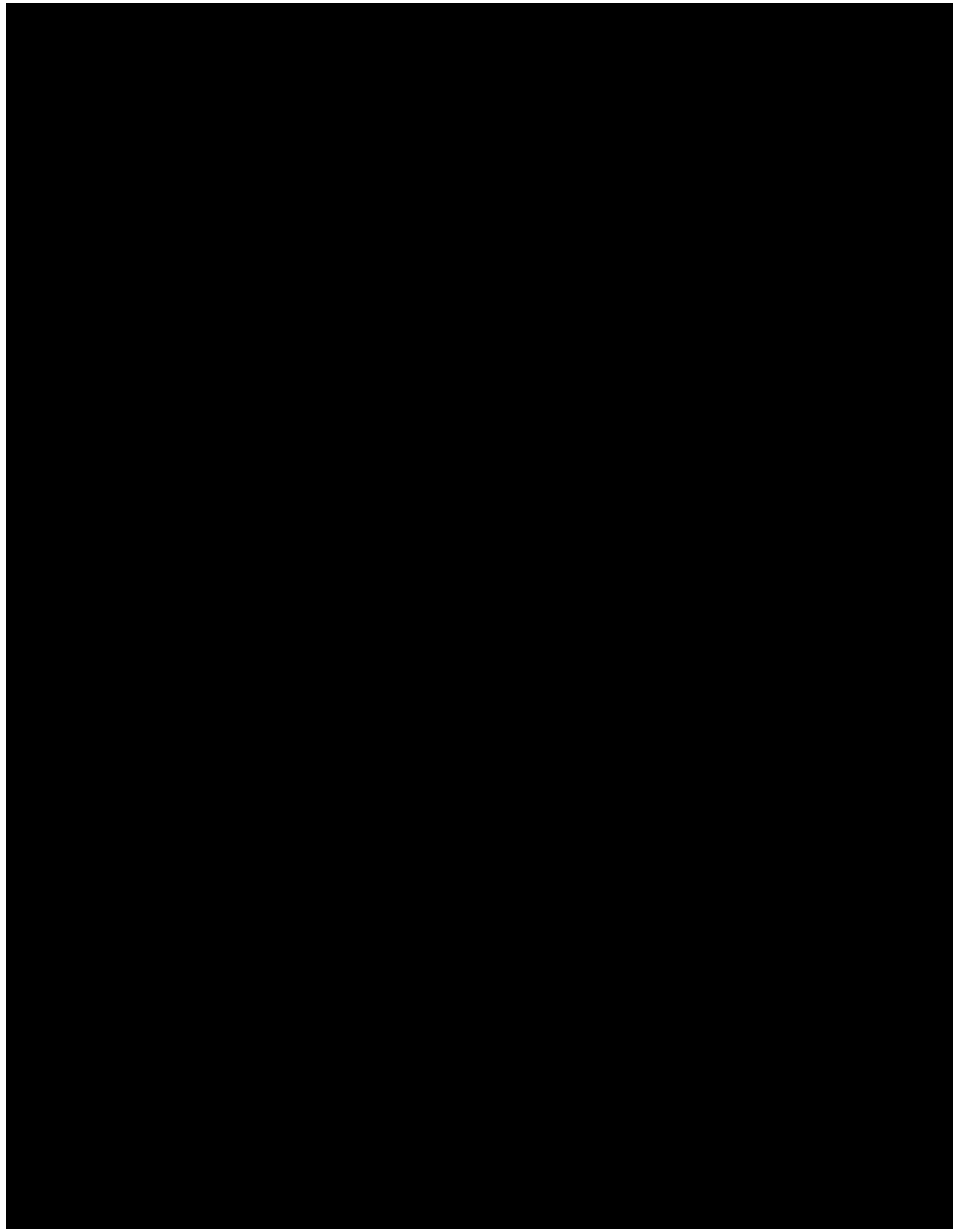

"Bowl of Donuts" by Maria Sheehy

Sensors and Samples: A Homological Approach

Nick Cavanna, Kirk Gardner, and Don Sheehy
UConn

Topological Data Analysis Challenges

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Complex (Complicated) Data

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collected as a

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collected as a

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Unions of Balls

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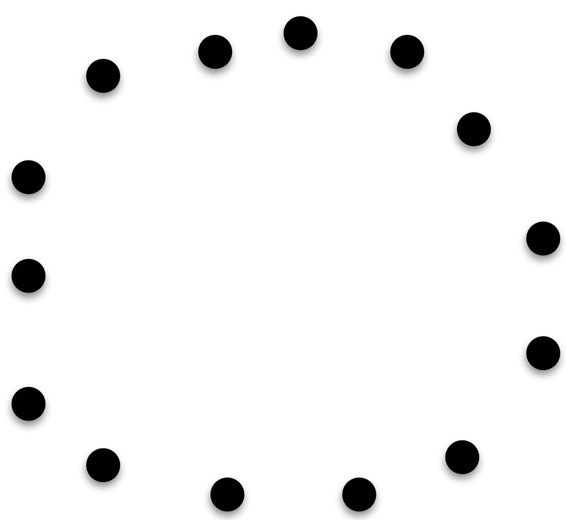
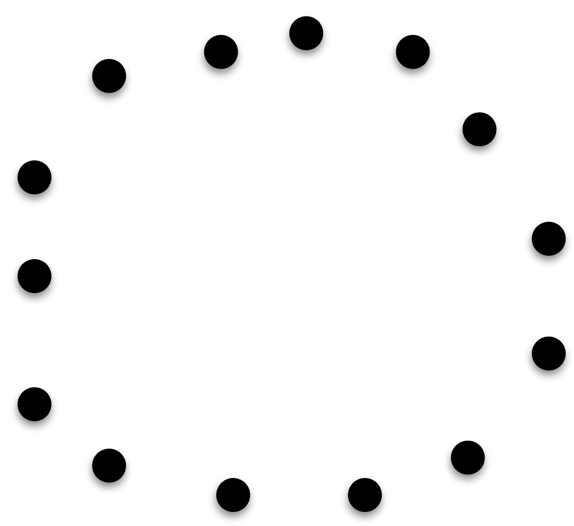
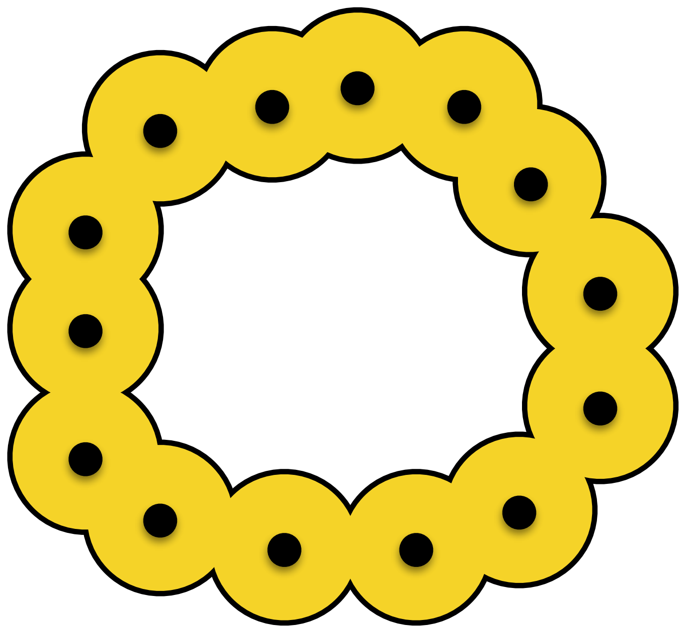
Finite Point Set



Unions of Balls

Finite Point Set → Union of Balls





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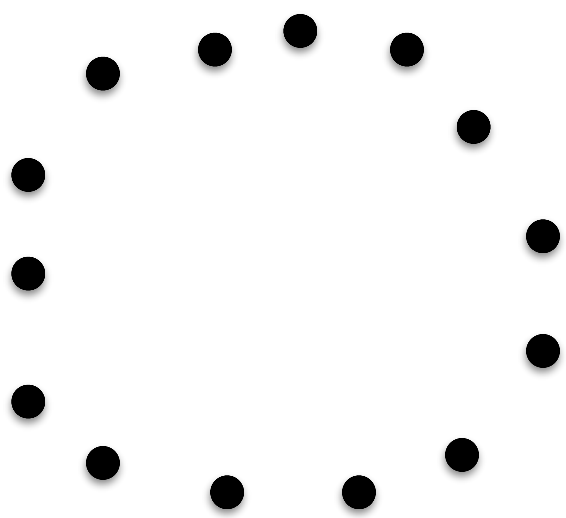
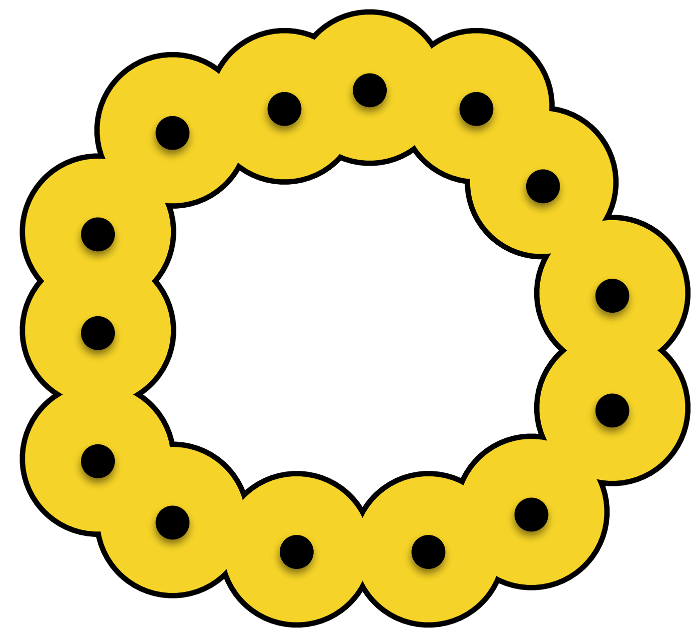


Topologically uninteresting



Potentially Interesting

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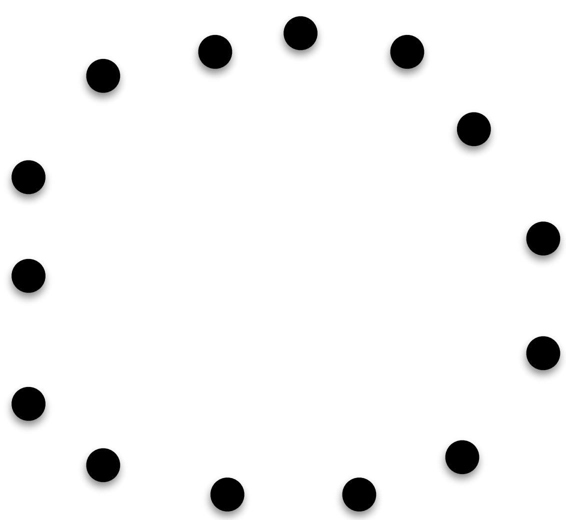

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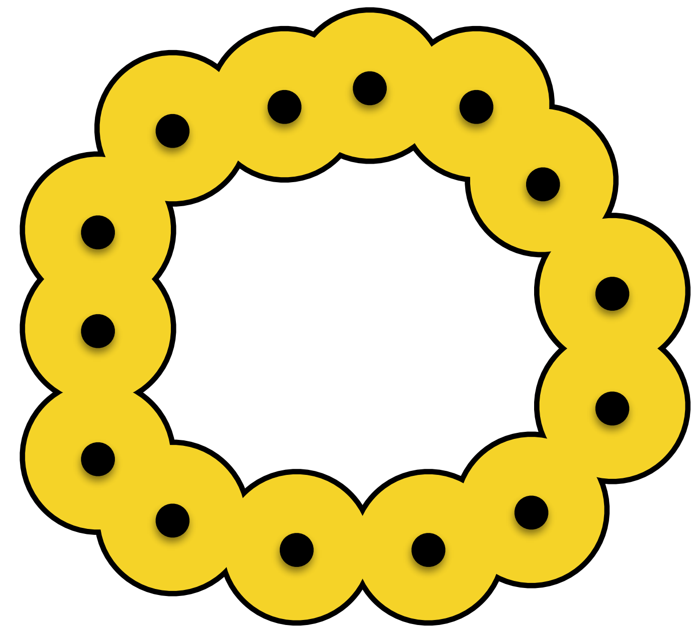
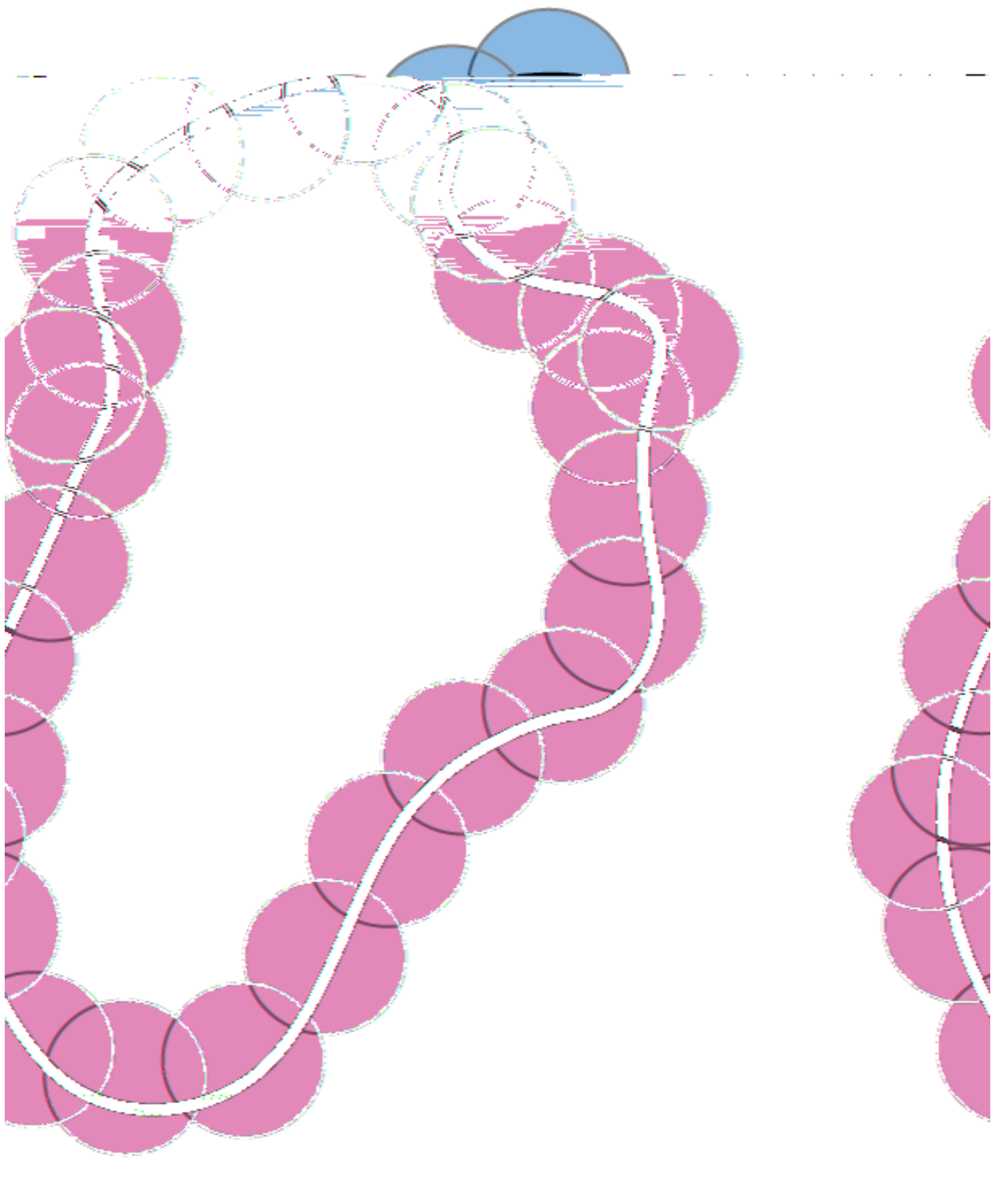


