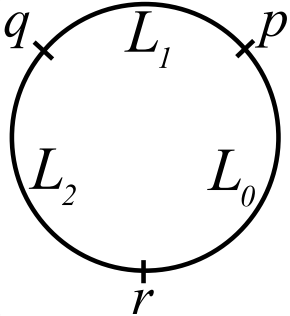
# Open-string mirror symmetry

- 1994: Kontsevich proposed **homological mirror symmetry.**
- In HMS, $A(M)$ is the **Fukaya category** $\mathcal{F}(M)$:
  - Coefficient ring is $R = \mathbb{C}[\![T]\!]$
  - Objects are Lagrangian submanifolds $L \subset M$;
  - Morphism spaces are $\mathrm{Hom}(L_0, L_1) := R\langle L_0 \cap L_1 \rangle$;
  - Composition maps count holomorphic maps of disks into $M$, with boundary on the Lagrangian objects.
- $B(M) := D^b Coh(M)$, the **derived category of coherent sheaves** on $M$.

# Composition in the Fukaya category

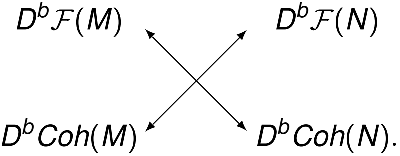
Composition sends

$$\mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L_0, L_2),$$

$$\langle p \bullet q, r \rangle = \# \left( \vphantom{\int} \right. $$  $$ \left. \vphantom{\int} \right) T^{u \cdot D}$$

# Statement of homological mirror symmetry

Kontsevich conjectured that, if $M$ and $N$ are mirror, then there are equivalences of categories




**Questions:**

1. Is it true? How generally?
2. Can you use it to prove results about symplectic topology?
3. Does it imply closed-string mirror symmetry? (Kontsevich conjectured that it does, via **noncommutative Hodge theory**).

# Homological mirror symmetry for the quintic threefold

## Theorem (S.)

*If $M^5$ is the quintic three-fold, and $N^5_t$ is its mirror, then there is a $\mathbb{C}[\![T, T^{-1}]\!]$-linear equivalence of categories*

$$D^\pi \mathcal{F}(M^5) \cong D^b Coh(N^5),$$

*after a change of variables in the base field (the mirror map).*

**Question:** Can we re-prove closed-string mirror symmetry (hence compute curve counts on $M^5$) from this result?

# Answer (joint work in progress with Tim Perutz and Sheel Ganatra):

- We hope to prove that the answer is **yes**.
- On the B-model, there is a procedure for reconstructing the variation of Hodge structures from $D^b Coh(N)$.
- The aim of our project is to carry this out on the A-model: i.e., show that the same procedure, when applied to the Fukaya category, gives back the A-model variation of Hodge structures.