Filling metric spaces

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Theorem

(M. Gromov) Let M^n be a closed manifold. Consider its (isometric) Kuratowski embedding in $L^\infty(M)$. Then M^n bounds in its $c(n)vol^{\frac{1}{n}}(M)$-neighbourhood. In other terms, $FillRad(M^n) \leq c(n)vol^{\frac{1}{n}}(M^n)$.

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Essential manifolds: A class of closed non-simply connected manifolds that includes all non-simply connected closed surfaces, $\mathbb{R}P^n$, aspherical manifolds (including tori), etc.
M^n is *essential* if the classifying map $f : M^n \longrightarrow B\pi_1(M^n)$ satisfies $f_*([M^n]) \neq 0 \in H_n(B\pi_1(M^n))$.
Here: $B\pi_1(M^n)$ is the aspherical space with the fundamental group $\pi_1(M^n)$; f_* homomorphism of homology groups induced by f; if M^n is non-orientable, one considers homology grouos with $\mathbb{Z}_2$ coefficients; $[M^n]$ is the fundamental homology class.

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(M. Gromov) Let M^n be an essential manifold. Then the length of the shortest non-contractible closed curve does not exceed $6FillRad(M^n) \leq const(n)vol^{\frac{1}{n}}(M^n)$.

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2. The assumption that the volume of all balls of radius 1 are small yields more: One can conclude that the $(n-1)$-dimensional Urysohn width of M^n is less than 1.
Urysohn $(n-1)$-dimensional width, UW_{n-1}, of X: the infimum of d such that there exists a continuous map $X \longrightarrow K^{n-1}$ such that K^{n-1} is a $(n-1)$-dim CW-complex, and for each $y \in K^{n-1}$ $diam(F^{-1}(y)) \leq d$.

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Theorem

(L. Guth). There exists $\epsilon_n > 0$ with the following property: Assume that each metric ball of radius 1 in a closed Riemannian manifold M^n has volume less than ϵ_n. Then the Urysohn $(n-1)$-width of M^n is less than or equal to 1.

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2. (Guth) Is it true that there exists $\epsilon_m > 0$ with the following property: Let X be a compact metric space such that the m-dimensional Hausdorff content of each metric ball of radius 1 is less than ϵ_m. Then $UW_{m-1}(X) \leq 1$.

Main result: Guth's conjecture is true:

Theorem

(Y. Liokumovich, B. Lishak, A.N., R. Rotman) There exists a positive ϵ_m with the following property. Let X be a compact (or even boundedly compact) metric space. Assume that for some positive R $HC_m(B) \leq \epsilon_m R^m$ for each metric ball B of radius R in X. Then $UW_{m-1}(X) \leq R$.

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Motivation(s):
1.Even for Riemannian manifolds this gives an intrinsic metric criterion when a n-dimensional closed or complete non-compact manifold is close not merely to a $(n-1)$-dimensional CW-complex but a $(m-1)$-dimensional one for any $m \leq n$.

2. If m-dimensional Hausdorff measure $(X) = 0$, then $HC_m(X) = 0$. Now the corollary implies that $UW_{m-1}(X) = 0$. Thus, the covering dimension of $X \leq m - 1$. We obtained the Szpilrajn theorem.

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3. Even for m-dimenional Riemannian manifolds this is a strengthening of previous results by Gromov and Guth, as $HC_m(M^m) \leq vol(M^m)$, and, in fact, can be much smaller. (Think about hyperbolic disc of radius 1 but with a very large negative curvature.)

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Proof of Gromov's theorem:
Pseudo-proof. Pretend that $L^\infty(M^n)$ is $\mathbb{R}^N$. Fill M^n by a minimal surface W^{n+1}.
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Monotonicity formula mplies that W^{n+1} is in the $\sim vol(W)^{\frac{1}{n+1}} = const(n) vol(M^n)^{\frac{1}{n}}$-neighbourhood of M^n.

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Improve M: Cover a significant part of volume of M by disjoint metric balls $B_i(r_i)$ in the ambient Banach space.
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$vol_{n-1}(\partial B_i \bigcap M) \leq \frac{r_i^{n-1}}{1000^n}$
(Use coarea inequality).

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Improve $M_1 = M^n$ inductively obtaining $M_2, M_3, \ldots$.
On each step the volume drops by a constant factor.
When it becomes very small, just cone off M_N in the ambient space.

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$$vol(\phi(X) \bigcap StarF) \leq const(n)\epsilon_m r_i^m \exp(-Const(n)\ dim(F)),$$

where ϕ is the map to the nerve, F is a face, r_1 is the smallest radius of a ball in the intersection of metric balls corresponding to F.

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By-product of Guth's construction: If B is a good ball of radius r, ϵ_n small, then $vol(B) \leq (\frac{r}{1000})^n$ (in fact even $\leq (\frac{r}{1000)}^{n+1})$.

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Idea: $HC_m(B_i) < (\frac{r_i}{1000})^m$ implies one need to use balls of radius $< \frac{r}{100}$ to cover $B_1 \bigcup B_2$. The sum of mth powers of radii is greater than HC_m of $(1 - \frac{1}{100})B_1 \bigcup (1 - \frac{1}{100})B_2$. It remains to ensure that HC_m of the somewhat smaller balls is comparable with a content of larger balls.

Lemma

(Co-area inequality) Let $U \subset B(R_2) \setminus B(R_1)$ *be a closed set. Then,*

$$\int_{R_1}^{*R_2} HC_{m-1}(S_R \bigcap U)\; dR \leq 2HC_m(U).$$

Therefore, there exists $R \in [R_1, R_2]$*, such that* $HC_{m-1}(S_R \cap U) \leq \frac{2}{R_2 - R_1} HC_m(U)$.

Proof.

Let $\{B_{r_i}(p_i)\}$ be a covering of U with $\sum_i r_i^m \le HC_m(U) + \epsilon$, where $i \in \{1, \dots, N\}$ for some N. The desired inequality would follow from the inequality $\int_{R_1}^{*R_2} HC_{m-1}(S_R \bigcap U)\ dR \le 2 \sum_i r_i^m$. We are going to prove a stronger inequality, where $HC_{m-1}(S_R \bigcap U)$ is replaced by the following quantity that is obviously not less than $HC_{m-1}(S_R \bigcap U)$, namely, $\sum_{i \in I(R)} r_i^{m-1}$, where $I(R)$ denotes the set of all indices i such that the intersection of $B_{r_i}(p_i)$ and $S(R)$ is not empty. The left hand side of the desired inequality becomes

$$\int_{R_1}^{R_2} \sum_{i \in I(R)} r_i^{m-1} dR = \int_{R_1}^{R_2} \sum_{i=1}^{N} r_i^{m-1} \chi_i(R) dR = \sum_{i=1}^{N} r_i^{m-1} \int_{R_1}^{R_2} \chi_i(R) dR,$$

where the characteristic function $\chi_i(R)$ is equal to 1 for all $R \in [R_1, R_2]$ such that S_R and $B_{r_i}(p_i)$ have a non-empty intersection, and to 0 otherwise. Finally, observe that $\int_{R_1}^{R_2} \chi_i(R) dR \le 2 r_i$, which implies the desired inequality. □