Potentially Interesting

Idea: Fill in the gaps in the ambient space.
Examples: Molecules and Manifolds

Homology Inference



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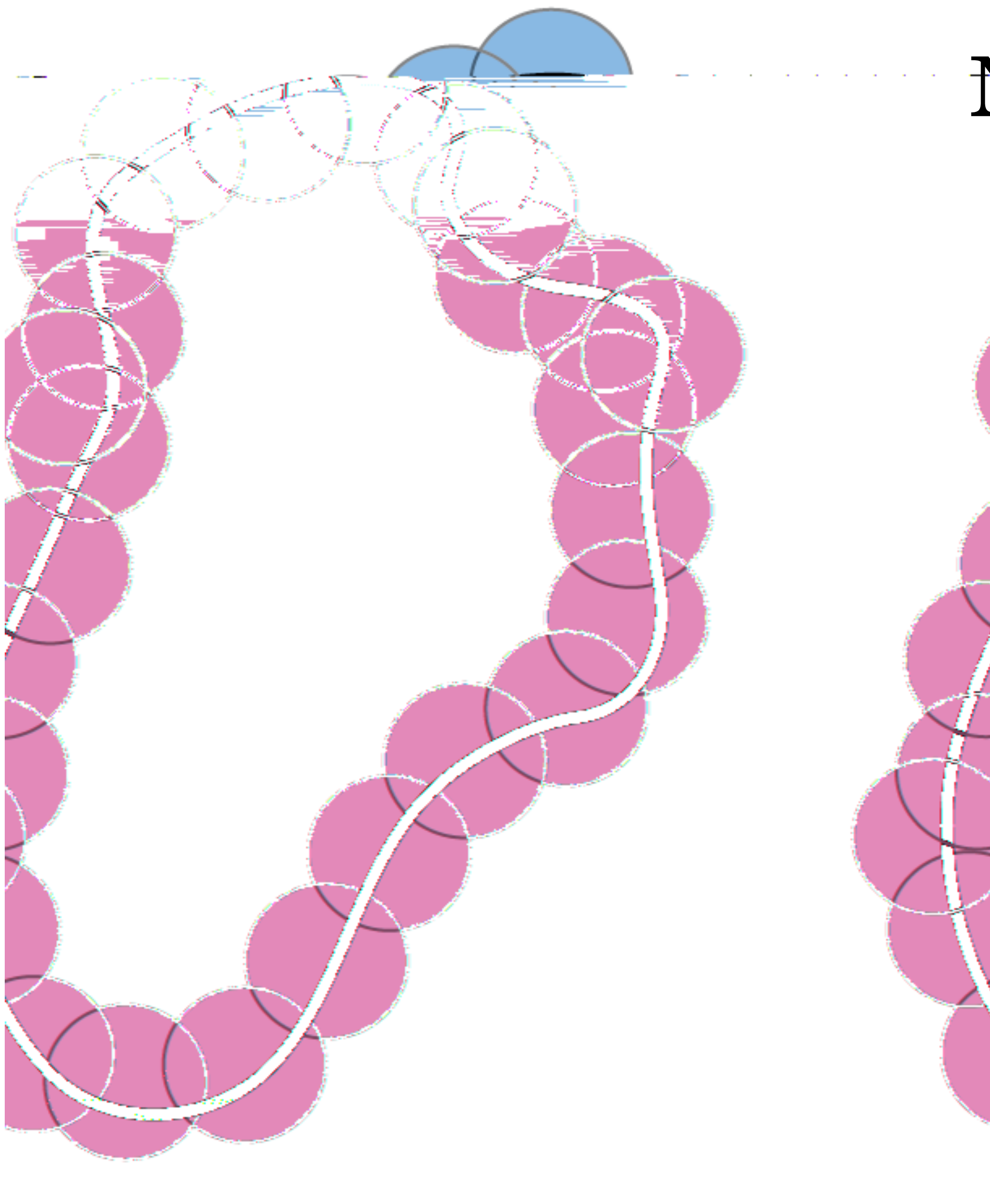

Niyogi, Smale, Weinberger

The homology of a smooth manifold can be computed from a sufficiently dense finite sample by considering the union of balls centered at the samples.

Compute the homology by looking at the nerve (or Cech) complex.

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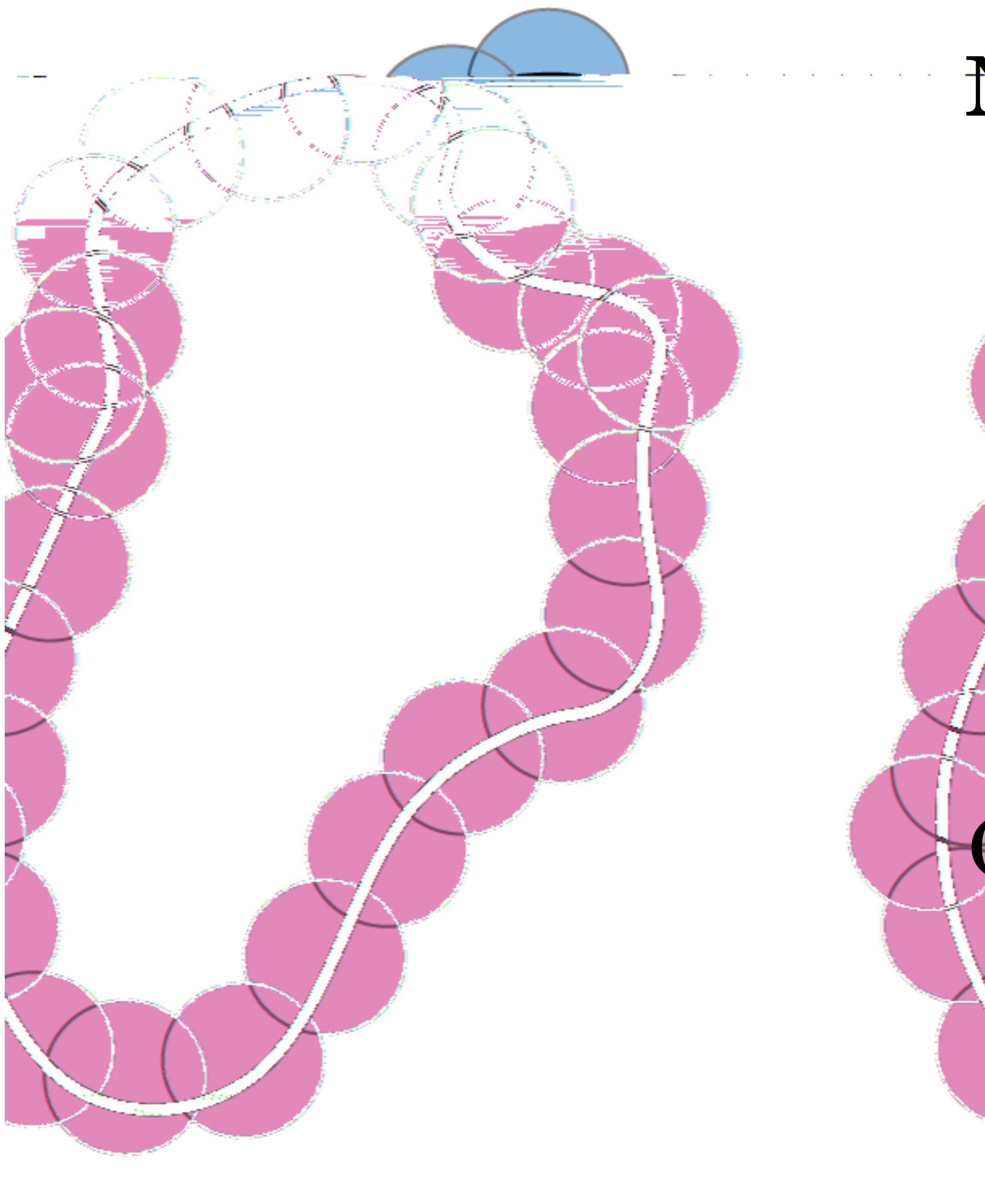

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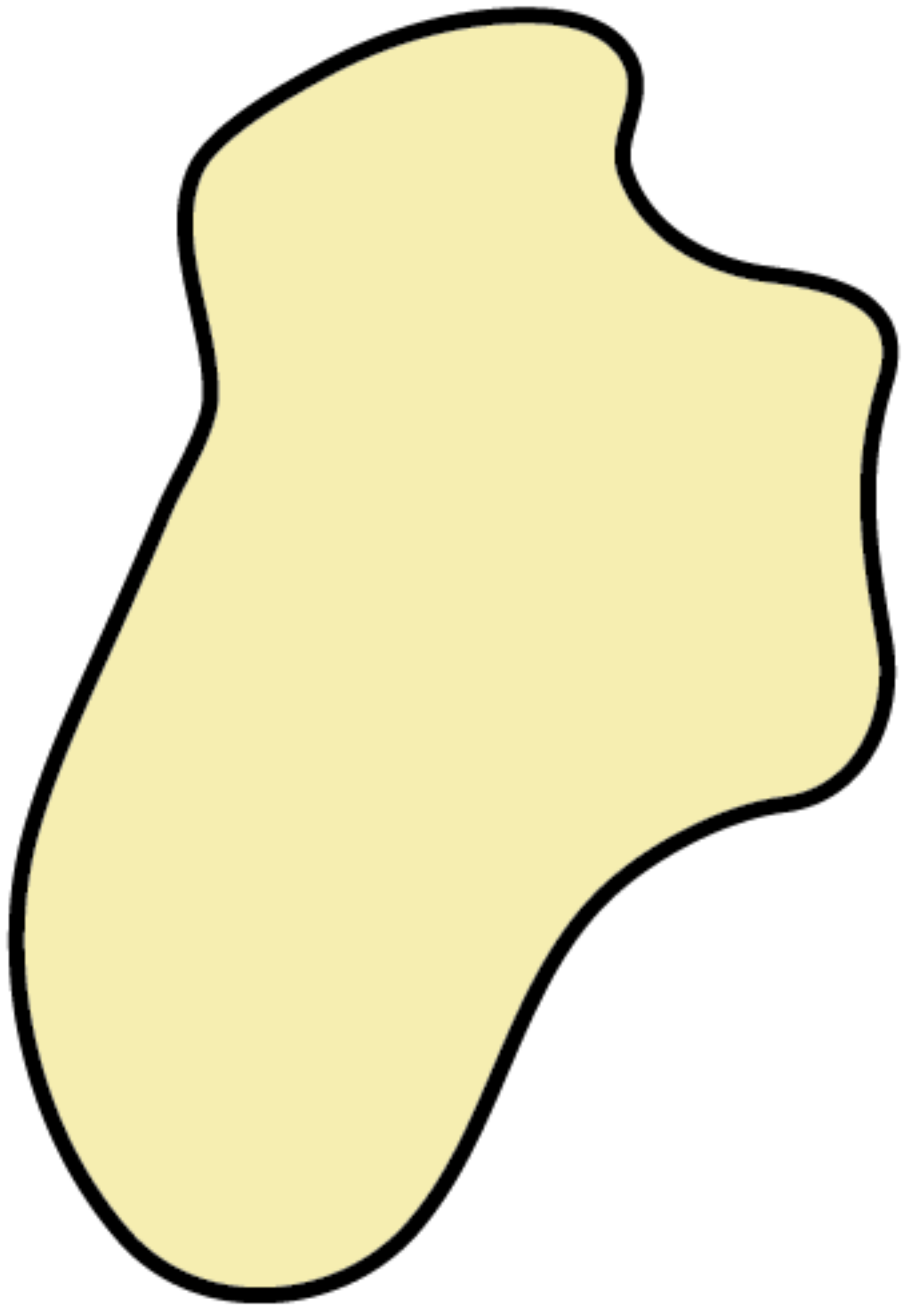
Questions:

What are the radii?
Is smoothness really necessary?

Sampling Compact Sets



Chazal and Lieutier

Infer the homology of compact sets assuming there are no critical points of the distance function nearby.

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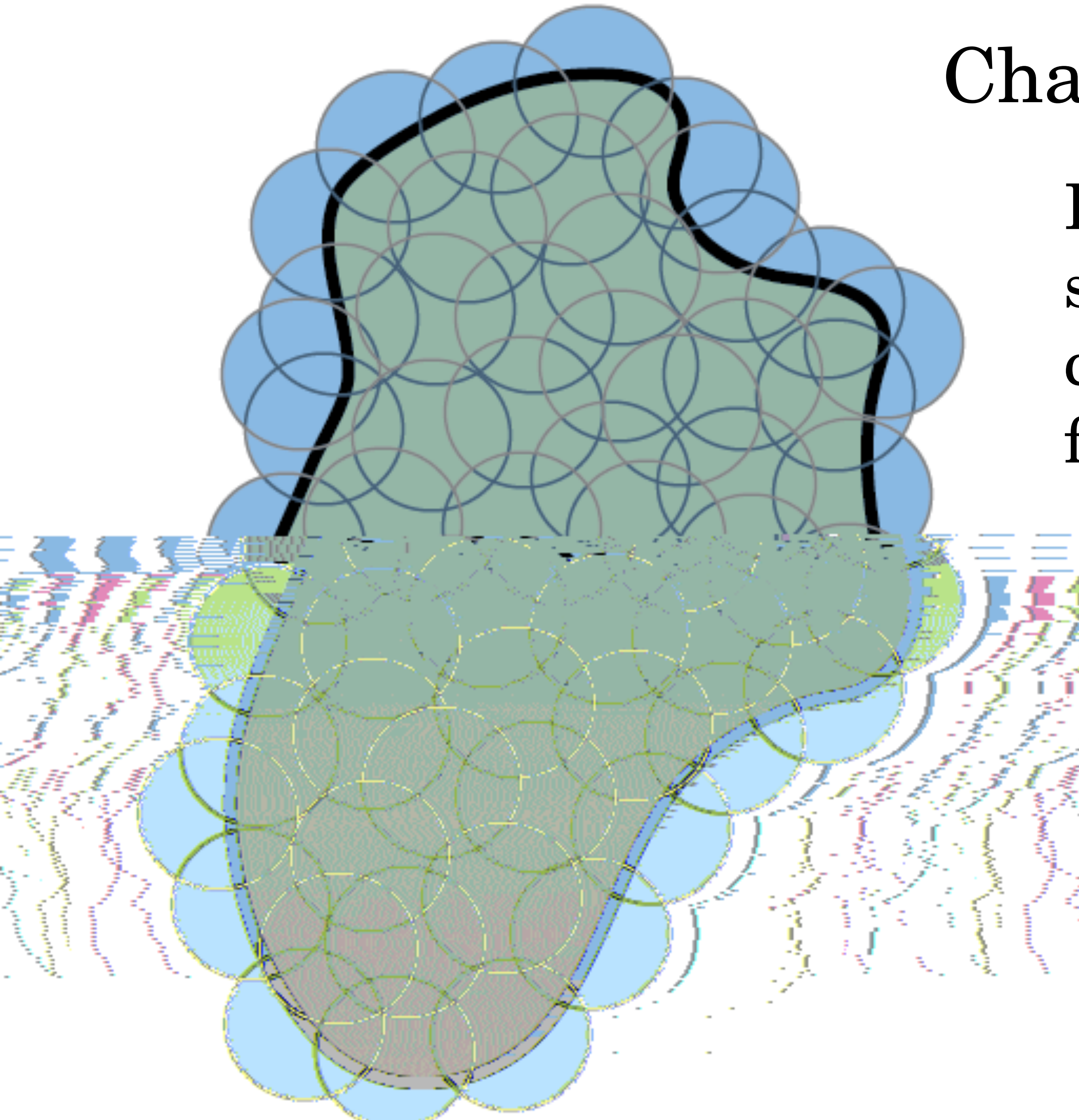

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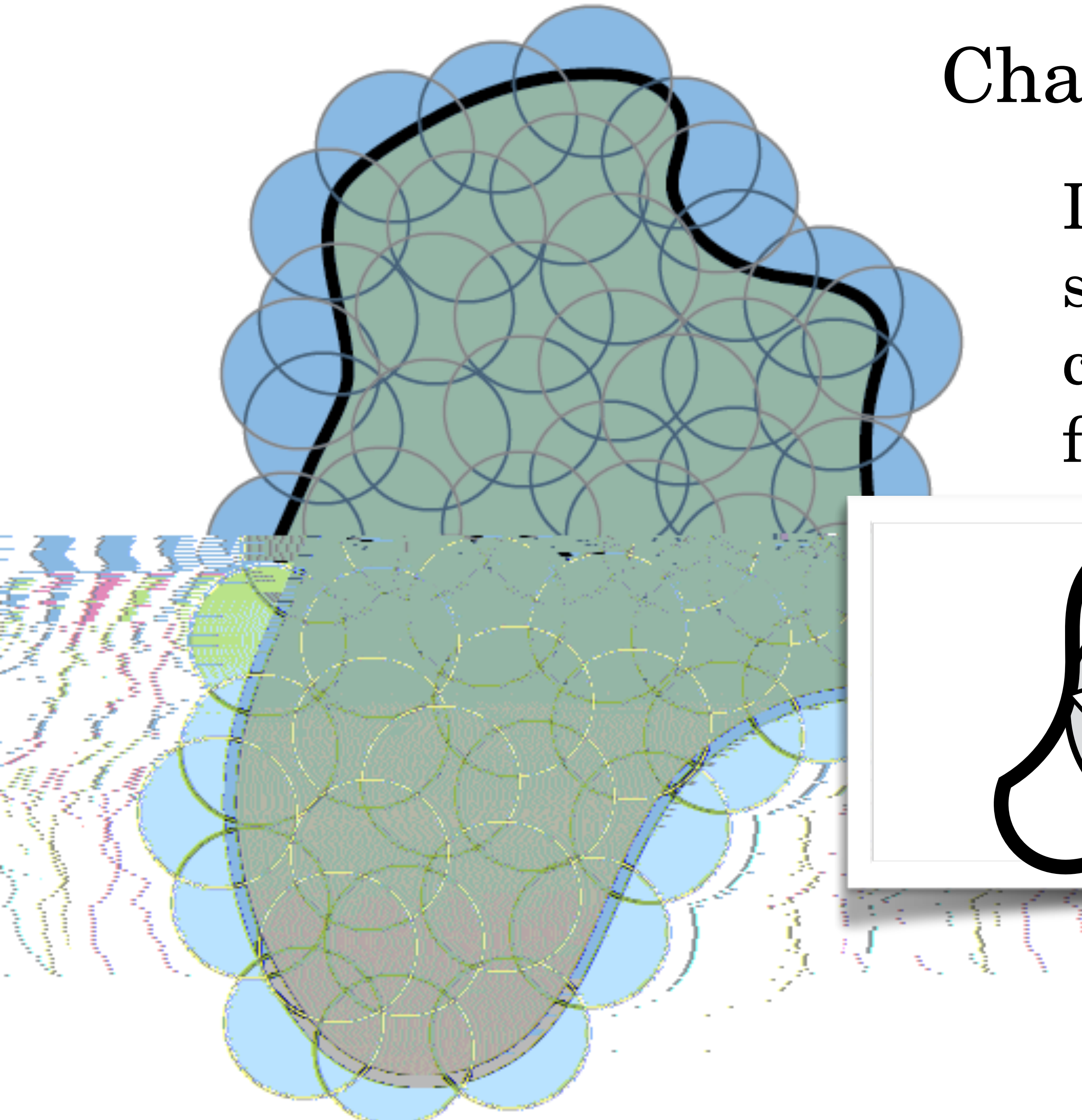

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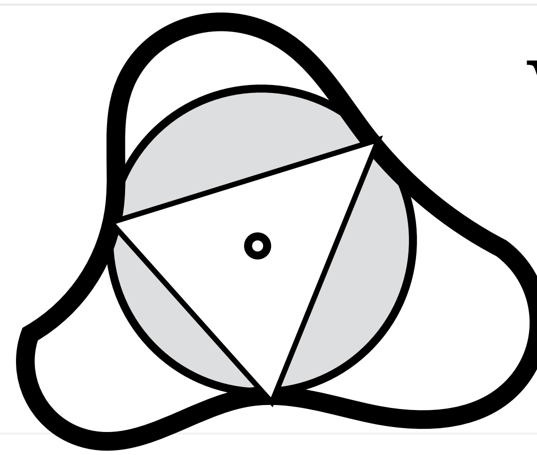

Weak Feature Size: distance to nearest critical point.

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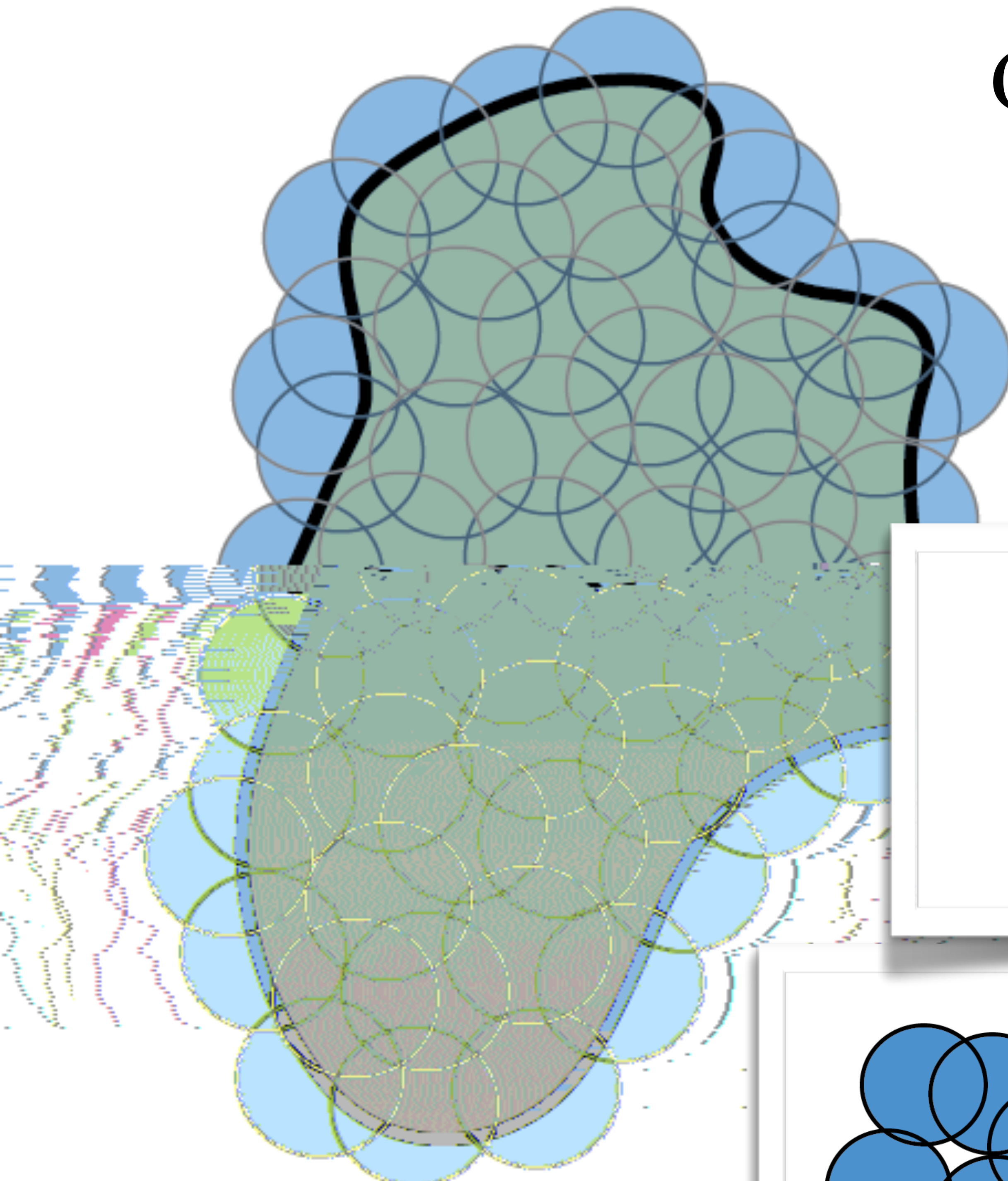

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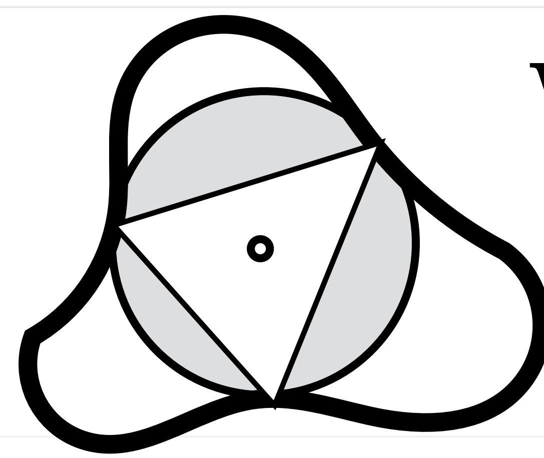

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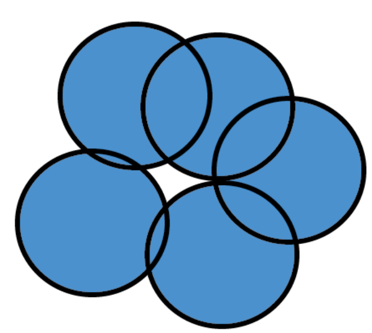
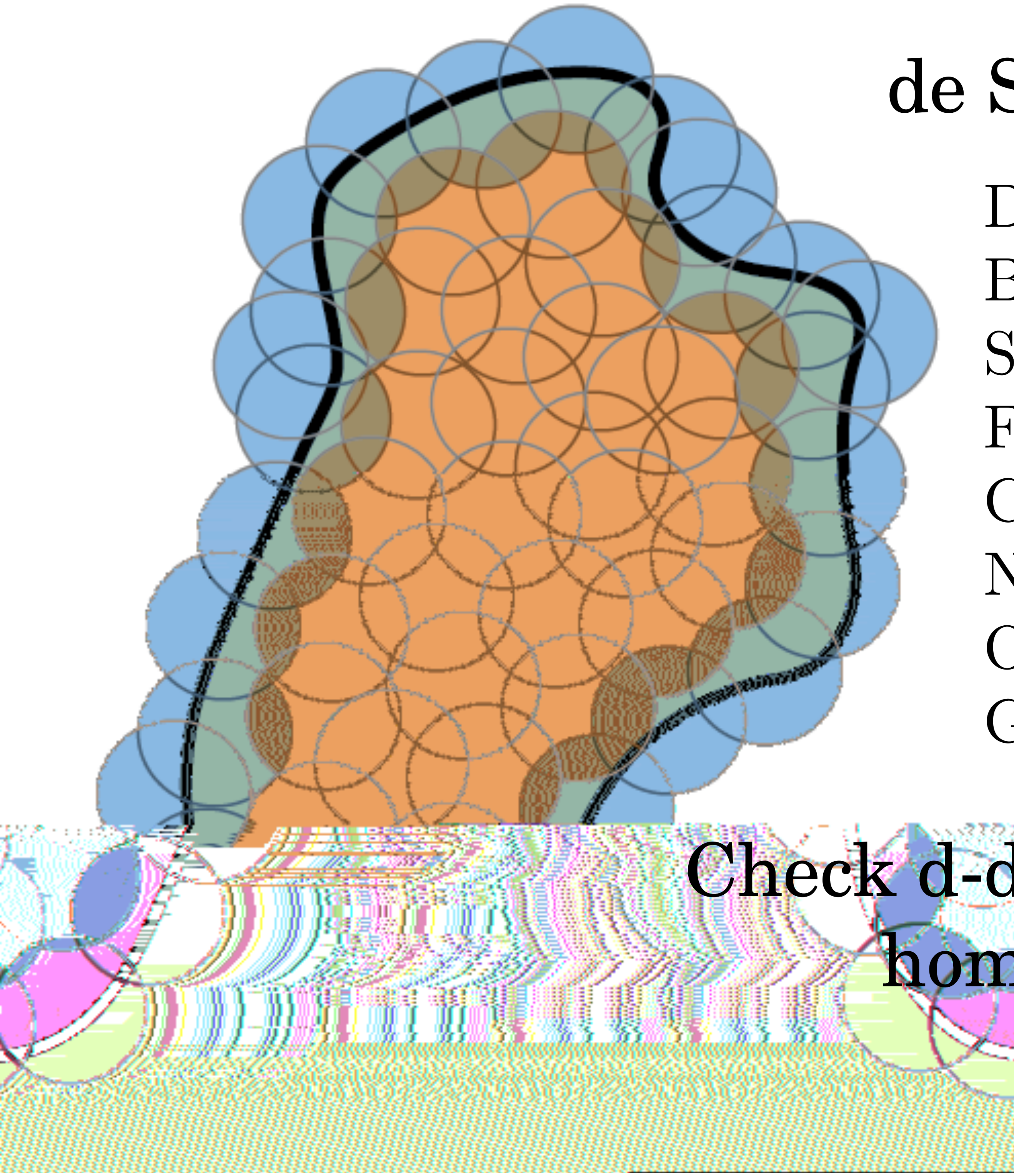


Key idea: Use persistence to eliminate noise.

Homological Sensor Networks



de Silva and Ghrist

Domain: compact $\mathcal{D} \subset \mathbb{R}^d$
Boundary: $\mathcal{B} = \mathrm{bdy}\mathcal{D}$
Sensors: $P \subset \mathcal{D}$
Fence nodes: $Q = P \cap \mathcal{B}^\alpha$
Coverage Area: P^α
No Coordinates
Only know nbhd at radii $\alpha/\sqrt{2}$ and 3α.
Goal: Certify $\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^\alpha$.

Check d-dimensional persistent relative homology of Rips complexes.

The (Vietoris-)Rips Filtration encodes the topology of a metric space when viewed at different scales.

Input: A finite metric space $(P, \mathbf{d})$.
Output: A sequence of simplicial complexes $\{R_\alpha\}$ such that $\sigma \in R_\alpha$ iff $\mathbf{d}(p, q) \leq 2\alpha$ for all $p, q \in \sigma$.

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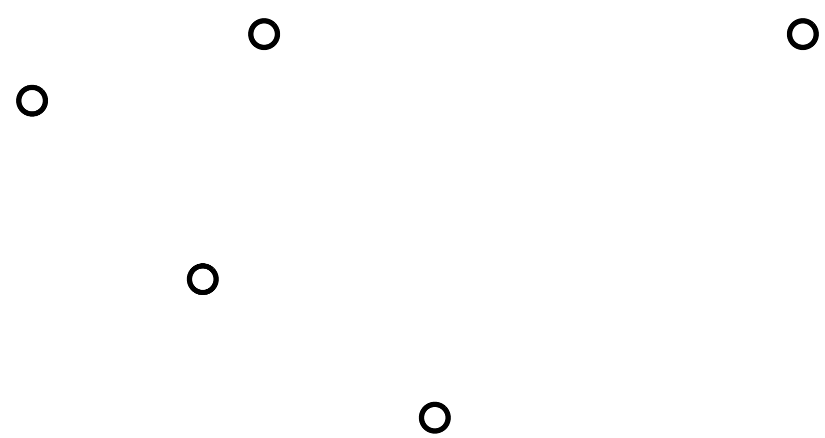

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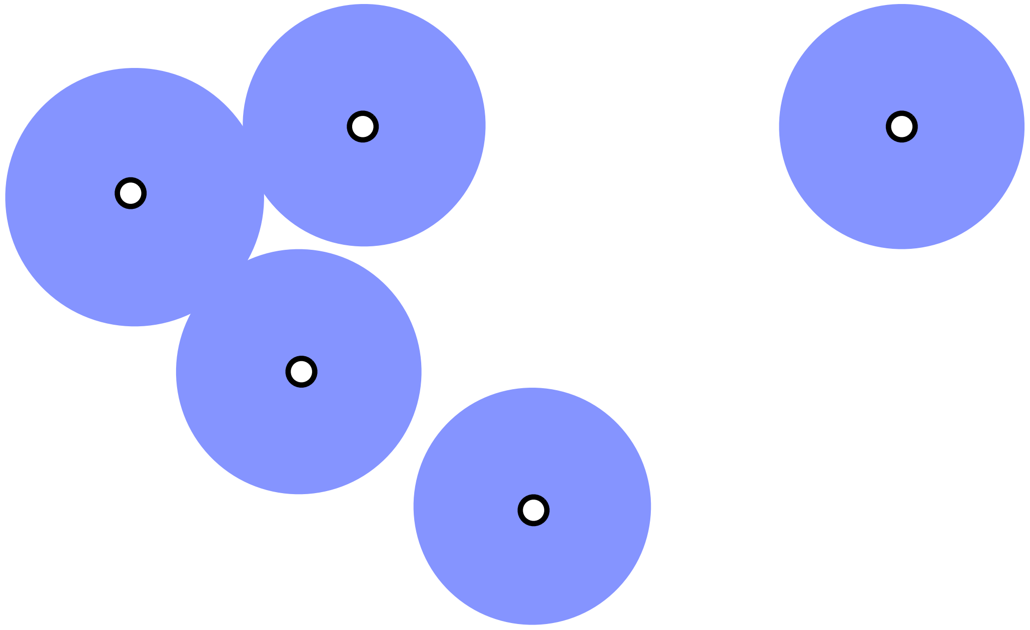

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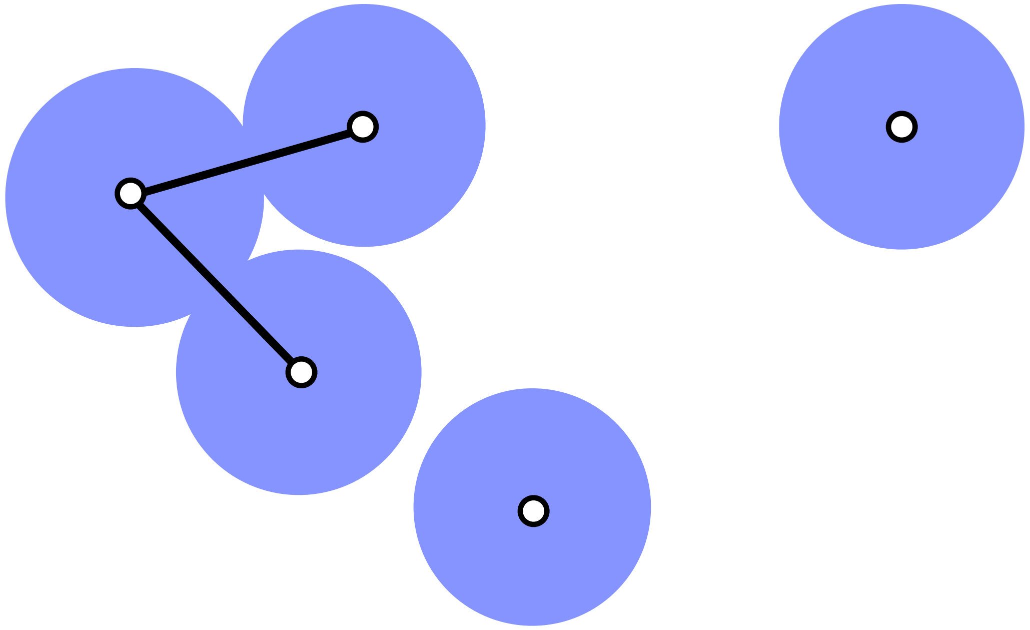

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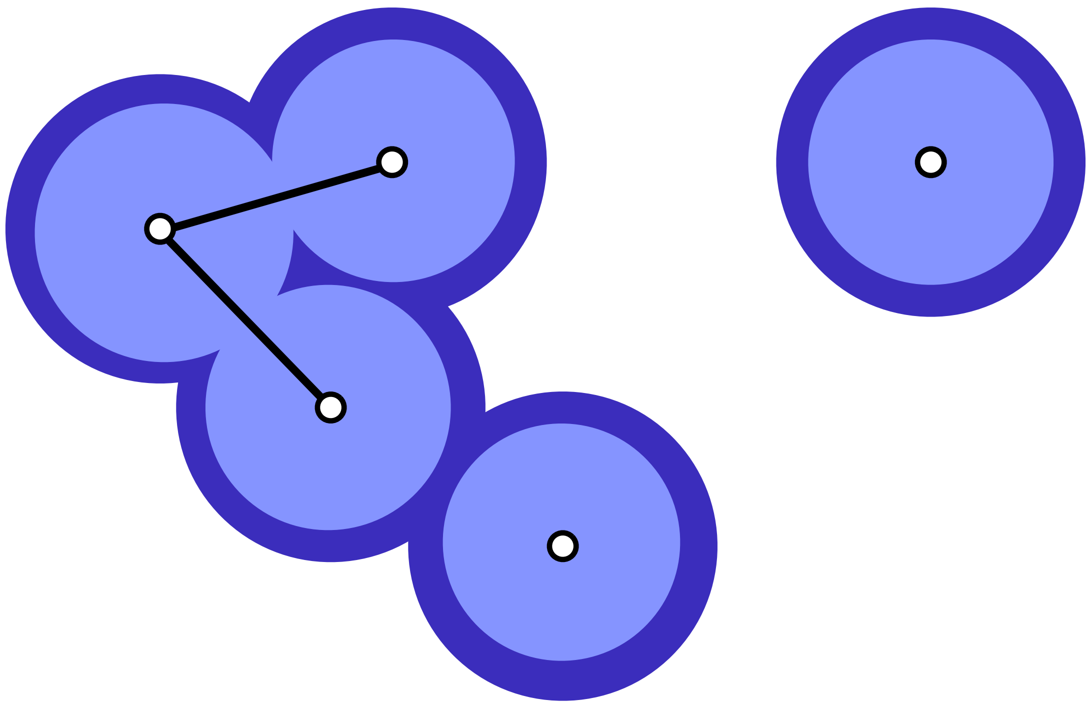

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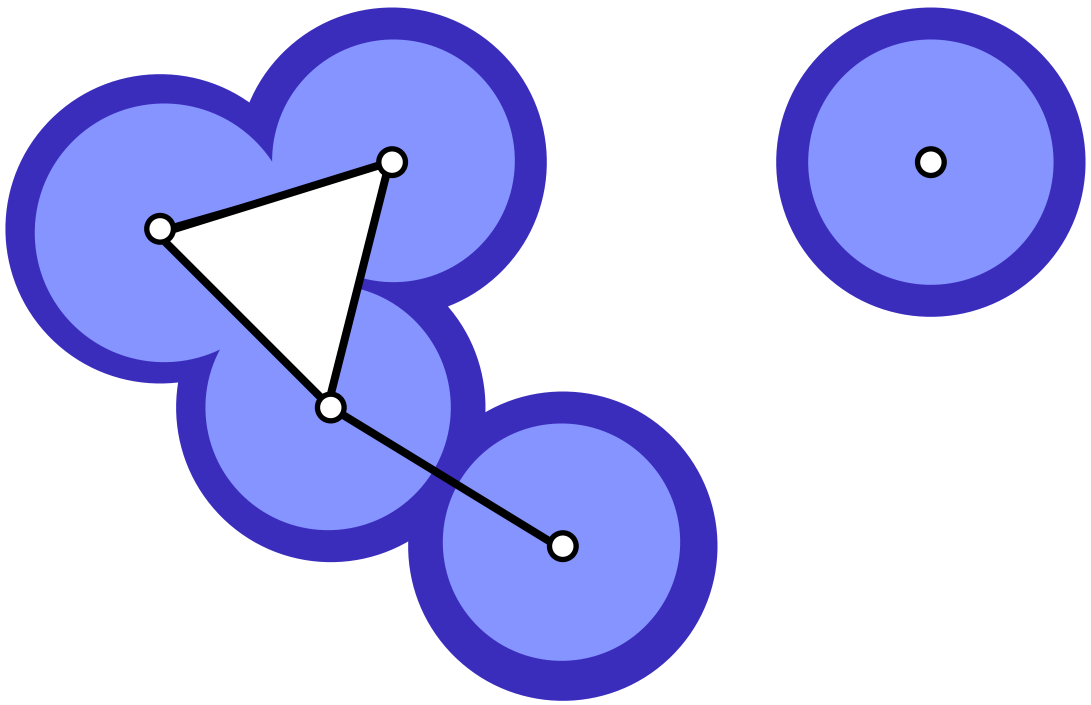

The Topological Coverage Criterion

Theorem 1* *Let $\mathcal{D} \subset \mathbb{R}^d$ be a connected set whose boundary $\mathcal{B}$ is a smooth manifold with injectivity radius at least 4α. If*

$$\mathrm{H}_d((Rips_{\alpha/\sqrt{2}}(P), Rips_{\alpha/\sqrt{2}}(Q)) \hookrightarrow (Rips_{3\alpha}(P), Rips_{3\alpha}(Q))) \neq 0,$$

then $\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^{\alpha}$.

*based on de Silva and Ghrist 07

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TCC

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then $\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^{\alpha}$. Coverage Guarantee

TCC

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Key tricks

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1 Alexander Duality - *coverage to connectivity*

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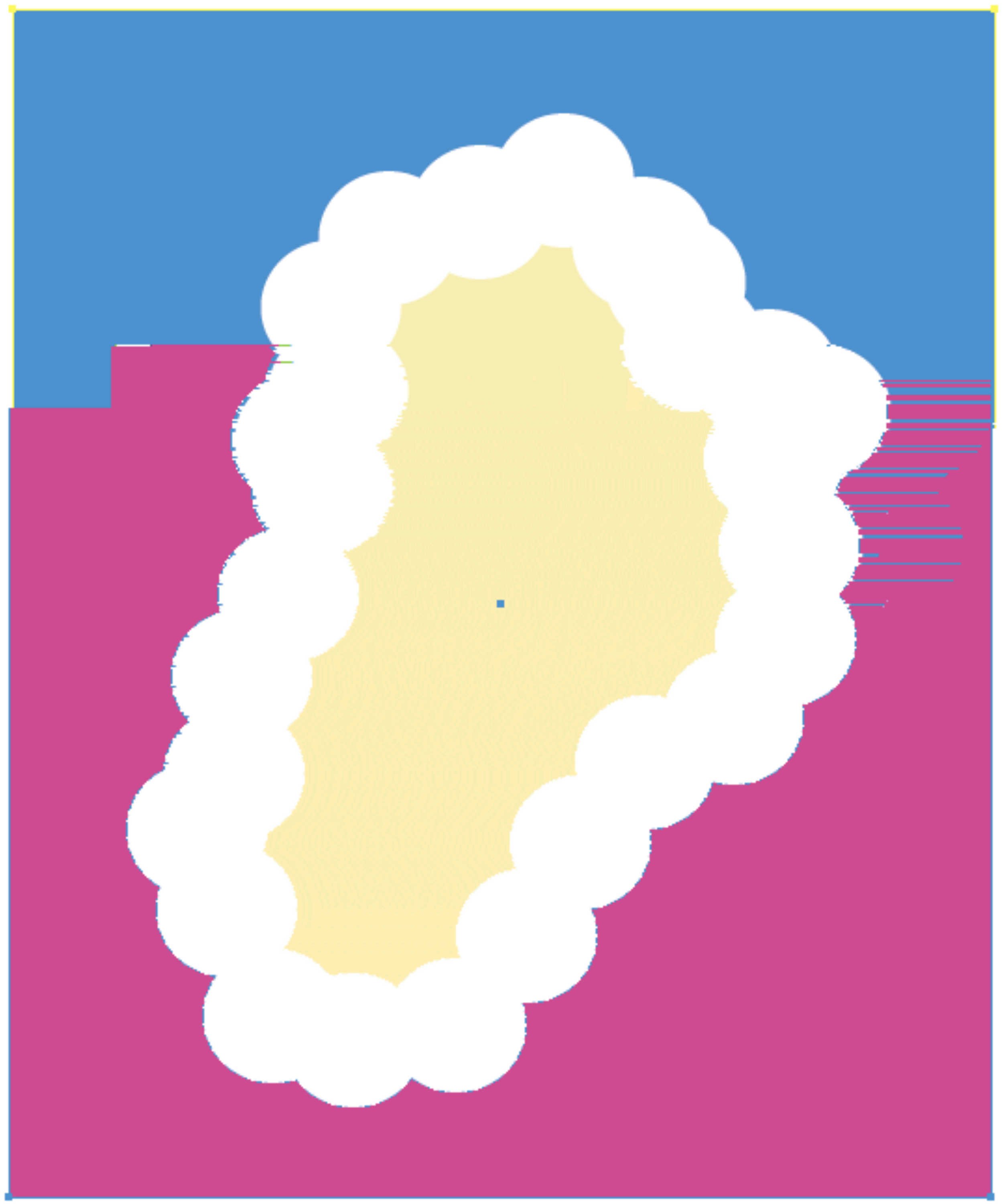
4 Persistent Homology - *eliminate noise*

Tricks: Alexander Duality



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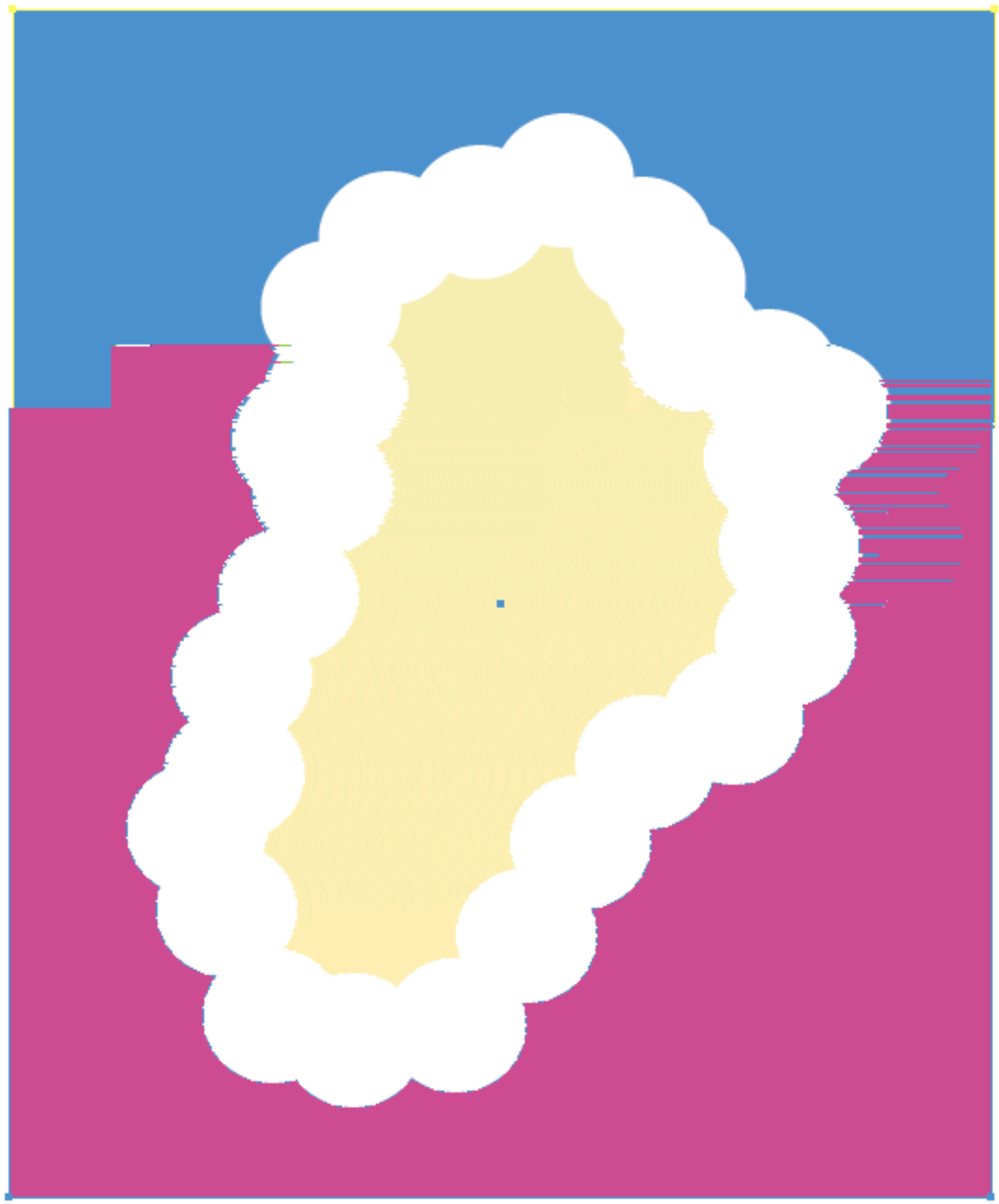
$$\mathrm{H}^d(D, B) \stackrel{\sqcup}{=} \mathrm{H}_0(\bar{B}, \bar{D})$$



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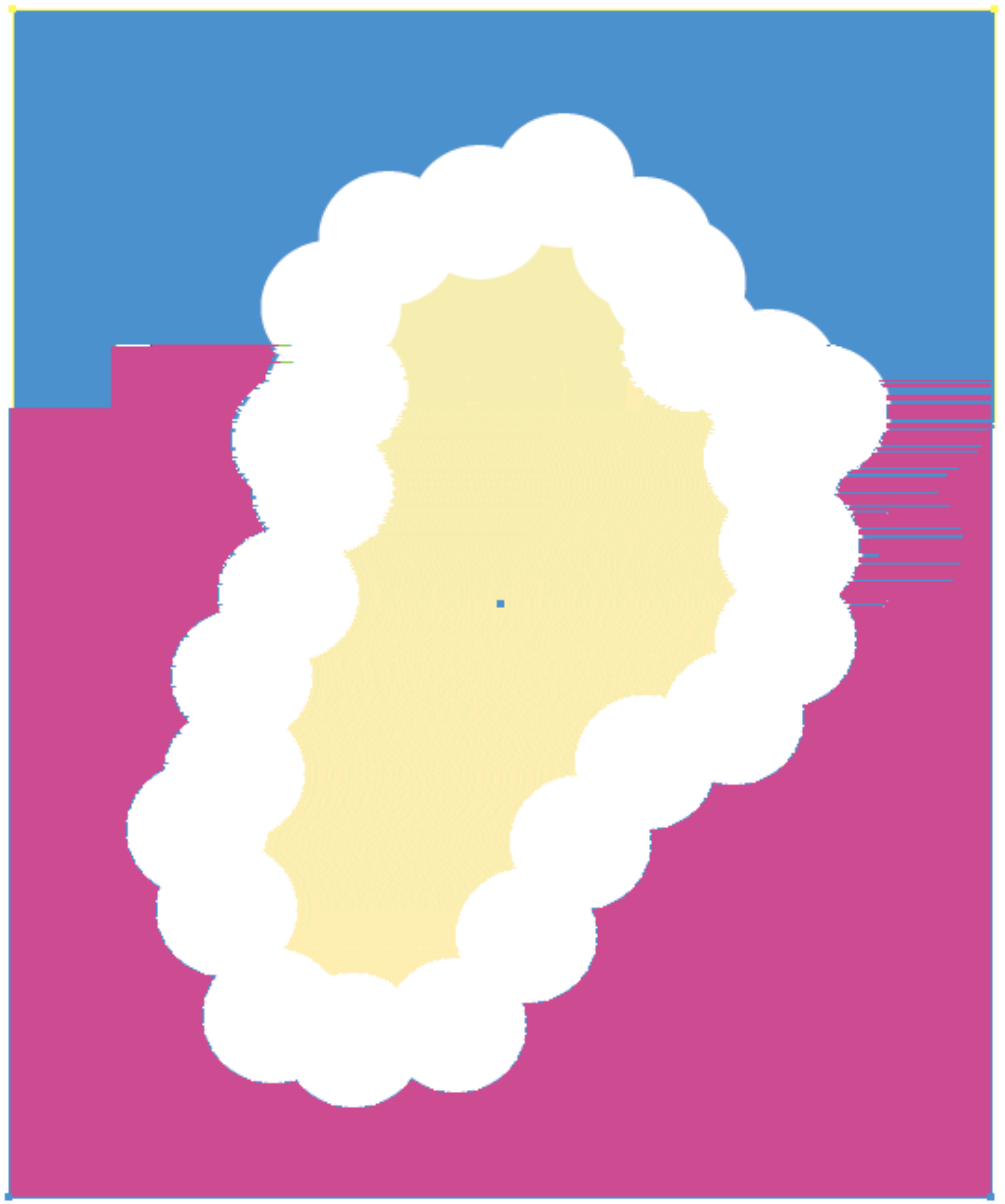

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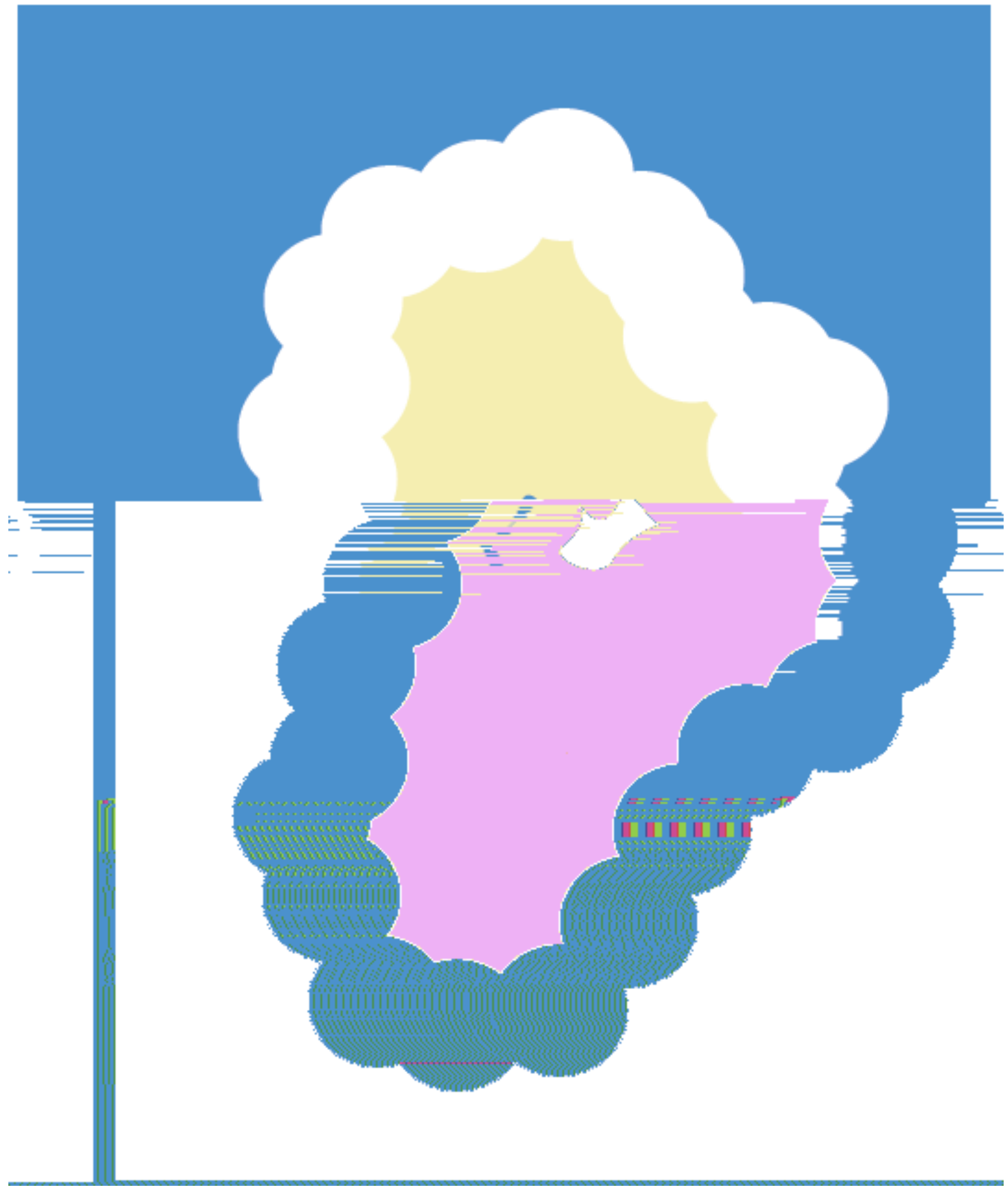

Interleaving Rips and Cech Filtrations

$$H_*(Rips_{\alpha/\sqrt{2}}(P), Rips_{\alpha/\sqrt{2}}(Q)) \longrightarrow H_{\square}(Rips_{3\alpha}(P), Rips_{3\alpha}(Q))$$

interleaving

$$H_{\square}(\check{C}ech_{\alpha}(P), \check{C}ech_{\alpha}(Q)) \to H_{\square}(\check{C}ech_{3\alpha}(P), \check{C}ech_{3\alpha}(Q))$$

Persistent Nerve Lemma $\cong$ $\cong$

$$H_{\square}(P^{\alpha}, Q^{\alpha}) \to H_{\square}(P^{3\alpha}, Q^{3\alpha})$$

Suffices to look at offsets.

Tricks: Persistent Homology

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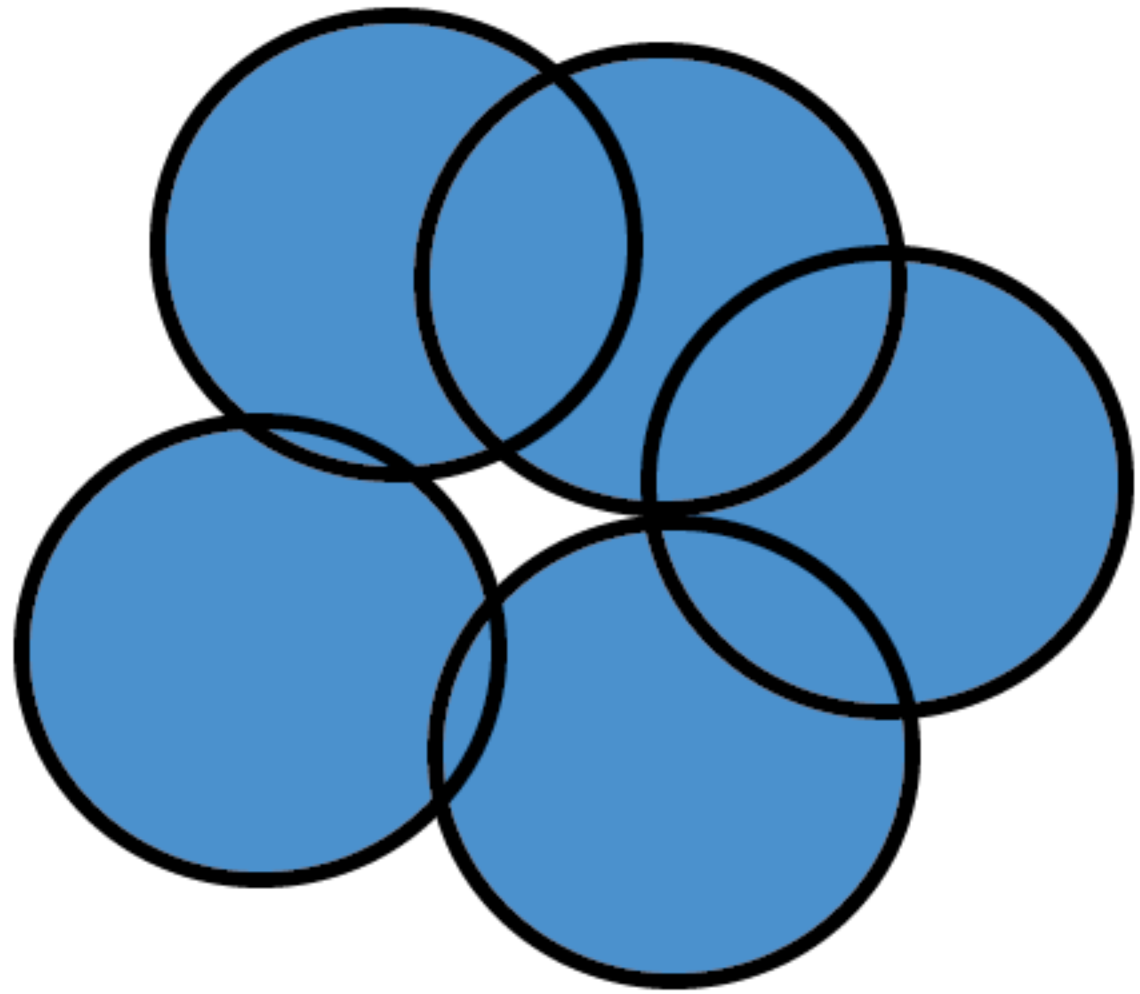

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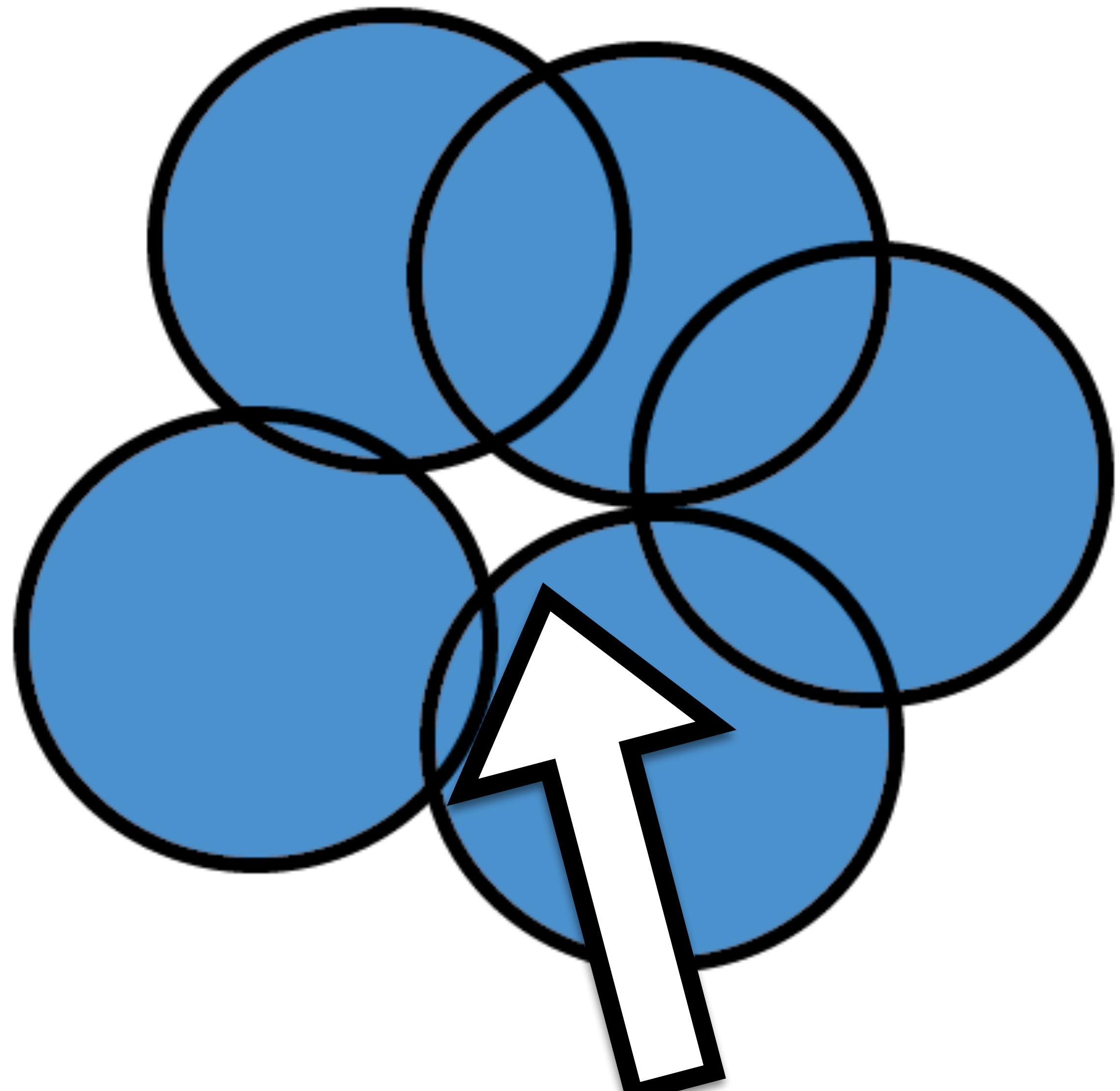

As with Chazal and Lieutier, persistence eliminates spurious features near the boundary.

TCC Proof Idea I

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$$P^{\alpha} \hookrightarrow \mathcal{D}^{2\alpha}$$

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$$a_{\square} : H_0(\overline{\mathsf{B}^{2\alpha}}, \overline{\mathsf{D}^{2\alpha}}) \to H_0(\overline{Q^\alpha}, \overline{P^\alpha})$$

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a injective implies "coverage", i.e.* $(\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^\alpha)$

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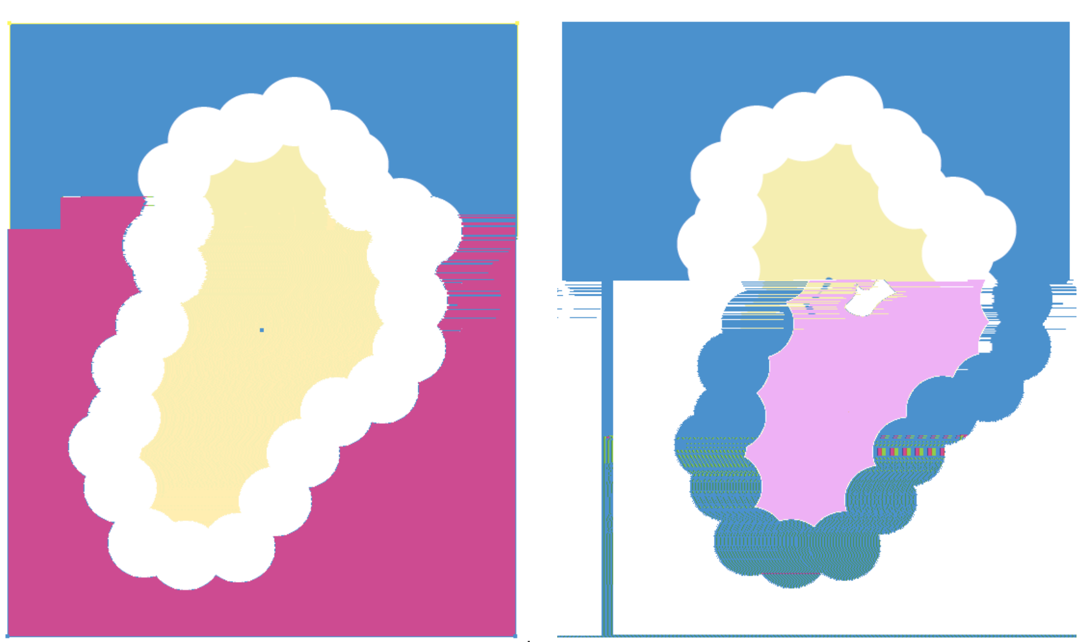
a_* injective implies "coverage", i.e. $(\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^{\alpha})$



Suppose there is an uncovered point x.

$$x \in \overline{P^{\alpha}} \cap (\mathcal{D} \setminus \mathcal{B}^{2\alpha})$$

$$[x] \neq 0 \text{ in } H_0(\overline{\mathcal{B}^{2\alpha}, \mathcal{D}^{2\alpha}})$$

However, $a_{\square}[x] = 0$.

TCC Proof Idea II

$$a_{\square} : H_0(\overline{B^{2\alpha}}, \overline{D^{2\alpha}}) \to H_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})$$

a∗ injective implies "coverage", i.e. $(\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^{\alpha})$

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If $D \setminus B^{2\alpha}$ is connected and $a_{\square}$ is surjective,
then we can infer coverage by computing $H_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})$.

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Problem: There could be "spurious" features.

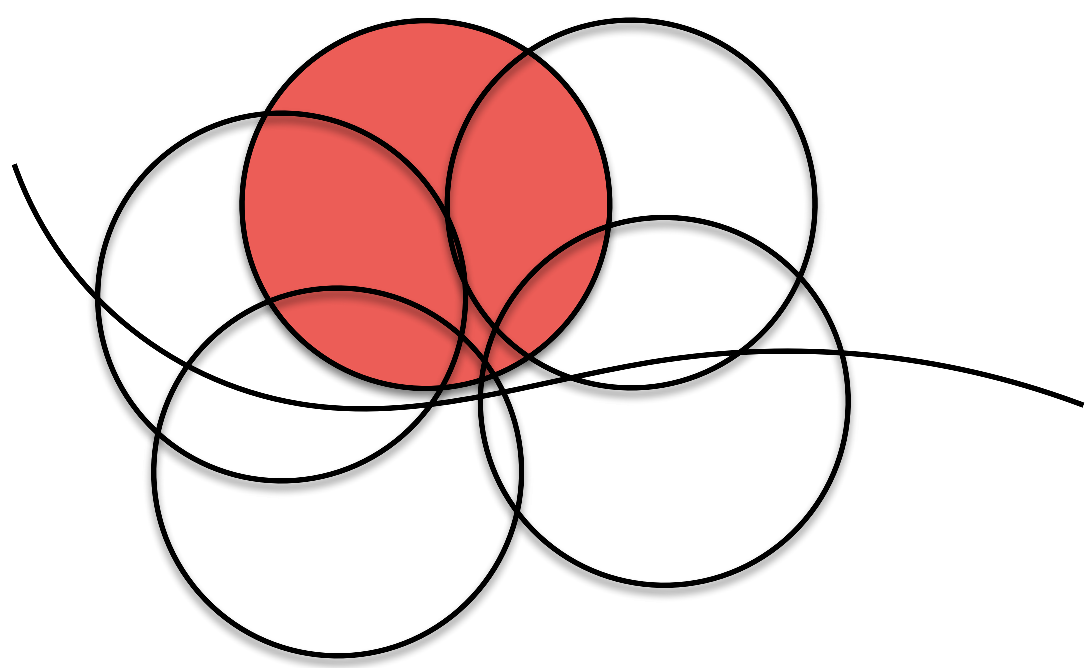
TCC Proof Idea II

$$a_{\square} : H_0(\overline{B^{2\alpha}}, \overline{D^{2\alpha}}) \to H_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})$$

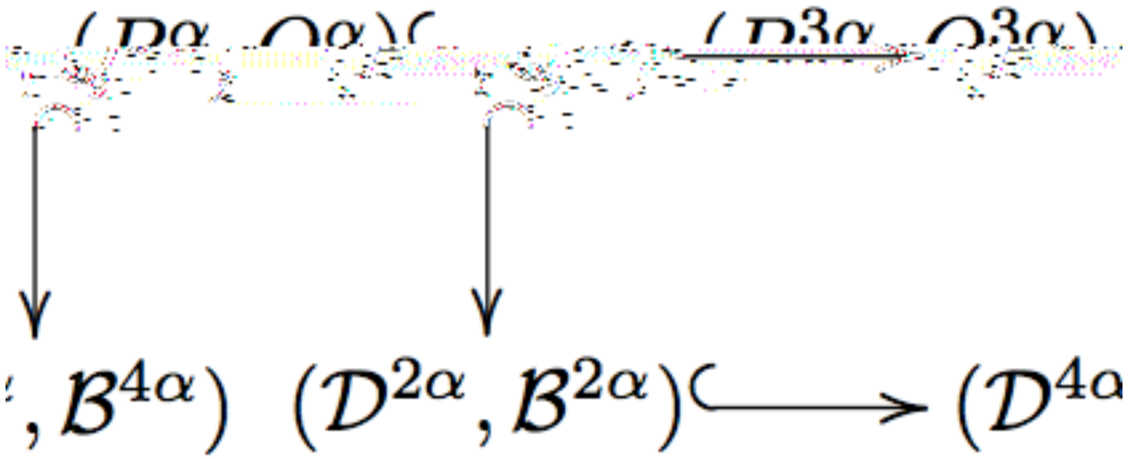
a injective implies "coverage", i.e.* $(\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^{\alpha})$

If $D \setminus B^{2\alpha}$ is connected and $a_{\square}$ is surjective,
then we can infer coverage by computing $H_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})$.

Problem: There could be "spurious" features.



TCC Proof of Correctness I



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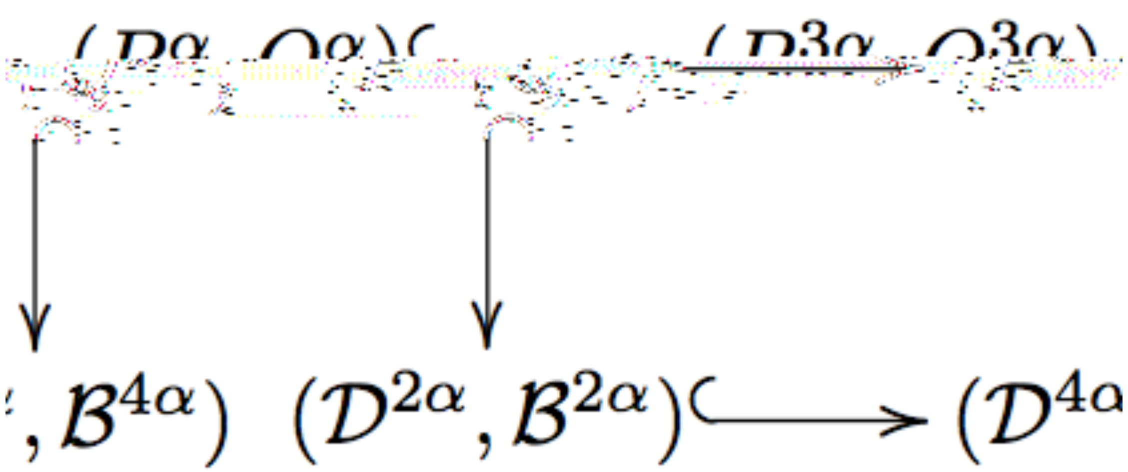
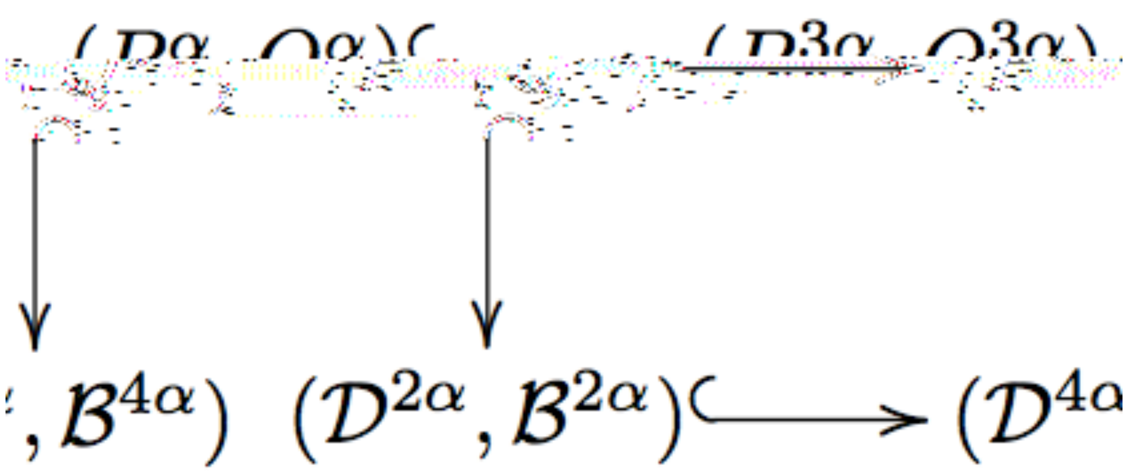


$$
\begin{array}{ccc}
\mathrm{H}_0(\overline{\mathcal{B}^{4\alpha}}, \overline{\mathcal{D}^{4\alpha}}) & \xrightarrow{j_*} & \mathrm{H}_0(\overline{\mathcal{B}^{2\alpha}}, \overline{\mathcal{D}^{2\alpha}}) \\
\downarrow & & \downarrow \\
\mathrm{H}_0(\overline{Q^{3\alpha}}, \overline{P^{3\alpha}}) & \xrightarrow{i_*} & \mathrm{H}_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})
\end{array}
$$

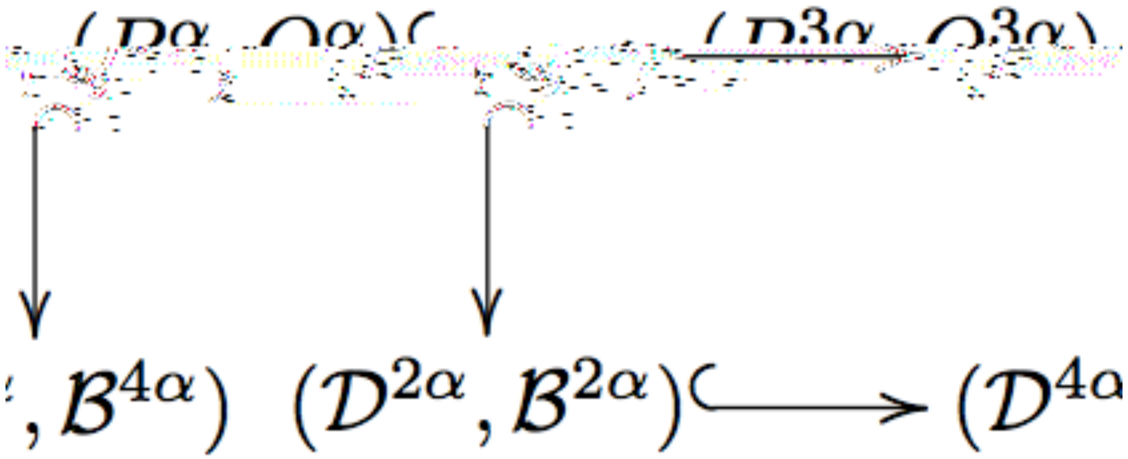
TCC Proof of Correctness I



$$
\begin{array}{ccc}
H_0(\overline{\mathcal{B}^{4\alpha}}, \overline{\mathcal{D}^{4\alpha}}) & \xrightarrow{j_*} & H_0(\overline{\mathcal{B}^{2\alpha}}, \overline{\mathcal{D}^{2\alpha}}) \\
\downarrow & & \downarrow \\
H_0(\overline{Q^{3\alpha}}, \overline{P^{3\alpha}}) & \xrightarrow{i_*} & H_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})
\end{array}
$$

Let $\varphi_* : \mathrm{im}\, j_* \to \mathrm{im}\, i_*$ be the homomorphsim induced by inclusion.

TCC Proof of Correctness I



$$\begin{array}{ccc}
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\downarrow & & \downarrow \\
\mathrm{H}_0(\overline{Q^{3\alpha}}, \overline{P^{3\alpha}}) & \xrightarrow{i_*} & \mathrm{H}_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})
\end{array}$$

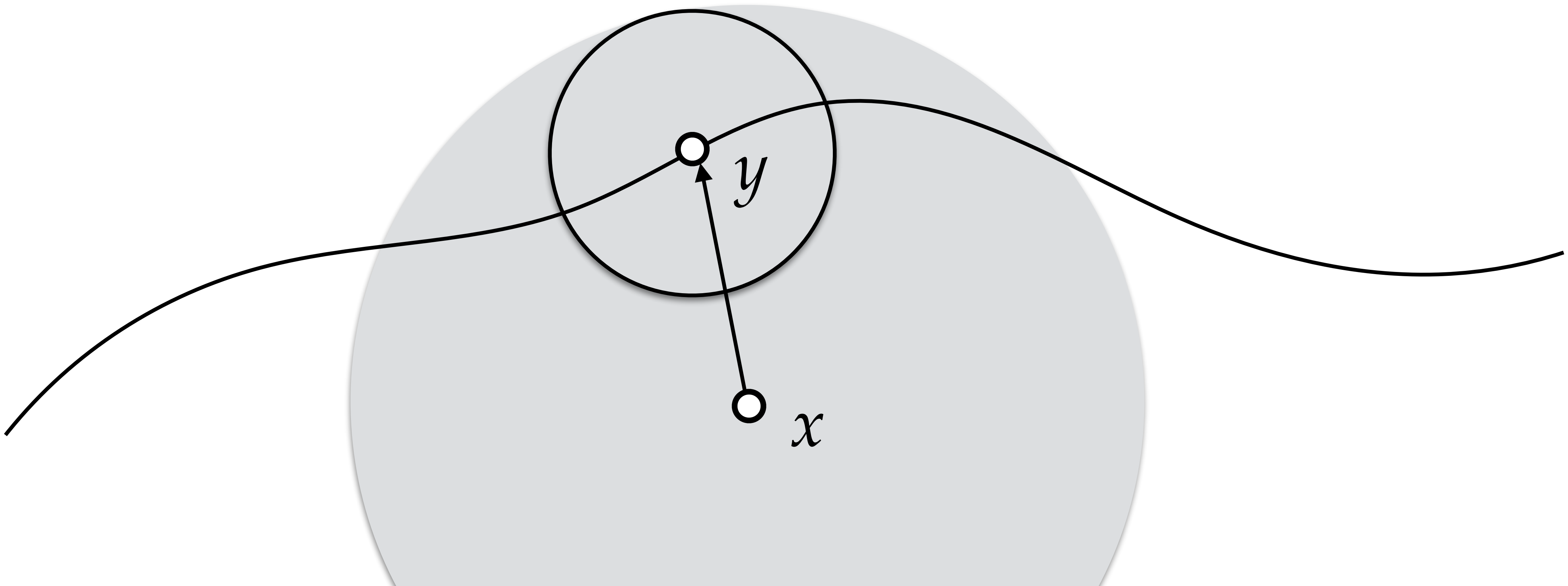
Let $\varphi_{\Box} : \mathrm{im} j_{\Box} \to \mathrm{im} i_{\Box}$ be the homomorphsim induced by inclusion.

Lemma 1 *If $\phi_{\Box}$ is injective, then $\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^{\alpha}$.*

Lemma 2 *If $j_{\Box}$ is an isomorphism, then $\phi_{\Box}$ is surjective.*

TCC Proof of Correctness II

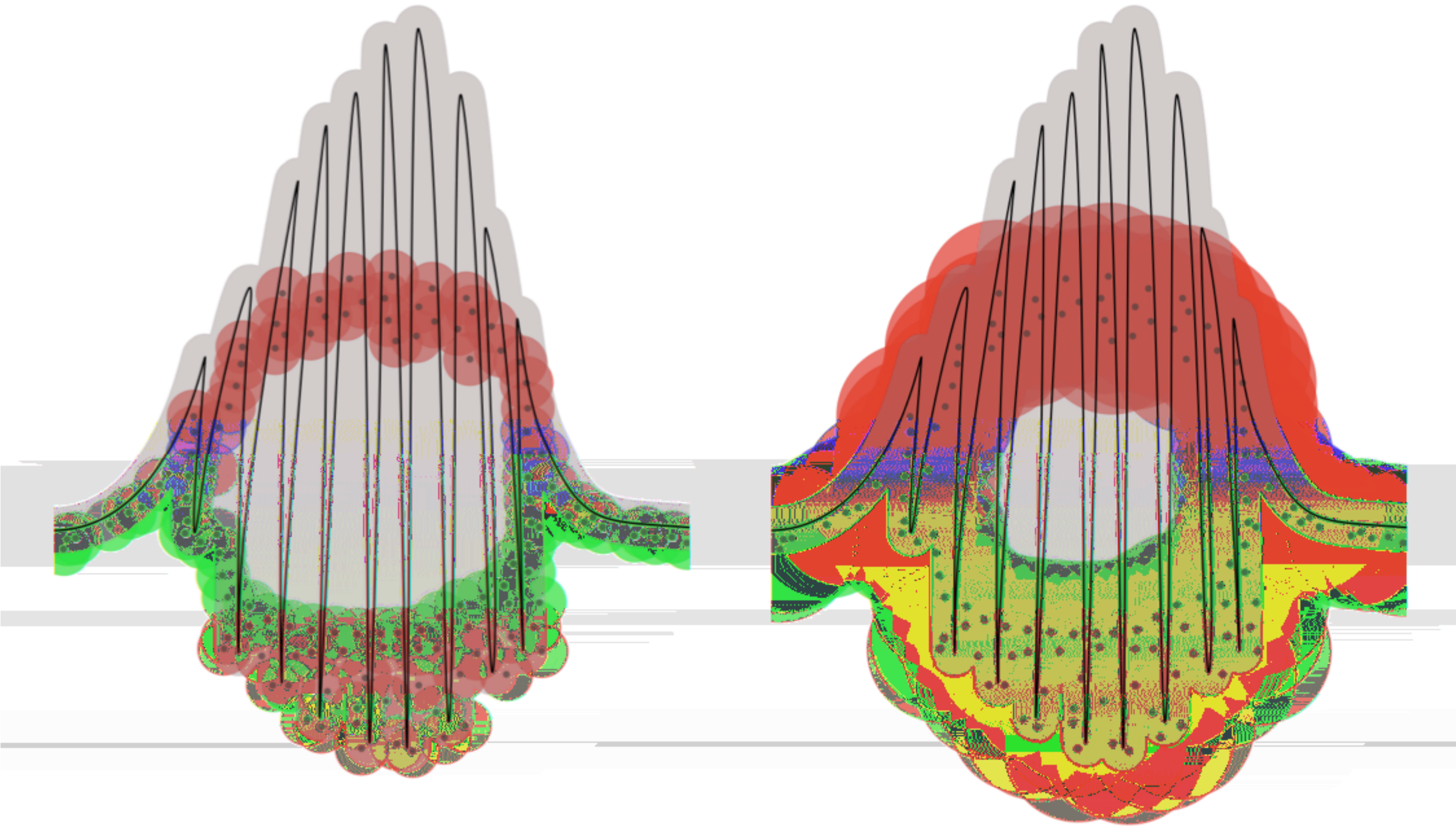
Lemma 2 *If $j_{\square}$ is an isomorphism, then $\phi_{\square}$ is surjective.*



Let $x \in P^{\alpha} \setminus Q^{3\alpha}$ so that $[x] \in \mathrm{im}\ i_{\square}$.
If $x \in \overline{\mathcal{B}^{2\alpha}}$, then $[x] \in \mathrm{im}\ \phi_{\square}$.
So, there is some $y \in \mathcal{B}$ s.t. $\|x - y\| \leq 2\alpha$.
By the triangle inequality, $y \in \overline{Q^{\alpha}}$ and so $y \in \overline{P^{\alpha}}$.
Thus, $[x] = 0$ in $\mathrm{H}_0(\overline{Q^{\alpha}}, \overline{P^{\alpha}})$ because $\partial(\overline{xy}) \equiv x$.

TCC Proof of Correctness III

Smoothness does matter in the de Silva-Ghrist proof.



TCC and WFS

Theorem 1 *Let $\mathcal{D} \subset \mathbb{R}^d$ be a locally contractible, compact set, and let $\mathcal{B}$ be the boundary of $\mathcal{D}$ with* $\mathrm{wfs}(\mathcal{B}) > 4\alpha$. *Let $P \subset \mathcal{D}^\alpha$, and let $Q = B^\alpha \cap P$. For any integer k, let h_k denote the homomorphism $h_k : \mathrm{H}_k(P^\alpha, Q^\alpha) \to \mathrm{H}_k(P^{3\alpha}, Q^{3\alpha})$ induced by inclusion. Then, the following two statements hold.*

1. *If $\mathcal{D} \subseteq P^\alpha$, then $\mathrm{im}\, h_k \cong \mathrm{H}_k(\mathcal{D}, \mathcal{B})$ for all integers k.*
2. *If $\mathrm{im}\, h_d \cong \mathrm{H}_d(\mathcal{D}, \mathcal{B})$, then $\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subset P^\alpha$.*

Almost, but not quite converses.

Certified Homology Inference

$$U_\beta = P \setminus \mathcal{B}^\beta = \{p \in P \mid d(p, \mathcal{B}) > \beta\}.$$

Lemma 3 *Suppose the sample $P \subset \mathcal{D}^\alpha$ is such that $\mathcal{D} \setminus \mathcal{B}^{2\alpha} \subseteq P^\alpha$ as asserted by the TCC. Let $\beta, \gamma, \varepsilon, \delta$ be constants such that $\varepsilon \geq \gamma \geq \alpha$ and $\beta \geq \varepsilon + \delta + \gamma$, we have If* $\mathrm{wfs}(\mathcal{B}) > \beta + \gamma$*, then*

$$\mathrm{rank}(\mathrm{H}_k(U_\beta^\gamma) \to \mathrm{H}_k(U_\delta^\varepsilon)) = \dim(\mathrm{H}_k(\mathcal{D})), \textit{ for all integers } k.$$

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$$\mathrm{rank}(\mathrm{H}_k(U_\beta^\gamma) \to \mathrm{H}_k(U_\delta^\varepsilon)) = \dim(\mathrm{H}_k(\mathcal{D})), \textit{ for all integers } k.$$

Key Idea:

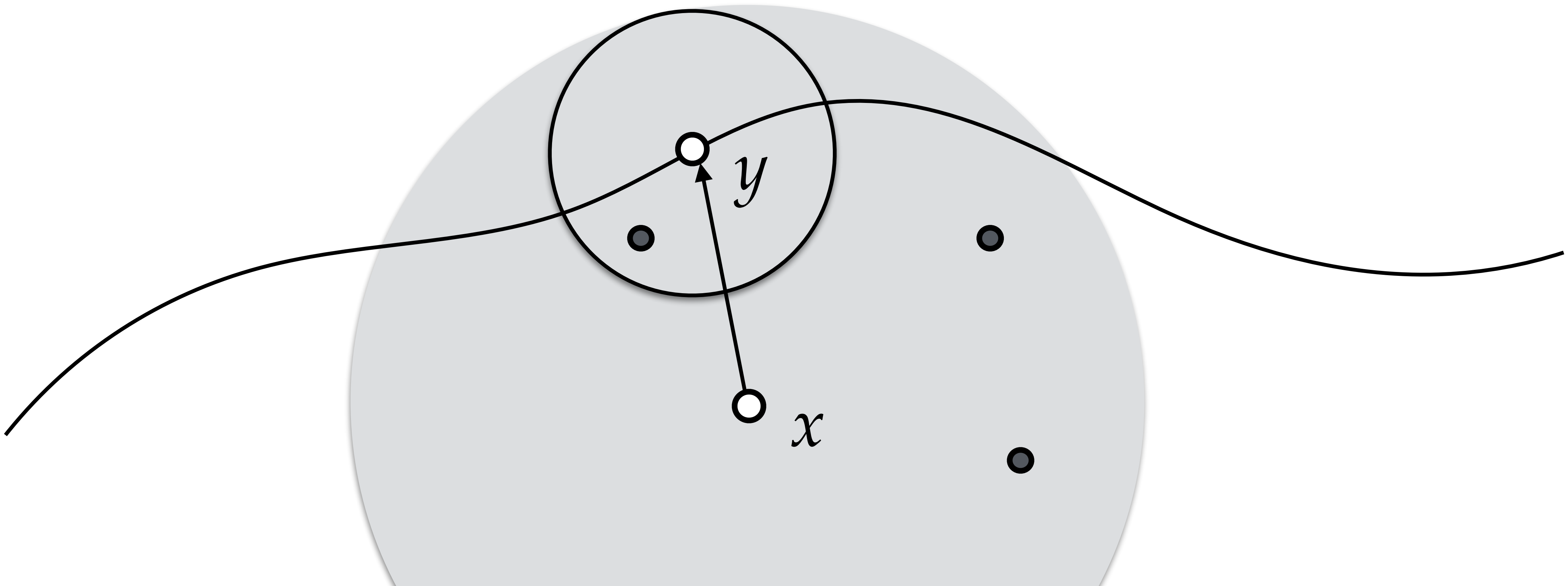
Use TCC to certify coverage assuming the number of connected components is known.

Then compute the higher Betti numbers by looking at the persistent homology of subsamples.

Throw out points too close to the boundary.

k-Coverage

Lemma 2 *If $j_{\square}$ is an isomorphism, then $\phi_{\square}$ is surjective.*



Let $x \in P_k^{\alpha} \setminus Q_k^{3\alpha}$ so that $[x] \in \text{im } i_{\square}$.

If $x \in \overline{\mathcal{B}^{2\alpha}}$, then $[x] \in \text{im } \phi_{\square}$.

So, there is some $y \in \mathcal{B}$ s.t. $\|x - y\| \leq 2\alpha$.

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Thus, $[x] = 0$ in $\mathrm{H}_0(\overline{Q_k^{\alpha}}, \overline{P_k^{\alpha}})$ because $\partial(\overline{xy}) \equiv x$.